Convexity and mass transport methods for solving problems in optics

Vladimir Oliker
Emory University, Atlanta
oliker@mathcs.emory.edu

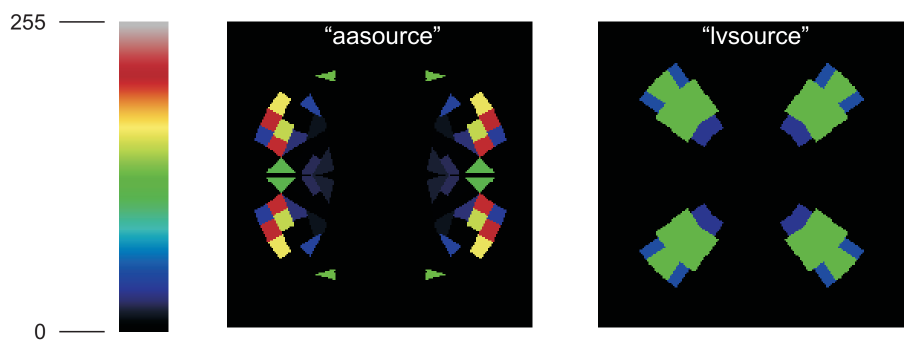
Workshop on Convex Geometry - Analytic Aspects
Cortona, Italy
June, 2011

Is it feasible to generate these patterns by reflective or refractive optics reshaping the irradiance distribution of a laser beam?

Irradiance distributions for mask illumination

Our source functions generally have 4-fold symmetry

Field shown is ~ 12 mm square



Is it feasible to generate these patterns by reflective or refractive laser beam shaping?

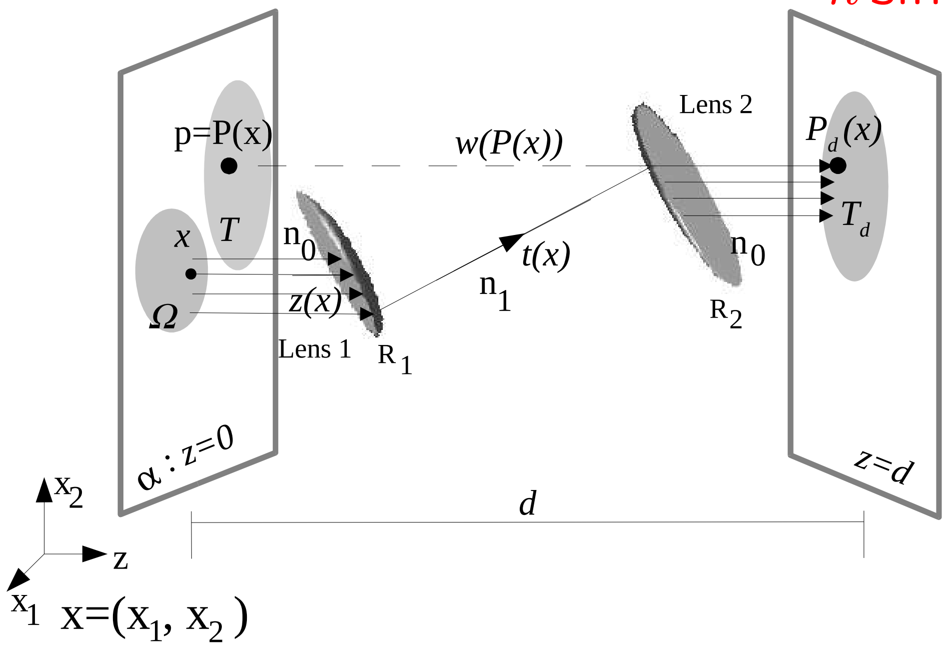
Input can be assumed to be uniform rectangular distribution with dimensions ~ 3 mm x 6 mm

Input and out are collimated, propagate in same direction

John Hoffnagle 16 May 2008

Applications: Computing lithography,...

$$n \sin\theta_i = \sin\theta_r \left(n = \frac{n_0}{n_1}\right)$$



Problem: Determine R_1 and R_2 such that for given refraction indices n_0, n_1, the incoming plane wave of cross-section $\bar{\Omega}$ with intensity distribution $I(x)$ is transformed into a plane wave irradiating at a given $\bar{T}_d$ with prescribed intensity distribution $L(p)$.

Problems of this type with lenses/mirrors have been usually solved under **a priori assumption of rotational/rectangular symmetry**.

Applications: materials processing (welding, cutting, drilling), energy concentrators, illumination, antennas, computing lithography, optical data/image processing, laser weapons, medicine (skin treatment, corneal surgery), planet detection, ...

In some important applications (lithography, illumination,...) the **assumption of rotational/rectangular symmetry** is overly restrictive.

Our goal:

Get rid of the symmetry assumption and design freeform lenses!

The focus of this talk is on freeform two-lens systems.

Some of the earlier (related) work on rotationally symmetric lenses:
B.R. Frieden ('65), J.L. Kreuzer ('69), P.W. Rhodes & D.L. Shealy ('80), W. Jiang, D.L. Shealy & J.C. Martin ('93), J. A. Hoffnagle & C.M. Jefferson ('00–'05).

The two-lens system designed by Hoffnagle & Jefferson was fabricated by QED Technologies ('03?). The authors received the 2003 Kingslake Medal and Prize for this work.

Previous (most relevant) work on freeform lenses:

H. Ries - J.Muschaweck, 2002 (no details),

J. Rubinstein - G. Wolansky, 2007-2008 (single lens, $0 < n < 1$), - Weighted least action,

V. Oliker, 2005- (two-lens and single lens systems, $n > 1$, $0 < n < 1$), - Geometric methods

Our mathematical framework is applicable in many other optics problems. This, in turn, can be traced to **H. Minkowski, A.D. Aleksandrov, A. V. Pogorelov**.

Current work has appeared online in

V. Oliker, Arch. Rational Mech. Anal. 2011.

Our main claims are:

- Freeform lenses can be designed under very general assumptions.
- Analytically, these problems can be formulated as:
  (a) PDE's of Monge-Ampère type, **(b)** Variational problems
- Two designs are available for the same data; one of them always consists of a concave and convex lenses.
- Practical computational approaches exist for calculating the solutions

Part I. FROM OPTICS TO PDE

Assuming **the geometrical optics approximation**, the following three laws are used to derive the equations for functions describing the lens surfaces:

- **Snell's (the refraction) law**
- **Conservation of Energy Along Infinitesimal Tubes of Rays**
- **Constancy of the Optical Path Length (OPL)**

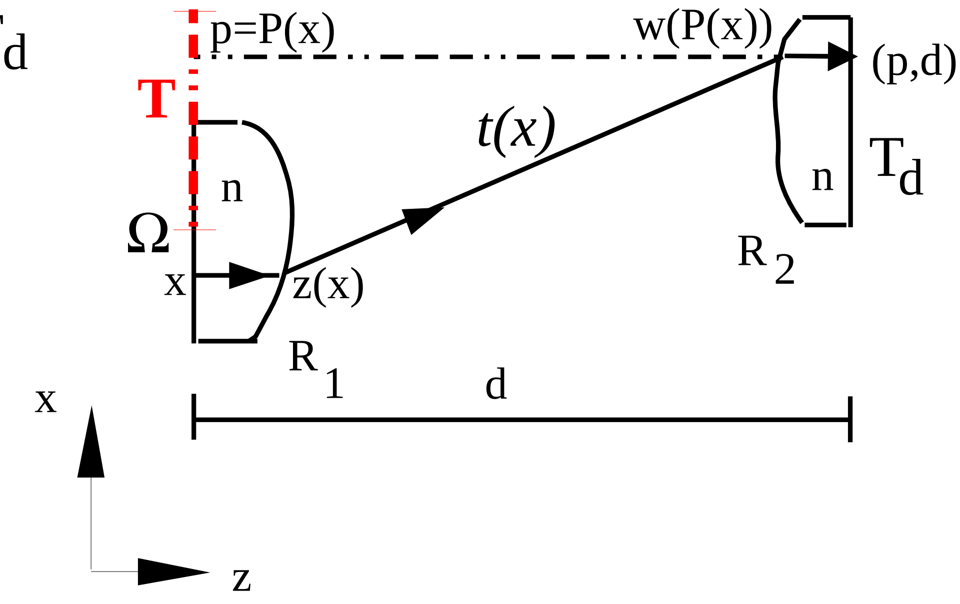
Notations:

Optical path length (OPL): $l = nz(x)+t(x)+n[d-w(P(x)]=\text{const}$

Reduced OPL: $\beta := l-nd = n[z(x)-w(p)]+\sqrt{(x-p)^2+[w(p)-z(x)]^2}$

$T = \text{proj}_{\{z=0\}} T_d$

$n = n_0/n_1$



Let $(R_1, R_2) = (z(x), w(p))$**. Put** $M(z) := \sqrt{1 + (1 - n^2)|\nabla z|^2}$.

The refraction law gives the refracted direction at the lens surface R_1**:**

$$\omega(x) = n\mathbf{k} + \frac{\mathbf{-n + M(z(x))}}{\sqrt{1 + |\nabla z(x)|^2}}\mathbf{N(x)}.$$

The refraction law and $OPL = const$ **give the refractor map:**

$$P(x) = x - \frac{\beta \nabla z(x)}{M(z(x))} : \bar{\Omega} \to \bar{T}$$

The energy conservation law:

$$L(P(x))|J(P(x)| = I(x), \quad (J \text{ is the Jacobian}).$$

Due to constancy of the OPL, **the second lens is given by**

$$w(P(x)) = -\frac{\beta}{n^2 - 1}\left[n + \frac{1}{M(z(x))}\right] + z(x).$$

The PDE problem

For bounded planar regions $\Omega, T \subset \alpha$, input intensity I, defined on $\bar{\Omega}$, and output intensity L, defined on $\bar{T}$, **find** $z \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that, the map

$$P(x) = x - \frac{\beta \nabla z(x)}{M(z(x))} : \bar{\Omega} \to \bar{T} \text{ is onto,}$$

and

$$L(P) \frac{\det\left\{M(z)\left[\mathrm{Id} + (1-n^2)\nabla z \otimes \nabla z\right] - \beta\,\mathrm{Hess}(z)\right\}}{M^{N+2}(z)} = I \text{ in } \Omega.$$

This PDE is of Monge-Ampère type (fully nonlinear).

BACK TO DESCARTES!

Part II. FROM PDE's TO GEOMETRY. WEAK SOLUTIONS

Refracting properties of quadrics. Main Lemma.

1. The constancy of the (reduced) optical path length

ROPL : $\sqrt{[w-z]^2 + (p-x)^2} - n[w-z] = \beta \neq 0, \;\; (x,z),\; (p,w) \in \mathbb{R}^3$

Implies $$\frac{\left\{z - \left[w + \frac{\beta n}{n^2-1}\right]\right\}^2}{\frac{\beta^2}{(n^2-1)^2}} - \frac{[x-p]^2}{\frac{\beta^2}{n^2-1}} = 1.$$

2. If $n > 1, \beta < 0$ and (p,w) **is fixed,** this is a **hyperboloid** of revolution of two sheets H^l, H^r with **eccentricity** n, **center** $(p, w + \frac{n\beta}{n^2-1})$, **foci:**

$$F^l = (p, w + \frac{n}{n^2-1}(\beta - |\beta|)), \quad F^r = (p, w + \frac{n}{n^2-1}(\beta + |\beta|)),$$

Left branch $H^l : z(x) = \dfrac{n\beta - c(x,p)}{n^2-1} + w, \;\; x \in \alpha.$

3. For $n > 1, \beta < 0$ and **fixed** (x,z) we get again a hyperboloid of revolution with branches H_1^l, H_1^r, **eccentricity** n, **center** $(x, z - \frac{n\beta}{n^2-1})$, **foci:**

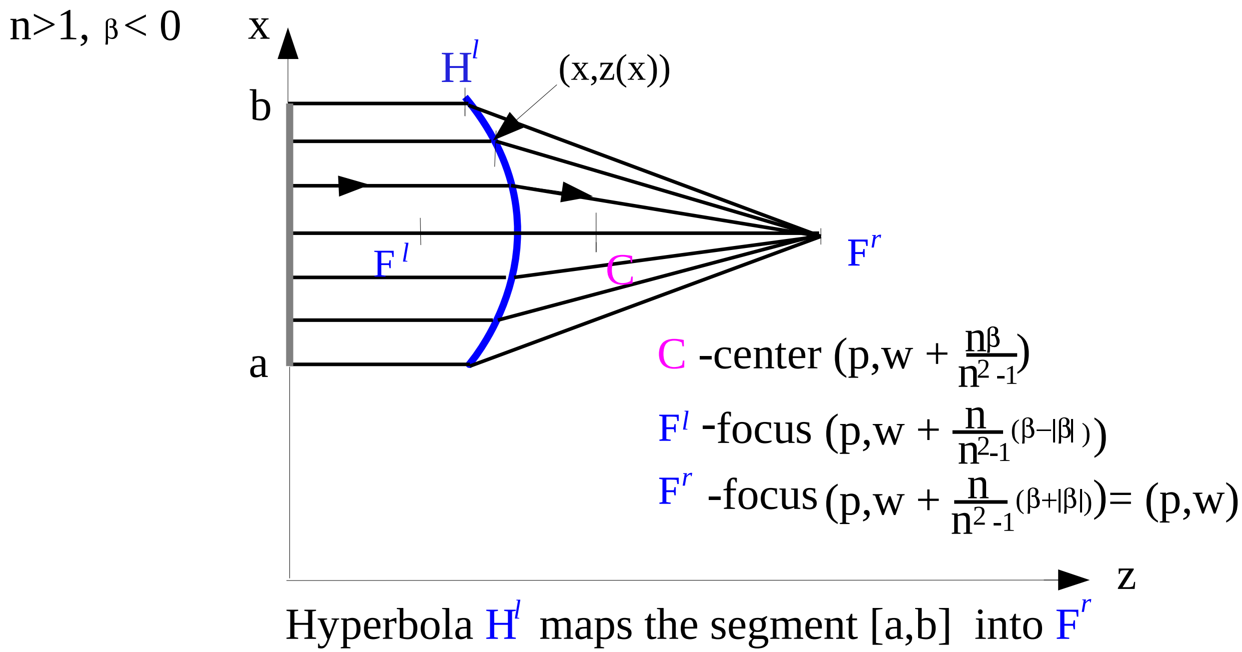
$$F_1^l = (x, z - \frac{n}{n^2-1}(\beta + |\beta|)), \quad F_1^r = (x, z - \frac{n}{n^2-1}(\beta - |\beta|)),$$

$$\textbf{Right branch} \quad H_1^r : w(p) = -\frac{n\beta - c(x,p)}{n^2-1} + z, \; p \in \alpha.$$

4. **When** $0 < n < 1, \; \beta > 0,$ **instead of hyperboloids we get branches of ellipsoids of revolution.**

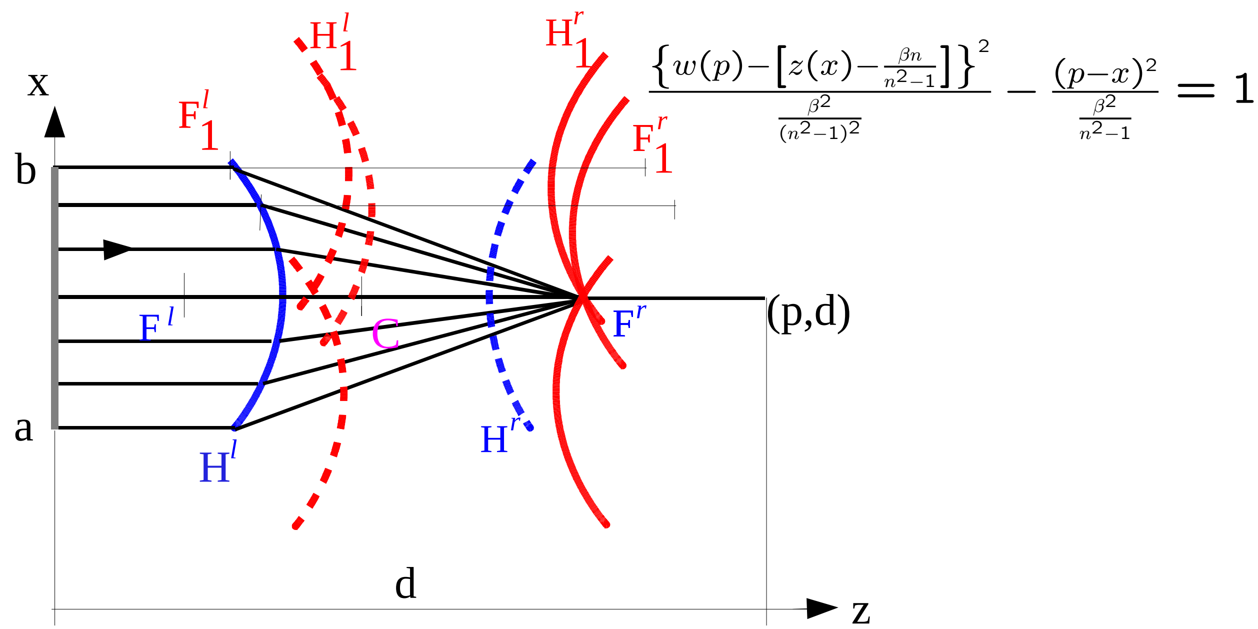
5. The case $n > 1, \beta > 0$ is not physically usefull.

Fact A: if $n > 1, \beta < 0$ all rays refracted by H^l pass through focus F^r.



Proof. Use $z(x)$ in the Main Lemma and calculate $\omega(x)$ or see R. Descartes and R. Luneburg.

FACT B: We can change the directions of rays at F^r



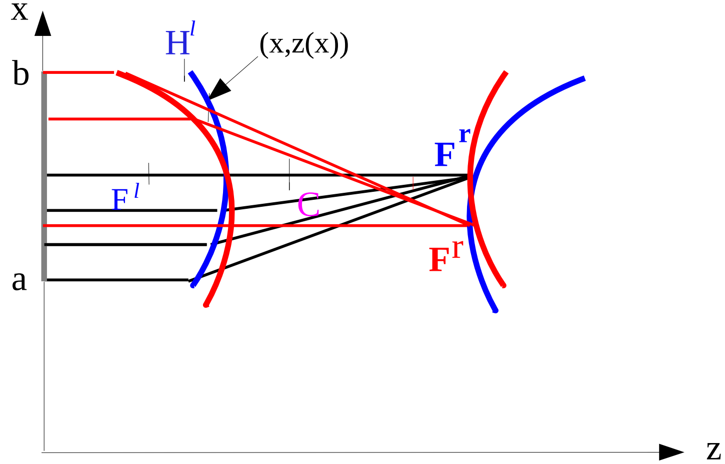
Mapping the segment [a,b] into point (p,d)

H^l, H^r are the left and right branches of the hyperboloid with foci F^l, F^r. To change direction of the ray $F_1^l F^r$ at F^r, we refract it in H_1^r with foci $F_1^l \in H^l$ and F_1^r. Repeat this for all rays refracted by H^l. The point F^r lies on all such appropriately translated H_1^r. **Thus, we mapped** $[a, b] \rightarrow (p, d)$**.**

Thus, we constructed a “two-lens” system with active area of the second lens being **ONE POINT** F^r. For that system each point $(x, z(x)) \in H^l$ is a focus of a branch passing through F^r and F^r is the focus of H^l.

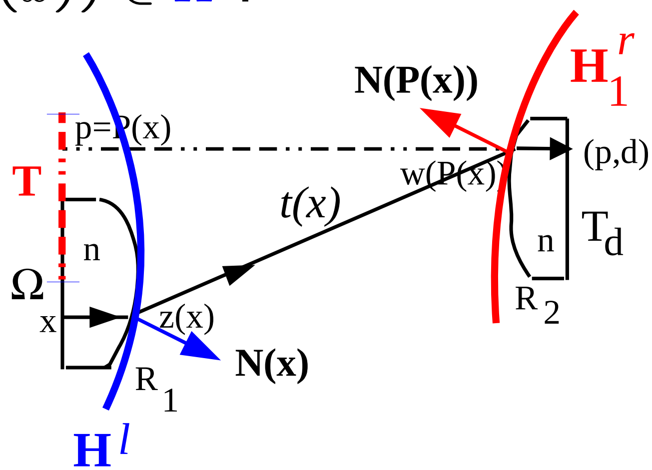
Using several $\{H^l\}$, we can construct a **a multifocal** system:

n>1, β< 0



Mapping the segment [a,b] into F^r and F^r

Main Lemma and Facts A and B imply:
Suppose we already have two lenses as required $(R_1, R_2) = (z(x), w(p))$. Then we have the map $(x, z(x)) \longrightarrow (p = P(x), w(P(x)))$. Since $ROPL = \beta$, it must be that R_1 at $(x, z(x))$ is tangent to H^l with right focus $(P(x), w(P(x))) \in H_1^r$ and R_2 at $(P(x), w(P(x)))$ is tangent to H_1^r with left focus $(x, z(x)) \in H^l$.



Main Idea: ($n > 1, \beta < 0$) Construct R_1, R_2 as envelopes of families of branches of two-sheeted hyperboloids of revolution with foci on R_1 and R_2.

How to construct such envelopes?

Definition: Lower and Upper envelopes (using branches of hyperboloids).

Assume $n > 1,\ \beta < 0$. Let $z \in C(\bar{\Omega}),\ w \in C(\bar{T})$. If

$$z(x) = \inf_{p \in \bar{T}} \left\{ \frac{\beta n - c(x,p)}{n^2-1} + w(p) \right\},\ x \in \bar{\Omega},$$

$$w(p) = \sup_{x \in \bar{\Omega}} \left\{ -\frac{\beta n - c(x,p)}{n^2-1} + z(x) \right\},\ p \in \bar{T}.$$

then $(R_1, R_2) = (z(x), w(p))$ is called a *two-lens system* of **type A**.

Notes. (0) Note the use of **c(x,p)**, **provided** by ROPL $\beta = const$!!

(1) Since $c(|x-p|)$ is convex, type $A \Longrightarrow R_1$ is concave and R_2 is convex;

(2) Switching inf and sup, systems of type B are defined. The type B may be neither convex nor concave.

Transfer of energy and weak solutions for type A (& B)

For a two-lens $(z(x), w(p))$ of type A, the **refractor map**, possibly multi-valued, is defined by:

$$P(x) = \left\{ p \in \bar{T} \mid z(x) = \frac{\beta n - c(x,p)}{n^2-1} + w(p) \right\},\ x \in \bar{\Omega} \text{ and}$$

$$P^{-1}(p) = \left\{ x \in \bar{\Omega} \mid w(p) = -\frac{\beta n - c(x,p)}{n^2-1} + z(x) \right\},\ p \in \bar{T}.$$

2. For input intensity $I(x), x \in \bar{\Omega}$, the function

$$G(z, w, \tau) := \int_{P^{-1}(\tau)} I(x)dx, \quad \tau \subset \bar{T}$$

is the **energy transfer** function.

Now, we define (WEAK) solutions.

Let $L(p), p \in \bar{T}$, be the prescribed output intensity. A two-lens system (z, w) of type A is a **weak solution** of type A of the two-lens problem if the map $P : \bar{\Omega} \to \bar{T}$ is onto and

$$G(z, w, \tau) = \int_{\tau} L(p)dp \;\; \forall \tau \subset \bar{T}$$

Part III. Finding Weak Solutions with Calculus of Variations

The designer chooses $n, l, d > 0$ such that $\beta := l - nd \neq 0$.
Consider the two-lens case when $n > 1$ and $\beta < 0$.

Define the set of admissible functions:

$$Adm_A = \Big\{(z,w) \in C(\bar{\Omega}) \times C(\bar{T}) \text{ s. t. } \quad \text{for all } (x,p) \in \bar{\Omega} \times \bar{T}$$

$$ROPL := \sqrt{|x-p|^2 + |w(p) - z(x)|^2} - n[w(p) - z(x)] \leq \beta\Big\}.$$

It is easier to work with an equivalent form:

$$Adm_A = \Big\{(z,w) \in C(\bar{\Omega}) \times C(\bar{T}) \text{ s. t. } \quad \text{for all } (x,p) \in \bar{\Omega} \times \bar{T}$$

$$w(p) - z(x) \geq -\frac{1}{n^2-1}\Big[n\beta - c(x,p)\Big]\Big\}.$$

On a two-lens system of type A we have equality for x, p when $p = P(x)$.

Theorem. Assume $\int_{\bar\Omega} I(x)dx = \int_{\bar T} L(p)dp \neq 0$.
The Fermat-like minimization problem

$$\mathcal{F}(z,w) := \int_{\bar T} w(p)L(p)dp - \int_{\bar\Omega} z(x)I(x)dx \longmapsto \text{min on } Adm_A$$

admits a unique minimizing pair $(z_{\text{min}}[concave], w_{\text{min}}[convex])$ which is a weak solution of type A of the two-lens problem. It is given by

$$z_{\text{min}}(x) = \inf_{p\in\bar T}\left\{\frac{\beta n - c(x,p)}{n^2-1} + w_{\text{min}}(p)\right\},$$

$$w_{\text{min}}(p) = \sup_{x\in\bar\Omega}\left\{-\frac{\beta n - c(x,p)}{n^2-1} + z_{\text{min}}(x)\right\}.$$

In addition,
$z \in Lip(\bar\Omega)$, $w \in Lip(\bar T)$, $|\nabla z|, |\nabla w| < 1/\sqrt{n^2-1}$ and a.e. in $\bar\Omega$

$$P(x) = x - \frac{\beta\nabla z(x)}{M(z(x))} : \bar\Omega \to \bar T \quad \left(M(z) := \sqrt{1+(1-n^2)|\nabla z|^2}\right).$$

Notes. (i) The solution that we found is NOT an approximate solution optimizing some merit function!!! The pair $(z_{\text{min}}, w_{\text{min}}$ is the exact solution of the two-lens problem.

(ii) z_{min} is concave and w_{min} is convex which is useful for fabrication.

(iii) The Fermat-like functional $\mathcal{F}(z,w)$ is the mean horizontal distance between the lenses with the average weighted by intensities.

(iv) The refractor maps are “built-in” into the constraints (Adm_A).

(v) The problem $\mathcal{F} \longmapsto \text{min}$ on Adm_A is a linear programming problem.

Notes. (1) Solutions of type B (not necessarily convex/concave!) are obtained when $n > 1, \beta < 0$, and Adm_B defined with $\geq \beta$.

(2) The same approach applies to the case $\beta > 0$, $0 < n < 1$ and generates single lenses of type C (convex/concave) and type D (may be neither convex nor concave).

(3) When $\beta > 0$ and $0 < n < 1$, J. Rubinstein and G. Wolansky obtained, in a very different way, single lens designs using their **weighted least action** concept. Their design generates lenses which are neither convex nor concave (type D in our terminology) but can be manufactured by a convex tool.

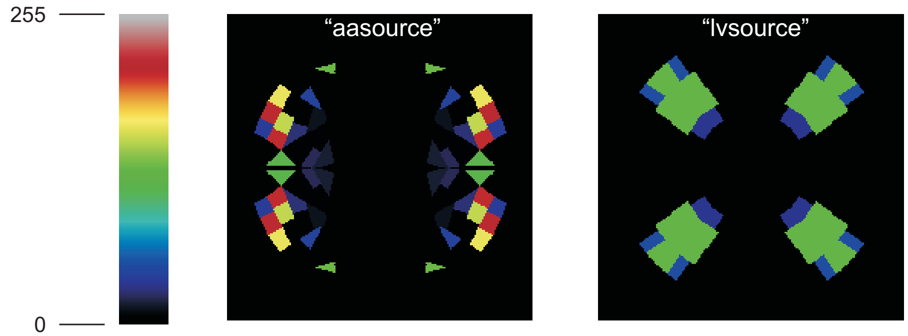
Part IV. TEST DESIGN

Two-lens optical system for reshaping the irradiance distribution of a laser beam

Irradiance distributions for mask illumination

Our source functions generally have 4-fold symmetry

Field shown is ~ 12 mm square

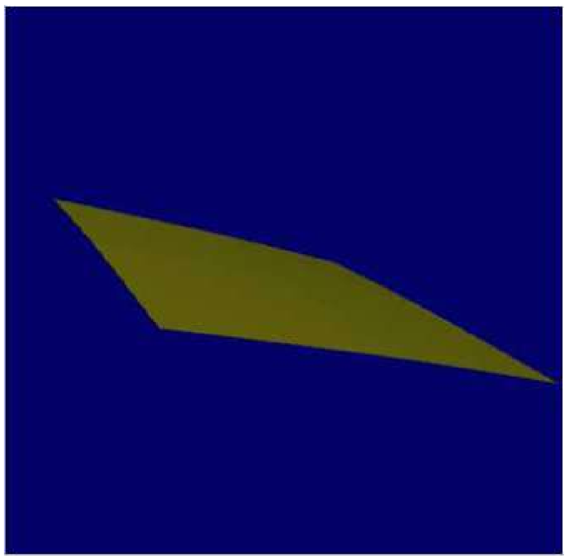


Is it feasible to generate these patterns by reflective or refractive laser beam shaping?

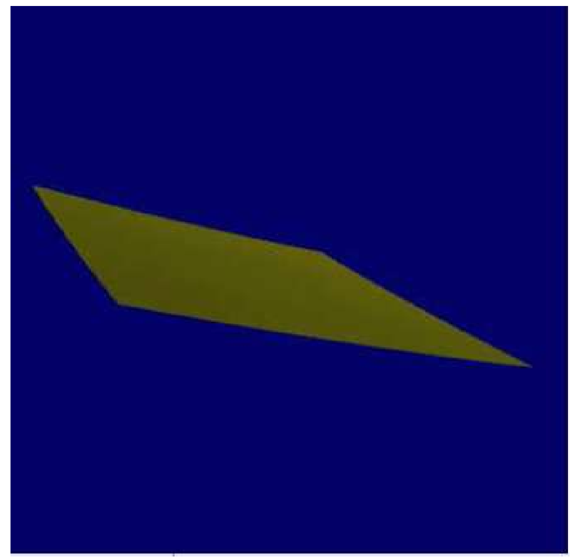
Input can be assumed to be uniform rectangular distribution with dimensions ~ 3 mm x 6 mm

Input and out are collimated, propagate in same direction

John Hoffnagle 16 May 2008

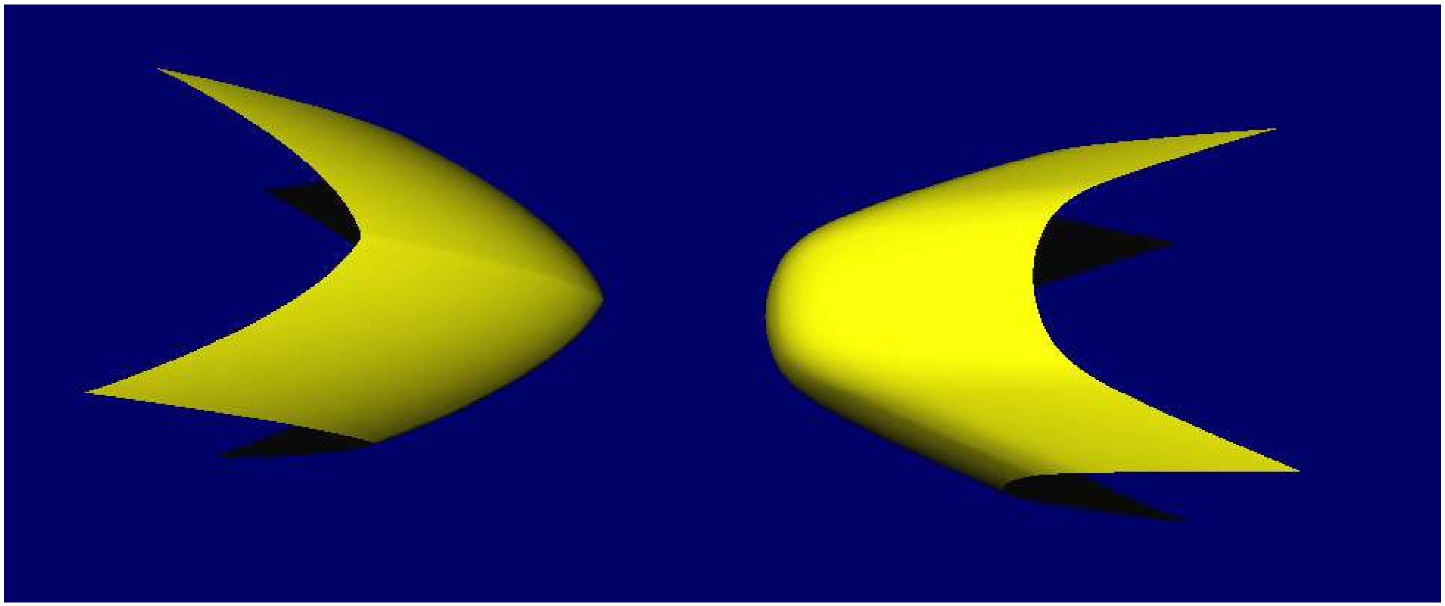


Lens 1



Lens 2

Grids: 198×198 square grids over $\bar{\Omega}$ and over $\bar{T}$.
About $40,000$ grid points on each set.



Lenses 1 and 2 (times 100)

Philosophy of applying geometry and calculus of variations to mirror/lens design:

1. Recognize the quadric(s) (or other functions) suitable for the problem (These usually solve the problem if one of the intensities is a sum of Dirac masses)

2. Describe the mirrors/lenses/surfaces by expressions for lower and upper envelopes of such quadrics (This also defines convex/non-convex solutions, the admissible functions and the needed functional!)

3. Formulate a Fermat-like functional to be maximized or minimized on the appropriate class of admissible functions. This problem is usually easier to study and solve numerically than the original nonlinear PDE's.


Galileo
vedeva
così

The End