The convex intersection body of a convex body

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Cortona, June 2011

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In general, $I(L)$ and $C(L)$ are not convex bodies, although they are identical and convex when L is centrally symmetric. This follows from Brunn-Minkowski theorem and Busemann's theorem (1950). We define a new convex body associated with L, generalizing $I(L)$ and $C(L)$, the **convex intersection body $CI(L)$ of L** by its radial function

$$\rho_{CI(L)}(u) = \min_{z \in P_u(L^{*g(L)})} \mathrm{vol}_{n-1}\Big(\big[P_u(L^{*g(L)})\big]^{*z}\Big). \tag{1}$$

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$g(L)$ is the centroid of L, P_u denotes the orthogonal projection from $\mathbb{R}^n$ onto $u^\perp$, and if $E \subset R^n$ is an affine subspace, $M \subset E$ and $z \in E$, $M^{*z} = \{y \in E; \langle y - z, x - z\rangle \leq 1 \text{ for every } x \in M\}$.

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As we shall see, this convex intersection body of L is actually convex !

Notation

If L a convex set in $\mathbb{R}^n$, let $[L]$ be the affine space spanned by L and $z \in \mathrm{relint}(L)$, **the polar body of L with respect to z** is

$$L^{*z} = \{y \in [L]; \langle y - z, x - z\rangle \leq 1 \text{ for all } x \in L\} = \big((L - z)^* + z\big) \cap [L],$$

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The Santaló point $s(L)$ is the unique point $z \in \mathrm{int}(L)$ which is the centroid of L^{*z} : one has $z = s(L)$ iff $g(L^{*z}) = z$, where $g(M)$ denotes the centroid of M in $[M]$.

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Theorem

If K is a convex body in $\mathbb{R}^n$. Define $N_K : \mathbb{R}^n \to \mathbb{R}_+$ by :

$$N_K(u) = \frac{1}{\mathrm{vol}\big((\mathrm{P_u K})^{*s}\big)} = \frac{1}{\min_{z \in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)} \text{ for } u \in S^{n-1},$$

and $N_K(ru) = rN_K(u)$ for $r \geq 0$. Then N_K is a norm on $\mathbb{R}^n$.

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Definition. The preceding theorem associates to any convex body K a centrally symmetric convex body $J(K)$ in $\mathbb{R}^n$ defined by

$$J(K) = \{x \in \mathbb{R}^n ; N_K(x) \leq 1\}.$$

Its radial function is $r_{J(K)}(u) = \mathrm{vol}\big((\mathrm{P_u K})^{*s}\big)$.

Why is $J(K)$ convex ?

I recall some facts.

Definition. Let $v \in S^{n-1}$, $B \subset \mathbb{R}^n$ bounded and $V : B \to \mathbb{R}$ bounded. The **shadow system (L_t), $t \in [0,1]$, of convex bodies in $\mathbb{R}^n$, with direction v, basis B and speed V**, is the family of convex bodies

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The following, due to Shephard, was also used by Campi-Gronchi.

Proposition Let K be a convex body in $\mathbb{R}^n$. Then, for $u, v \in S^{n-1}$, such that $\langle u, v \rangle = 0$, the family $L_t = \Pi_{u+tv, u^\perp} K$, $t \in \mathbb{R}$, is a shadow system of convex bodies in $u^\perp$, in the direction v.

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Here $\Pi_{x,y^\perp} : \mathbb{R}^n \to y^\perp$ denotes *the linear projection onto $y^\perp$ with direction parallel to $x \notin y^\perp := \{z \in \mathbb{R}^n; \langle z, y\rangle = 0\}$*

The converse statement

Observe that the converse statement of the last proposition is true :

Every shadow system L_t in $\mathbb{R}^n$ can be seen as $L_t = \Pi_{u+tv,u^\perp}(K)$ for some convex body $K \subset \mathbb{R}^{n+1}$ and $u, v \in S^n$.

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$$\Pi_{u+tv,u^\perp}(b - V(b)u) = b + tV(b)v \in u^\perp$$

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$$\Pi_{u+tv,u^\perp} M = L_t.$$

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The following result was proved by S. Reisner and M. Meyer (07):

Theorem

Let $t \in [0,1] \to L_t$ be a shadow system in $\mathbb{R}^n$; define

$$\phi(t) = \frac{1}{\mathrm{vol}\big((\mathrm{L_t})^{*\mathrm{s}}\big)} = \frac{1}{\min_z \mathrm{vol}\big((\mathrm{L_t})^{*\mathrm{z}}\big)} \ .$$

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We use also :

Lemma Suppose that $N : \mathbb{R}^n \to \mathbb{R}$ satisfies $N(x) > 0$ for $x \neq 0$, $N(\alpha x) = |\alpha| N(x)$ for all $\alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$ and that for all $u, v \in S^{n-1}$ with $\langle u, v \rangle = 0$, $t \mapsto N(u + tv)$ is convex. Then N is a norm on $\mathbb{R}^n$.

Proof of the convexity of $J(K)$.

We want to prove the following :

Theorem For a convex body K, $r_{J(K)}(u) = \min_{z \in u^\perp} \mathrm{vol}\big((\mathrm{P_u K})^{*z}\big)$ is the radial function of a centrally symmetric convex body.

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Hence

$$\min_{z\in\{u+tv\}^\perp} \mathrm{vol}\big((\mathrm{P_{u+tv}K})^{*z}\big) = \sqrt{1+t^2}\,\min_{z\in u^\perp} \mathrm{vol}\big((\Pi_{\mathrm{u+tv,u^\perp}}\mathrm{K})^{*z}\big)\big)$$

Proof of the theorem about $J(K)$.

It follows that

$$N(u+tv) = \frac{|u+tv|}{\min_{z\in\{u+tv\}^\perp} \mathrm{vol}\big((\mathrm{P_{u+tv}K})^{*z}\big)\big)}$$

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By the proposition, $t \to \Pi_{u+tv,u^\perp} K$ is a shadow system. Thus by the last theorem, $g_{u,v}$ is convex.

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2) Let $J(K) = \{x \in \mathbb{R}^n; N_K(x) \leq 1\}$. One has $J(K+x) = J(K)$ and for all linear isomorphism A, $J\big((AK)\big) = |\det(A)|\,(A^*)^{-1}\big(J(K)\big)$.

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3) If $n = 2$ and if R is the rotation with angle $\pi/2$ in $\mathbb{R}^2$, then

$$\mathrm{vol}(\mathrm{P_u K}) = \mathrm{h_K(Ru)} + \mathrm{h_K(-Ru)} = \mathrm{h_K(Ru)} + \mathrm{h_{-K}(Ru)},$$

so that $J(K) = \frac{1}{4}R(K-K)$.

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The convex intersection bodies $IC(L,z)$ of a convex body L.

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$$\rho_{I(L,z)}(u) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L}; \langle \mathrm{x}-\mathrm{z}, \mathrm{u}\rangle = 0\}\big) = \mathrm{vol}\big(\mathrm{L} \cap (\mathrm{z}+\mathrm{u}^\perp)\big).$$

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$$CI(L,z) = J(L^{*z}) \ .$$

If $z = g(L)$, the centroid of L, we set $CI(L) = CI\big(L, g(L)\big)$.

The radial function of $CI(L, z)$ is thus given for $u \in S^{n-1}$ by

$$\rho_{CI(L,z)}(u) = \min_{x \in u^\perp} \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L}^{*z})\big)^{*x} \Big) = \mathrm{vol}\Big(\big(\mathrm{P_u}(\mathrm{L}^{*z})\big)^{*s} \Big).$$

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In view of the first theorem, one has

Theorem

Let L be a convex body. Then for every $z \in \mathrm{int}(L)$, the convex intersection body $CI(L,z)$ of L with respect to z is a centrally symmetric convex body such that $CI(L,z) \subset I(L,z)$.

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Proposition $CI(L,z) = I(L,z)$ iff L is centrally symmetric about z.

This follows from the following lemma:

Lemma

Let L be a convex body and $z \in L$. Then z is the centroid of every hyperplane section of L through itself iff $L - z$ is centrally symmetric.

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Fix some $z_0 \in \mathrm{int}(L)$, $z_0 \neq z$. Define $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(y) = \mathrm{vol}\big(\{\mathrm{x} \in \mathrm{L} - \mathrm{z}_0; \langle \mathrm{x}, \mathrm{y} \rangle \geq 1\}\big).$$

By Meyer-Reisner (89), F is C^1 on $\{F > 0\} = \mathbb{R}^n \setminus \{0\}$ and for $y \neq 0$

$$\nabla F(y) = \langle \nabla F(y), y \rangle \big(g(\{x \in L; \langle x - z_0, y \rangle = 1\}) - z_0 \big).$$

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$$F(y') - F(y) = \int_0^1 \langle y' - y, \nabla F\big((1 - t)y + ty'\big)\rangle dt = 0.$$

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Additional comments and some open problems

The bodies $C(L)$ and $I(L,z)$ are not in general convex : $C(L)$ is always convex only for $n \leq 3$ (Meyer) and Brehm proved that if Δ_n is a simplex in $\mathbb{R}^n$, $n \geq 4$, $C(\Delta_n)$ is not convex.

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$$\frac{d}{\mathrm{vol(L)}^{\frac{3}{2}}}\Big(\int_{L-g(L)} \langle x,u\rangle^2 dx\Big)^{\frac{1}{2}} \leq \frac{1}{\max_t \mathrm{vol}\big(\mathrm{L}\cap(\mathrm{tu}+\mathrm{u}^\perp)\big)} = \rho_{C(L)}(u)$$

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For centrally symmetric L, this was proved by Hensley (Ball for sharp constants), and in the general case by Schütt (Fradelizi for sharp constants) (see also Milman-Pajor).

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An other formulation of this problem is the following : Let K be a convex body in $\mathbb{R}^n$ with Santaló point is at 0. Does there exist an absolute constant $C > 0$, independent on n and K such that

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Or, given a convex $M \subset u^\perp$, with Santaló point $s(M)$, and a convex body K in $\mathbb{R}^n$, with Santaló point $s(K)$, such that $P_u K = M$, does

$$\mathrm{vol}(\mathrm{M}^{*\mathrm{s(M)}}) \geq \mathrm{Cvol}(\mathrm{M}^{*\mathrm{P_u s(K)}})$$

for some universal constant $C > 0$?

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If one could prove that in this situation, for some universal constant $c > 0$, it is true that $P_u s(K) - s(M) \in \frac{c}{n}\big(M - s(M)\big)$, If one could prove that in this situation, for some universal constant $c > 0$, it is true that $P_u s(K) - s(M) \in \frac{c}{n}\big(M - s(M)\big)$, then an affirmative answer could be given, using the following lemma :

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Lemma

Let V be a convex body in $\mathbb{R}^n$ and $x, y \in \mathrm{int}(V)$. Then

$$(1 - ||x - y||_{V-y})^n \ \mathrm{vol}(\mathrm{V}^{*\mathrm{x}}) \leq \mathrm{vol}(\mathrm{V}^{*\mathrm{y}}) \leq \frac{\mathrm{vol}(\mathrm{V}^{*\mathrm{x}})}{\big(1 - ||\mathrm{y} - \mathrm{x}||_{\mathrm{V}-\mathrm{x}}\big)^{\mathrm{n}}}$$

It is known (see Milman-Pajor) that for some affine mapping $A : \mathbb{R}^n \to \mathbb{R}^n$, $M := AL$ is *isotropic*, that is satisfies $\mathrm{vol(M)} = 1$ and

$$\Big(\int_{M-g(M)} \langle x, u\rangle^2 dx\Big)^{\frac{1}{2}} = c_M \text{ for all } u \in S^{n-1}.$$

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Open problem 3. Let Δ_n be a simplex in $\mathbb{R}^n$ with $g(\Delta_n) = 0$. Is there a constant c such that for every $n \geq 2$ and every $u \in S^{n-1}$

$$\mathrm{vol(\Delta_n \cap u^\perp) \leq c\,vol\Big(\big(P_u(\Delta_n^{*g})\big)^{*s} \Big) = c\,vol\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s} \Big)} \text{ ?}$$

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$$\mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big) = \frac{1}{n^{n-1}} \mathrm{vol}\big((P_u\Delta_n)^{*s}\big)\,.$$

Let $e_1, \ldots, e_{n+1}$, $|e_i| = 1$, be the vertices of Δ_n so that $0 = e_1 + \cdots + e_{n+1}$ and for $1 \leq i \neq j \leq n+1$, $\langle e_i, e_j \rangle = -\frac{1}{n}$.

Additional comments and some open problems

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Fact. *Let* $A \subset \{1, \ldots, n+1\}$ *satisfy* $1 \le k := \mathrm{card}(A) \le n$. *Define*

$$u_A = \frac{\sum_{i \in A} e_i}{|\sum_{i \in A} e_i|} = \sqrt{\frac{n}{k(n+1-k)}} \sum_{i \in A} e_i \in S^{n-1}.$$

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We get thus:

Proposition For every $A \subset \{1, \dots, n+1\}$, with $1 \le \mathrm{card}(A) \le n$, one has : $||u_A||_{CI(\Delta_n, 0)} = ||u_A||_{I(\Delta_n, 0)}$.

Additional comments and some open problems

When $u^\perp \cap \Delta_n$ is a simplex, one can also conclude :

Proposition Let $u \in S^{n-1}$, and if $u = \sum_{i=1}^{n+1} u_i e_i \in S^{n-1}$ with $\sum_{i=1}^{n+1} u_i = 0$ and $u_1, \dots, u_n \geq 0 > u_{n+1}$, then $u^\perp \cap \Delta_n$ is a simplex and

$$\rho_{I(\Delta_n,0)(u)} = \mathrm{vol}(\Delta_n \cap u^\perp) = \frac{1}{(n-1)!} \frac{(n+1)^{\frac{n+1}{2}}}{n^{\frac{n}{2}-1}} \frac{1}{\prod_{i=1}^n (u_i + \sum_{j=1}^n u_j)}$$

and

$$\rho_{CI(\Delta_n,0)(u)} = \mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0} \big)^{*s} \Big) = \frac{1}{(n-1)!} \frac{n^{\frac{n}{2}+1}}{(n+1)^{\frac{n+1}{2}}} \frac{1}{\sum_{i=1}^n u_i}.$$

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$$\rho_{CI(\Delta_n,0)(u)} = \mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big) = \frac{1}{(n-1)!} \frac{n^{\frac{n}{2}+1}}{(n+1)^{\frac{n+1}{2}}} \frac{1}{\sum_{i=1}^n u_i}.$$

Thus $CI(\Delta_n, 0)$ has $2n+2$ small faces around $u = \pm e_i$, $1 \leq i \leq n+1$. It is easy to check that for such directions $u \in S^{n-1}$ one has

$$1 \leq \frac{\mathrm{vol}(\Delta_n \cap u^\perp)}{\mathrm{vol}\Big(\big((\Delta_n \cap u^\perp)^{*0}\big)^{*s}\Big)} \leq \frac{e}{2}.$$

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THE END