Valuations on Sobolev Spaces

Monika Ludwig

Technische Universität Wien

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Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|\, dx \geqslant v_n^{1/n} \left|f\right|_{n/(n-1)}$$

- $f \in W^{1,1}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n), |\nabla f| \in L^1(\mathbb{R}^n)\}$
- $\|x\|$ Euclidean norm of $x \in \mathbb{R}^n$
- $\left|f\right|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dx\right)^{1/p}$
- v_n volume of n-dimensional unit ball

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- v_n volume of n-dimensional unit ball
- Equality for indicator functions of balls
- Equivalent to Euclidean isoperimetric inequality
- Federer & Fleming 1960, Maz′ya 1960

General Sobolev Inequality

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx \geqslant v_n^{1/n}\, |f|_{n/(n-1)}$$

- $f \in W^{1,1}(\mathbb{R}^n)$
- $\mathcal{K}_c^n$ origin-symmetric convex bodies (compact convex sets) in $\mathbb{R}^n$
- $K \in \mathcal{K}_c^n$ with $V(K) = v_n$
- $K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1 \text{ for all } y \in K\}$ polar body of K
- $\|\cdot\|_L$ norm with unit ball L

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- $\|\cdot\|_L$ norm with unit ball L
- Equality for $f = \mathbb{1}_K$
- Equivalent to Minkowski inequality
- Gromov 1986

Optimal Sobolev Inequality

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Question (Lutwak, Yang & Zhang 2006)

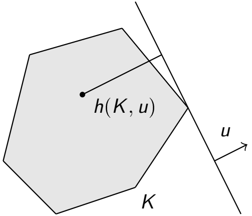
For given $f \in W^{1,1}(\mathbb{R}^n)$, which convex body K (of volume v_n) minimizes

$$\frac{1}{n}\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx?$$

Which norm is optimal?

Origin-symmetric Convex Bodies $\mathcal{K}_c^n$

- Support function $h(K,\cdot): \mathbb{R}^n \to \mathbb{R}$



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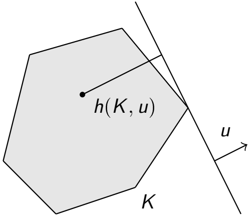

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  - $h(K, u + v) \leqslant h(K, u) + h(K, v)$
    sublinear and even

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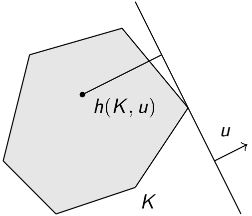

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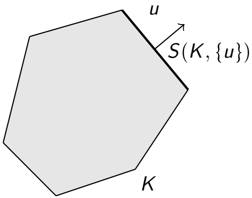
- Surface area measure $S(K,\cdot):\mathcal{B}(S^{n-1}) \to [0,\infty)$



  - $S(K,\omega) = \mathcal{H}^{n-1}(\{x \in \mathrm{bd}\, K : n(K,x) \in \omega\})$
  - $n(K,x)$ outer unit normal vector to K at $x \in \mathrm{bd}\, K$

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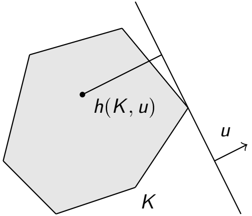

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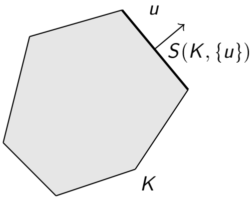
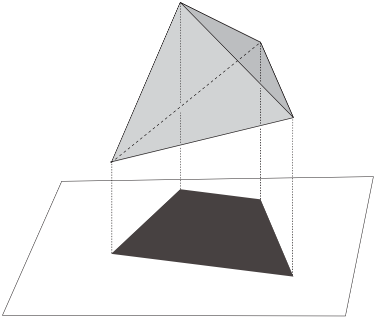


  - $S(K,\omega) =$
    $\mathcal{H}^{n-1}(\{x \in \mathrm{bd}\, K : n(K,x) \in \omega\})$
  - $n(K,x)$ outer unit normal vector to K at $x \in \mathrm{bd}\, K$
  - $S(K,\cdot)$ even measure,
    not concentrated on a great sphere

Projection Body, ΠK, of K



- $u^\perp$ hyperplane orthogonal to u
- $K|u^\perp$ projection of K to $u^\perp$
- V_{n-1} $(n-1)$-dimensional volume

Definition (Minkowski 1901)

$$h(\Pi K, u) = V_{n-1}(K|u^\perp) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \, dS(K, v)$$

Optimal Sobolev Body

Definition (LYZ 2006)

For $f \in W^{1,1}(\mathbb{R}^n)$, the optimal Sobolev body, $\langle f \rangle$, of f is defined as the unique origin-symmetric convex body such that

$$\int_{S^{n-1}} g(u)\, dS(\langle f \rangle, u) = \int_{\mathbb{R}^n} g(\nabla f(x))\, dx$$

for all even and positively 1-homogeneous functions $g \in C(\mathbb{R}^n)$.

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for all even and positively 1-homogeneous functions $g \in C(\mathbb{R}^n)$.

Theorem (LYZ 2006)

For $f \in W^{1,1}(\mathbb{R}^n)$, the infimum over all origin-symmetric convex bodies K of volume $V(K) = v_n$ over

$$\int_{\mathbb{R}^n} \|\nabla f(x)\|_{K^*}\, dx$$

is attained if and only if K is a dilate of $\langle f \rangle$.

Affine Sobolev inequality

Theorem (Gaoyong Zhang (JDG 1999))

For $f \in W^{1,1}(\mathbb{R}^n)$,

$$\frac{1}{n}\int_{S^{n-1}}\Big(\int_{\mathbb{R}^n}|u\cdot\nabla f(x)|\,dx\Big)^{-n}du \leqslant \Big(\frac{v_n}{2\,v_{n-1}}\Big)^n |f|_{\frac{n}{n-1}}^{-n}.$$

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- Left hand side is multiple of $V(\Pi^* \langle f \rangle)$.
- Affine isoperimetric inequality which implies Sobolev inequality
- Equality for indicator functions of ellipsoids.

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- Left hand side is multiple of $V(\Pi^*\langle f\rangle)$.
- Affine isoperimetric inequality which implies Sobolev inequality
- Equality for indicator functions of ellipsoids.
- $\Pi\langle f\rangle$ Sobolev-Zhang body of f

$$h(\Pi\langle f\rangle, v) = \frac{1}{2}\int_{\mathbb{R}^n}|v\cdot\nabla f(x)|\,dx$$

Valuations on Convex Bodies

- $\mathcal{K}^n$ space of convex bodies (compact convex sets) in $\mathbb{R}^n$
- $\langle \mathbb{A}, + \rangle$ Abelian semigroup
- A function $\mathrm{z} : \mathcal{K}^n \to \mathbb{A}$ is a *valuation* $\iff$

$$\mathrm{z}(K) + \mathrm{z}(L) = \mathrm{z}(K \cup L) + \mathrm{z}(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$.

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- Theory of valuations:

Blaschke 1937, **Hadwiger** 1949, Sallee 1966, Schneider 1971, Groemer 1972, McMullen 1977, Goodey & Weil 1984, Betke & Kneser 1985, Klain 1995, Alesker 1999, Ludwig 1999, Reitzner 1999, Dulio & Peri 2000, R. Schuster 2002, Hug 2005, Fu 2006, Bernig 2006, Haberl 2006, F. Schuster 2006, Abardia 2010, Wannerer 2010, Parapatits 2011, ...

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- $(\mathcal{L}, \vee, \wedge)$ lattice
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- Valuations on L^p stars: Dan Klain (AIM 1996, 1997)
- Valuations on L^p spaces: Andy Tsang (PhD Thesis NYU Poly 2010, IMRN 2010, . . .)
- Valuations on Orlicz spaces: Hassane Kone (2011+)
- Valuations on $\mathrm{BV}(\mathbb{R}^n)$: Tuo Wang (2011+)
- Valuations on Sobolev spaces: L. (AIM 2011, AJM 2011+)

Sobolev Space $W^{1,1}(\mathbb{R}^n)$

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- $\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is **affinely contravariant** $\Leftrightarrow$
  z is GL(n) contravariant, translation invariant and homogeneous

Valuations on Sobolev Spaces

- $\Pi\langle\cdot\rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is affinely contravariant (LYZ 2006).
- $\Pi\langle\cdot\rangle : W^{1,1}(\mathbb{R}^n) \to \mathcal{K}_c^n$ is *continuous*.
- $\Pi\langle\cdot\rangle : W^{1,1}(\mathbb{R}^n) \to \langle\mathcal{K}_c^n, +\rangle$ is a valuation.

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Theorem (L. (AJM, to appear))

$\mathrm{z} : W^{1,1}(\mathbb{R}^n) \to \langle\mathcal{K}_c^n, +\rangle$ *is a continuous, affinely contravariant valuation*

$$\Longleftrightarrow$$

$\exists\, c \geqslant 0$*:*

$$\mathrm{z}(f) = c\,\Pi\langle f\rangle$$

for every $f \in W^{1,1}(\mathbb{R}^n)$*.*

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  $S(K \# L, \cdot) = S(K, \cdot) + S(L, \cdot)$ Blaschke addition

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Sketch of the Proof

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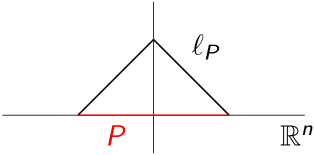
- $L^{1,1}(\mathbb{R}^n) \subset W^{1,1}(\mathbb{R}^n)$ piecewise linear continuous functions
- $P^{1,1}(\mathbb{R}^n) \subset L^{1,1}(\mathbb{R}^n)$ 'linear elements'



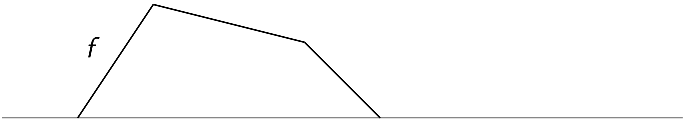
$\ell_P \in L^{1,1}(\mathbb{R}^n)$

$P \in \mathcal{P}_0^n$

$\mathcal{P}_0^n$ convex polytopes in $\mathbb{R}^n$ containing the origin in their interiors

Sketch of the Proof, cont.

- $f \in L^{1,1}(\mathbb{R}^n)$



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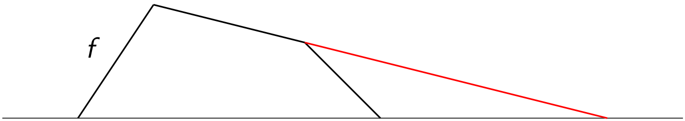

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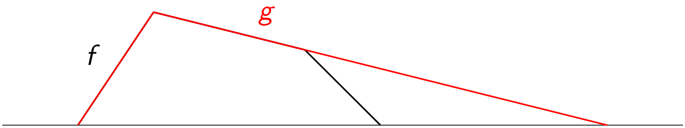

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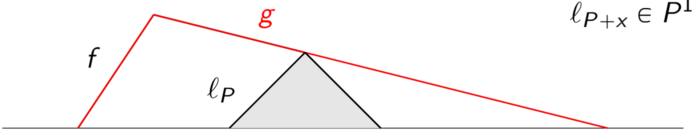

$\ell_{P+x} \in P^{1,1}(\mathbb{R}^n)$

Sketch of the Proof, cont.

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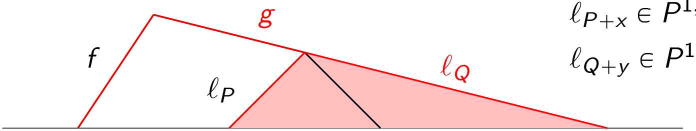

$\ell_{P+x} \in P^{1,1}(\mathbb{R}^n)$

$\ell_{Q+y} \in P^{1,1}(\mathbb{R}^n)$

Sketch of the Proof, cont.

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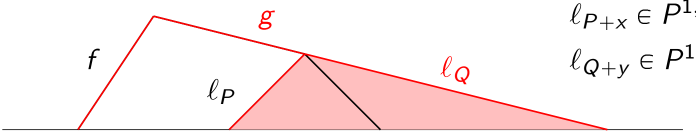

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$$f \vee \ell_Q = g, \; f \wedge \ell_Q = \ell_P$$
$$\mathrm{z}(f) + \mathrm{z}(\ell_Q) = \mathrm{z}(g) + \mathrm{z}(\ell_P)$$

Sketch of the Proof, cont.

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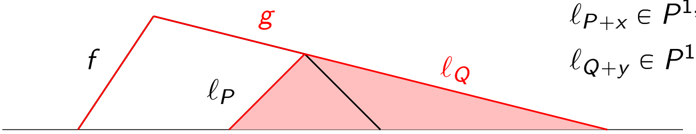

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$$\mathrm{z}(f) + \mathrm{z}(\ell_Q) = \mathrm{z}(g) + \mathrm{z}(\ell_P)$$

- $\mathrm{Z} : \mathcal{P}_0^n \to \langle \mathcal{K}_c^n, + \rangle, \quad \mathrm{Z}(P) = \mathrm{z}(\ell_P)$
  Classification of $\mathrm{SL}(n)$ contravariant valuations on $\mathcal{P}_0^n$
  (L. (JDG 2010))

Thank you !!!