Stability in volume comparison problems.

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A typical comparison problem for the volume of convex bodies asks whether inequalities

$$f_K(\xi) \le f_L(\xi), \qquad \forall \xi \in S^{n-1}$$

imply

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)$$

for any K, L from a certain class of origin-symmetric convex bodies in $\mathbb{R}^n$, where f_K is a certain geometric characteristic of K and Vol_n is the n-dimensional volume.

Busemann-Petty problem (1956); $f_K(\xi) = S_K(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp)$

Suppose K and L are origin symmetric convex bodies in $\mathbb{R}^n$ such that

$$S_K(\xi) \le S_L(\xi), \qquad \forall \xi \in S^{n-1}.$$

Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)?$$

Yes if $n \le 4$, no if $n \ge 5$; solution completed in the end of the 90's

Ball, Bourgain, Gardner, Giannopoulos, K., Larman, Lutwak, Papadimitrakis, Rogers, Schlumprecht, Zhang

Examples

Busemann-Petty problem (1956); $f_K(\xi) = S_K(\xi) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp)$

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Shephard's problem (1964); $f_K(\xi) = P_K(\xi) = \mathrm{Vol}_{n-1}(K|\xi^\perp)$

Suppose K and L are origin symmetric convex bodies in $\mathbb{R}^n$ such that

$$P_K(\xi) \le P_L(\xi), \qquad \forall \xi \in S^{n-1}.$$

Does it follow that

$$\mathrm{Vol}_n(K) \le \mathrm{Vol}_n(L)?$$

Yes if $n = 2$, no if $n \ge 3$; Petty and Schneider, independently

Modified Busemann-Petty problem; K., Yaskin, Yaskina (2006)

For two origin-symmetric infinitely smooth bodies K, L in $\mathbb{R}^n$ and $\alpha \in [n-4, n-1)$ the inequalities

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi), \qquad \forall \xi \in S^{n-1}$$

imply

$$\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L),$$

while for $\alpha < n-4$ this is not necessarily true.

Here Δ is the Laplace operator on $\mathbb{R}^n$ and the functions S_K and S_L are extended to homogeneous functions of degree -1 on the whole $\mathbb{R}^n$.

More examples

Modified Busemann-Petty problem; K., Yaskin, Yaskina (2006)

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Modified Shephard's problem; Yaskin (2008)

For $\alpha \in [n, n+1)$ the inequalities

$$(-\Delta)^{\alpha/2} P_K(\xi) \geq (-\Delta)^{\alpha/2} P_L(\xi), \qquad \forall \xi \in S^{n-1}$$

imply $\mathrm{Vol}_n(K) \leq \mathrm{Vol}_n(L)$, where the projection functions are extended to homogeneous functions of degree 1 on the whole $\mathbb{R}^n$. The latter result is no longer true for $\alpha < n$.

The Busemann-Petty problem for arbitrary measures; Zvavitch (2005)

Let f be an even continuous non-negative function on $\mathbb{R}^n$, and denote by μ the measure on $\mathbb{R}^n$ with density f. For every closed bounded set $B \subset \mathbb{R}^n$ define

$$\mu(B) = \int_B f(x)\ dx.$$

If $n \leq 4$, then for any convex origin-symmetric bodies K and L in $\mathbb{R}^n$ the inequalities

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp), \qquad \forall \xi \in S^{n-1}$$

imply

$$\mu(K) \leq \mu(L).$$

This is generally not true if $n \geq 5$ in the sense that for every strictly positive f there exist K, L providing a counterexample.

Stability

Let $\varepsilon > 0$. The inequalities

$$f_K(\xi) \le f_L(\xi) + \varepsilon, \qquad \forall \xi \in S^{n-1}$$

imply

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + C\varepsilon.$$

Stability and separation

Stability

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$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + C\varepsilon.$$

Separation

Let $\varepsilon > 0$. The inequalities

$$f_K(\xi) \le f_L(\xi) - \varepsilon, \qquad \forall \xi \in S^{n-1}$$

imply

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \mathrm{Vol}_n(L)^{\frac{n-1}{n}} - c\varepsilon.$$

Stability for sections

Suppose that $\varepsilon > 0$, K and L are origin-symmetric star bodies in $\mathbb{R}^n$, and K is an intersection body. If for every $\xi \in S^{n-1}$

$$S_K(\xi) \leq S_L(\xi) + \varepsilon,$$

then

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + \varepsilon.$$

Stability for sections and projections

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Stability for projections

Suppose that $\varepsilon > 0$, K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and L is a projection body. If for every $\xi \in S^{n-1}$

$$P_K(\xi) \leq P_L(\xi) + \varepsilon,$$

then

$$\text{Vol}_n(K)^{\frac{n-1}{n}} \leq \text{Vol}_n(L)^{\frac{n-1}{n}} + \sqrt{\frac{2\pi}{n}}\, R(L)\, \varepsilon,$$

where $R(L)$ is the normalized circumradius of L.

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$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} - \sqrt{\frac{2\pi}{n+1}}\, r(K)\varepsilon,$$

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then

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} - \frac{\varepsilon}{\sqrt{e}}.$$

The hyperplane problem.

Does there exist an absolute constant C such that for any origin-symmetric convex body K in $\mathbb{R}^n$

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le C \max_{\xi \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \xi^{\perp}).$$

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Dimension $n \le 4$

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le \frac{|B_2^n|^{\frac{n-1}{n}}}{\left|B_2^{n-1}\right|} \max_{\xi \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \xi^\perp).$$

Denote the constant by c_n; note that c_n is less than 1.

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Follows from the affirmative answer to the Busemann-Petty problem: if $\mathrm{Vol}_n(K) = |B_2^n|$, then it cannot happen that $\mathrm{Vol}(K \cap \xi^\perp) < \left|B_2^{n-1}\right|$ for every ξ, so

$$\frac{\max_{\xi \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \xi^\perp)}{\mathrm{Vol}_n(K)^{\frac{n-1}{n}}} \ge \frac{\left|B_2^{n-1}\right|}{\left|B_2^n\right|^{\frac{n-1}{n}}}.$$

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Stability for sections in dimensions $n \leq 4$

Suppose that $\varepsilon > 0$, K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, $n \leq 4$. If for every $\xi \in S^{n-1}$

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) \leq \mathrm{Vol}_{n-1}(L \cap \xi^\perp) + \varepsilon,$$

then

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \leq \mathrm{Vol}_n(L)^{\frac{n-1}{n}} + c_n \varepsilon.$$

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Switch K, L

$$\left| \mathrm{Vol}_n(K)^{\frac{n-1}{n}} - \mathrm{Vol}_n(L)^{\frac{n-1}{n}} \right| \le c_n \max_{\xi \in S^{n-1}} \left| \mathrm{Vol}_{n-1}(K \cap \xi^\perp) - \mathrm{Vol}_{n-1}(L \cap \xi^\perp) \right|.$$

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Put $L = \emptyset$:

$$\mathrm{Vol}_n(K)^{\frac{n-1}{n}} \le c_n \max_{\xi \in S^{n-1}} \mathrm{Vol}_{n-1}(K \cap \xi^\perp).$$

Hyperplane inequality for arbitrary measures, $n \leq 4$

Stability in Zvavitch's result

Let f be an even positive continuous function on $\mathbb{R}^n$, $2 \leq n \leq 4$, μ is the measure with density f, K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and $\varepsilon > 0$. Suppose that for every $\xi \in S^{n-1}$,

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp) + \varepsilon.$$

Then

$$\mu(K) \leq \mu(L) + \frac{n}{n-1} c_n \mathrm{Vol}_n(K)^{1/n} \varepsilon.$$

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Switch K, L

$$|\mu(K) - \mu(L)| \leq \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \left| \mu(K \cap \xi^\perp) - \mu(L \cap \xi^\perp) \right| \max \left(\mathrm{Vol}_n(K)^{\frac{1}{n}}, \mathrm{Vol}_n(L)^{\frac{1}{n}} \right).$$

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The constant is the best possible

Recall the inequality:

$$\mu(K) \leq \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^{\perp}) \ \mathrm{Vol}_n(K)^{1/n}.$$

The constant is the best possible

Recall the inequality:

$$\mu(K) \le \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) \, \mathrm{Vol}_n(K)^{1/n}.$$

The case of (asymptotic) equality

Let $K = B_2^n$ and, for every $j \in N$, let f_j be a non-negative continuous function on $[0,1]$ supported in $(1-\frac{1}{j}, 1)$ and such that $\int_0^1 f_j(t)dt = 1$. Let μ_j be the measure on $I\!R^n$ with density $f_j(|x|_2)$, where $|x|_2$ is the Euclidean norm. Then

$$\lim_{j\to\infty} \frac{\mu_j(B_2^n)}{\max_{\xi\in S^{n-1}} \mu_j(B_2^n \cap \xi^\perp) \, \mathrm{Vol}_n(B_2^n)^{1/n}} = \frac{n}{n-1} c_n.$$