Solution to the Auerbach Conjecture

Aldo Pratelli

Department of Mathematics, University of Pavia (Italy)

"Convex Geometry – Analytic Aspects",
Cortona, June 12–18 2011

The problem

The problem

Consider a convex set E with $|E| = 1$.

The problem

Consider a convex set E with $|E| = 1$.

Let $\ell(E)$ denote the shortest bisecting chord.

The problem

Consider a convex set E with $|E| = 1$.

Let $\ell(E)$ denote the shortest bisecting chord.

Which set maximizes $\ell(E)$?

The problem

Consider a convex set E with $|E| = 1$.

Let $\ell(E)$ denote the shortest bisecting chord.

Which set maximizes $\ell(E)$?

Equivalently, we can fix $\ell(E)$ and try to minimize $|E|$.

The problem

Consider a convex set E with $|E| = 1$.

Let $\ell(E)$ denote the shortest bisecting chord.

Which set maximizes $\ell(E)$?

Equivalently, we can fix $\ell(E)$ and try to minimize $|E|$.

The problem was set by Auerbach in the 1930's.

The problem

Consider a convex set E with $|E| = 1$.

Let $\ell(E)$ denote the shortest bisecting chord.

Which set maximizes $\ell(E)$?

Equivalently, we can fix $\ell(E)$ and try to minimize $|E|$.

The problem was set by Auerbach in the 1930's.

First intuition: all the bisecting chords should have the same length!

The problem

Consider a convex set E with $|E| = 1$.

Let $\ell(E)$ denote the shortest bisecting chord.

Which set maximizes $\ell(E)$?

Equivalently, we can fix $\ell(E)$ and try to minimize $|E|$.

The problem was set by Auerbach in the 1930's.

First intuition: all the bisecting chords should have the same length!

Such sets are called Zindler sets.

The problem

Consider a convex set E with $|E| = 1$.

Let $\ell(E)$ denote the shortest bisecting chord.

Which set maximizes $\ell(E)$?

Equivalently, we can fix $\ell(E)$ and try to minimize $|E|$.

The problem was set by Auerbach in the 1930's.

First intuition: all the bisecting chords should have the same length!

Such sets are called Zindler sets.

This has a strong connection with the Ulam floating property.

Something about Zindler sets (1/2)

Something about Zindler sets (1/2)

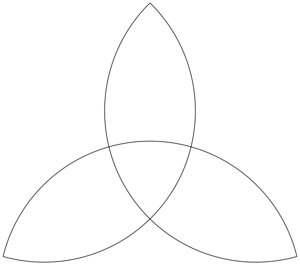
First question: is there any Zindler set, besides the disk?

Something about Zindler sets (1/2)

First question: is there any Zindler set, besides the disk?
The first example is by Zindler (1921).

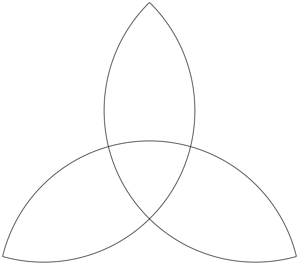
Something about Zindler sets (1/2)

First question: is there any Zindler set, besides the disk?
The first example is by Zindler (1921).



Something about Zindler sets (1/2)

First question: is there any Zindler set, besides the disk?
The first example is by Zindler (1921).



And what about convex Zindler sets?

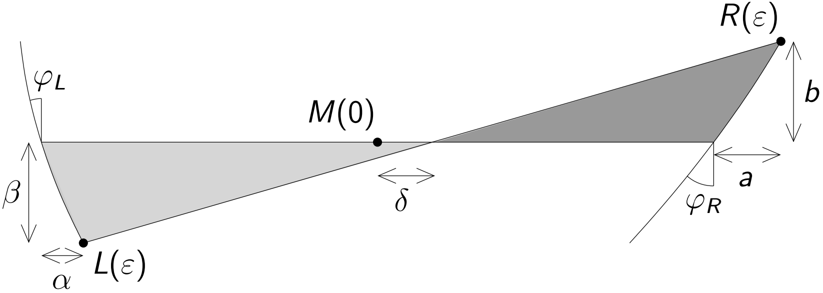
Something about Zindler sets (2/2)

Something about Zindler sets (2/2)

What can we say about a convex Zindler set?

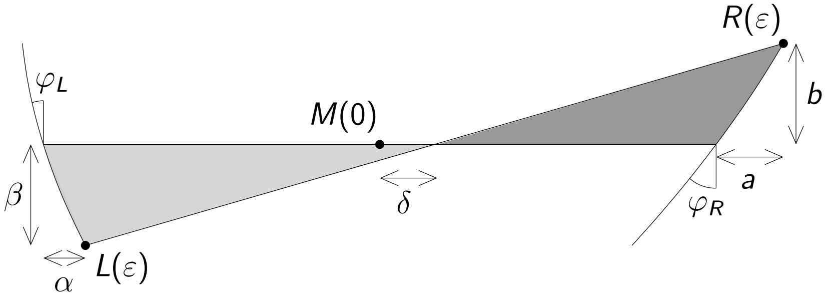
Something about Zindler sets (2/2)

What can we say about a convex Zindler set?



Something about Zindler sets (2/2)

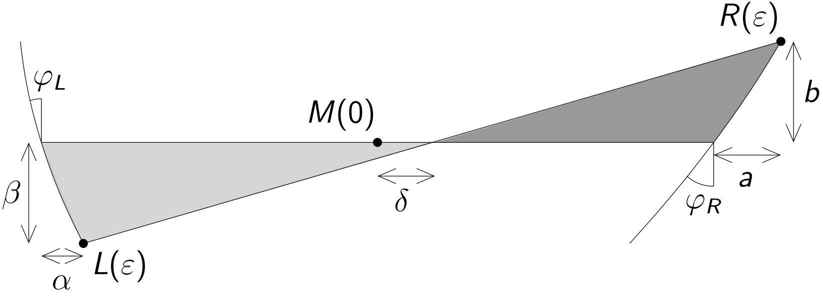
What can we say about a convex Zindler set?



$a \approx \alpha\,, \qquad b \approx \beta$

Something about Zindler sets (2/2)

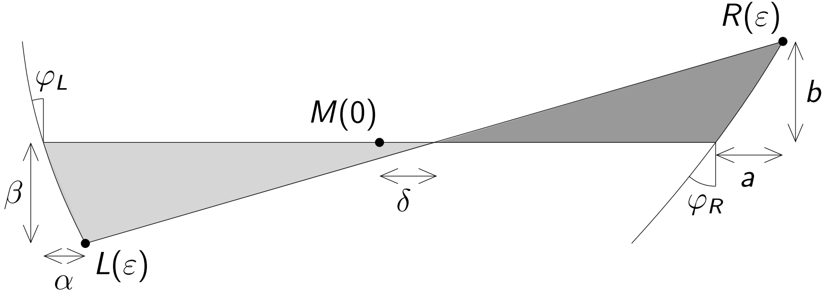
What can we say about a convex Zindler set?



$a \approx \alpha\,, \qquad b \approx \beta \qquad \Longrightarrow \varphi_L = \varphi_R$

Something about Zindler sets (2/2)

What can we say about a convex Zindler set?

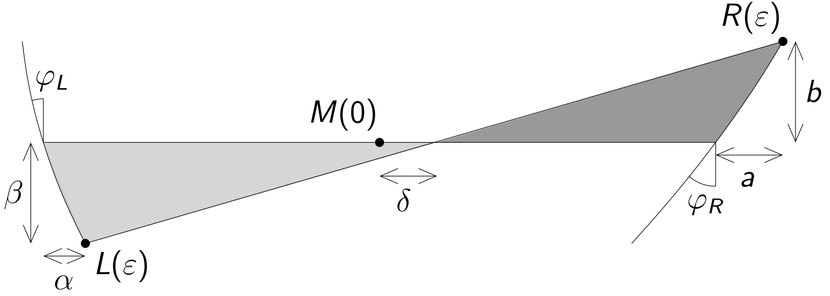


$$a \approx \alpha\,, \qquad b \approx \beta \qquad \Longrightarrow \varphi_L = \varphi_R$$

Then the velocity of the centre of the chord is in the direction of the chord

Something about Zindler sets (2/2)

What can we say about a convex Zindler set?



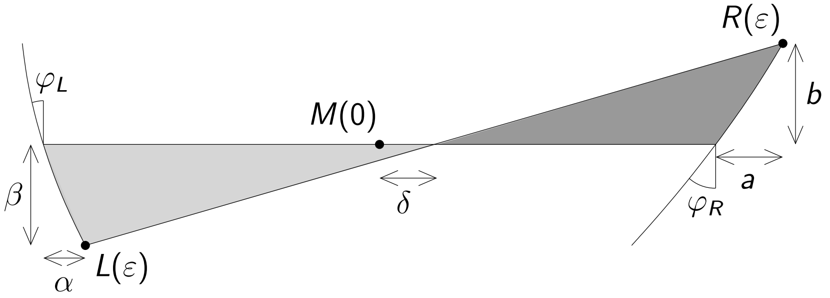
$a \approx \alpha\,, \qquad b \approx \beta \qquad \Longrightarrow \varphi_L = \varphi_R$

Then the velocity of the centre of the chord is in the direction of the chord

So we can describe the set with a function $c : [0, 2\pi] \to \mathbb{R}$. One has

Something about Zindler sets (2/2)

What can we say about a convex Zindler set?



$a \approx \alpha\,,\qquad b \approx \beta \qquad \Longrightarrow \varphi_L = \varphi_R$

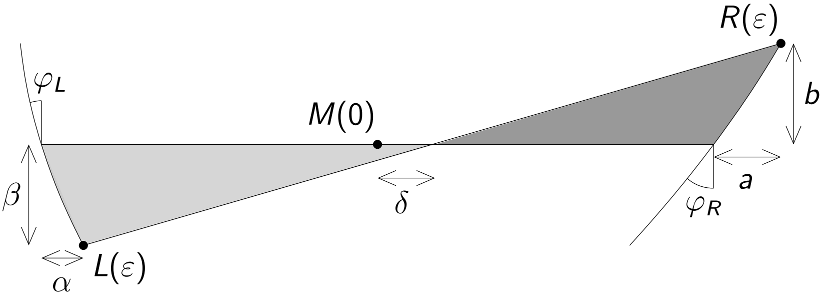
Then the velocity of the centre of the chord is in the direction of the chord

So we can describe the set with a function $c : [0, 2\pi] \to \mathbb{R}$. One has

$$c(\theta) = -c(\theta + \pi)$$

Something about Zindler sets (2/2)

What can we say about a convex Zindler set?



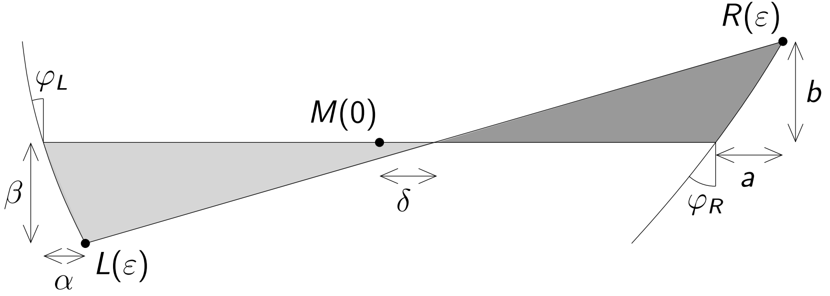
$a \approx \alpha\,, \qquad b \approx \beta \qquad \Longrightarrow \varphi_L = \varphi_R$

Then the velocity of the centre of the chord is in the direction of the chord
So we can describe the set with a function $c : [0, 2\pi] \to \mathbb{R}$. One has

$$c(\theta) = -c(\theta + \pi) \qquad \int_0^{\pi} c(\theta)(\cos(\theta), \sin(\theta))\, d\theta = 0$$

Something about Zindler sets (2/2)

What can we say about a convex Zindler set?



$a \approx \alpha\,,$ $\qquad b \approx \beta$ $\qquad \Longrightarrow \varphi_L = \varphi_R$

Then the velocity of the centre of the chord is in the direction of the chord

So we can describe the set with a function $c : [0, 2\pi] \to \mathbb{R}$. One has

$$c(\theta) = -c(\theta + \pi) \quad \int_0^\pi c(\theta)(\cos(\theta), \sin(\theta))\, d\theta = 0 \quad |c'(\theta)| \leq 1 + c(\theta)^2$$

Restating the problem

Restating the problem

We have rewritten the problem as follows:

Restating the problem

We have rewritten the problem as follows:
Among all functions $c : [0, 2\pi] \to \mathbb{R}$ for which

$$\begin{cases} c(\theta) = -c(\theta + \pi) \\ \displaystyle\int_0^\pi c(\theta)(\cos(\theta), \sin(\theta))\, d\theta = 0 \\ |c'(\theta)| \leq 1 + c(\theta)^2\,, \end{cases}$$

Restating the problem

We have rewritten the problem as follows:
Among all functions $c : [0, 2\pi] \to \mathbb{R}$ for which

$$\begin{cases} c(\theta) = -c(\theta + \pi) \\ \int_0^\pi c(\theta)(\cos(\theta), \sin(\theta))\, d\theta = 0 \\ |c'(\theta)| \leq 1 + c(\theta)^2\,, \end{cases}$$

which one minimizes the area, which is

Restating the problem

We have rewritten the problem as follows:
Among all functions $c : [0, 2\pi] \to \mathbb{R}$ for which

$$\begin{cases} c(\theta) = -c(\theta + \pi) \\ \int_0^\pi c(\theta)(\cos(\theta), \sin(\theta))\, d\theta = 0 \\ |c'(\theta)| \leq 1 + c(\theta)^2\,, \end{cases}$$

which one minimizes the area, which is

$$Area(c) = \pi - \int_0^\pi d\theta \int_0^\theta c(\theta)c(\varphi)\sin(\theta - \varphi)d\varphi \qquad ?$$

Playing with Fourier series (1/2)

Playing with Fourier series (1/2)

Let us look at the Fourier coefficients of c,

$$A_n = \int_0^{2\pi} c(\theta)\sin(n\theta)\,, \qquad\qquad B_n = \int_0^{2\pi} c(\theta)\cos(n\theta)\,.$$

Playing with Fourier series (1/2)

Let us look at the Fourier coefficients of c,

$$A_n = \int_0^{2\pi} c(\theta)\sin(n\theta)\,, \qquad\qquad B_n = \int_0^{2\pi} c(\theta)\cos(n\theta)\,.$$

$c(\theta) = -c(\theta+\pi) \Longrightarrow$ there are only coefficients of odd order

Playing with Fourier series (1/2)

Let us look at the Fourier coefficients of c,

$$A_n = \int_0^{2\pi} c(\theta)\sin(n\theta)\,, \qquad B_n = \int_0^{2\pi} c(\theta)\cos(n\theta)\,.$$

$c(\theta) = -c(\theta + \pi) \Longrightarrow$ there are only coefficients of odd order

$\int_0^{\pi} c(\theta)(\cos(\theta), \sin(\theta))\, d\theta = 0 \Longrightarrow$ there are no coefficients of order 1

Playing with Fourier series (1/2)

Let us look at the Fourier coefficients of c,

$$A_n = \int_0^{2\pi} c(\theta)\sin(n\theta)\,, \qquad B_n = \int_0^{2\pi} c(\theta)\cos(n\theta)\,.$$

$c(\theta) = -c(\theta+\pi) \Longrightarrow$ there are only coefficients of odd order

$\int_0^{\pi} c(\theta)(\cos(\theta),\sin(\theta))\,d\theta = 0 \Longrightarrow$ there are no coefficients of order 1

Minimizing the area means maximizing

$$\sum_{\substack{n \text{ odd} \\ n\geq 3}} \frac{A_n^2+B_n^2}{n^2-1}$$

Playing with Fourier series (1/2)

Let us look at the Fourier coefficients of c,

$$A_n = \int_0^{2\pi} c(\theta)\sin(n\theta)\,, \qquad B_n = \int_0^{2\pi} c(\theta)\cos(n\theta)\,.$$

$c(\theta) = -c(\theta+\pi) \Longrightarrow$ there are only coefficients of odd order

$\int_0^{\pi} c(\theta)(\cos(\theta),\sin(\theta))\,d\theta = 0 \Longrightarrow$ there are no coefficients of order 1

Minimizing the area means maximizing

$$\sum_{\substack{n \text{ odd} \\ n \geq 3}} \frac{A_n^2 + B_n^2}{n^2 - 1}$$

This implies that the disk is not optimal!

Playing with Fourier series (1/2)

Let us look at the Fourier coefficients of c,

$$A_n = \int_0^{2\pi} c(\theta)\sin(n\theta)\,, \qquad B_n = \int_0^{2\pi} c(\theta)\cos(n\theta)\,.$$

$c(\theta) = -c(\theta+\pi) \Longrightarrow$ there are only coefficients of odd order

$\int_0^{\pi} c(\theta)(\cos(\theta), \sin(\theta))\,d\theta = 0 \Longrightarrow$ there are no coefficients of order 1

Minimizing the area means maximizing

$$\sum_{\substack{n \text{ odd} \\ n \geq 3}} \frac{A_n^2 + B_n^2}{n^2 - 1}$$

This implies that the disk is not optimal!

On the countrary, it is the *biggest* Zindler set!

Playing with Fourier series (2/2)

Playing with Fourier series (2/2)

So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.

Playing with Fourier series (2/2)

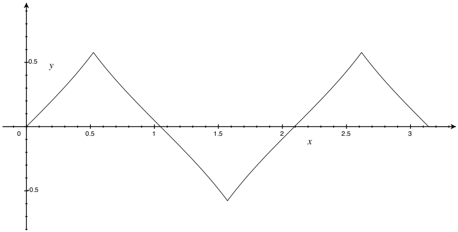
So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$?

Playing with Fourier series (2/2)

So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$? For $n =$

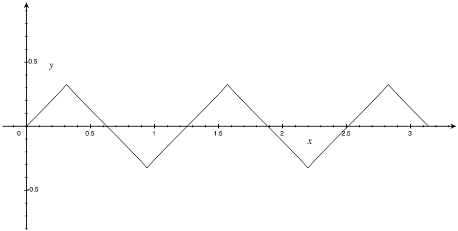
Playing with Fourier series (2/2)

So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$? For $n = 3$



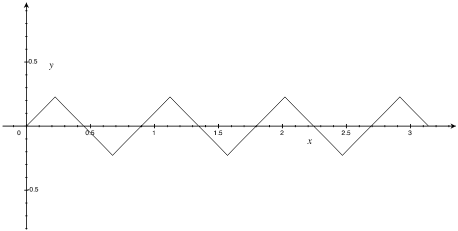
Playing with Fourier series (2/2)

So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$? For $n = 5$



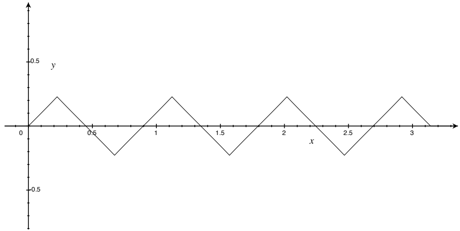
Playing with Fourier series (2/2)

So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$? For $n = 7$



Playing with Fourier series (2/2)

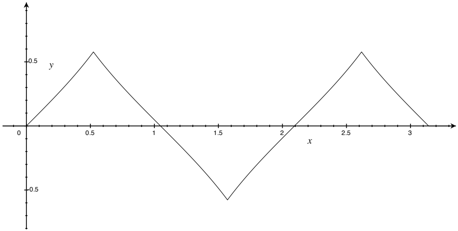
So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$? For $n = 7$



So one can understand that the term $A_3^2 + B_3^2$ is the most important.

Playing with Fourier series (2/2)

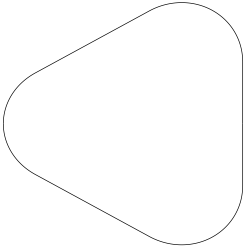
So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$? For $n = 3$



So one can understand that the term $A_3^2 + B_3^2$ is the most important.
Conjecture: the optimal function is $c = \hat{c}$ above.

Playing with Fourier series (2/2)

So we want to have big Fourier coefficients, but still with $|c'| \leq 1 + c^2$.
Which set maximizes $A_n^2 + B_n^2$? For $n = 3$



So one can understand that the term $A_3^2 + B_3^2$ is the most important.
Conjecture: the optimal function is $c = \hat{c}$ above.
The resulting set is the one above (the Auerbach triangle).

The results

The results

• The Auerbach triangle minimizes the area among Zindler sets of fixed "bisecting length" (Fusco–P., 2010)

The results

• The Auerbach triangle minimizes the area among Zindler sets of fixed "bisecting length" (Fusco–P., 2010)

• The best convex set must be a Zindler set (Esposito–Ferone–Kawohl–Nitsch–Trombetti, 2011)

The general case

The general case

What can we say about the general (not necessarily convex) case?

The general case

What can we say about the general (not necessarily convex) case?

- First (quite positive) point: an optimizer must trivially be Zindler.

The general case

What can we say about the general (not necessarily convex) case?

• First (quite positive) point: an optimizer must trivially be Zindler.

• Second (quite negative) point: there is no immediate condition to replace $|c'| \leq 1 + c^2$.

The general case

What can we say about the general (not necessarily convex) case?

• First (quite positive) point: an optimizer must trivially be Zindler.

• Second (quite negative) point: there is no immediate condition to replace $|c'| \leq 1 + c^2$.

• Third (very negative) point: it is no more true that the lowest Fourier coefficients are the most important.

The general case

What can we say about the general (not necessarily convex) case?

• First (quite positive) point: an optimizer must trivially be Zindler.

• Second (quite negative) point: there is no immediate condition to replace $|c'| \leq 1 + c^2$.

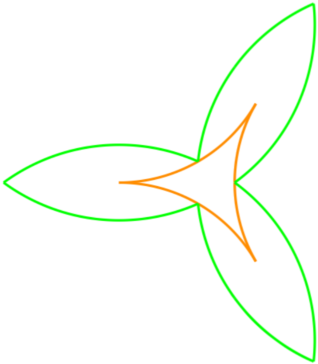
• Third (very negative) point: it is no more true that the lowest Fourier coefficients are the most important.

• Bad consequence: it is not even clear whether a minimizer exists!

Some non-convex examples

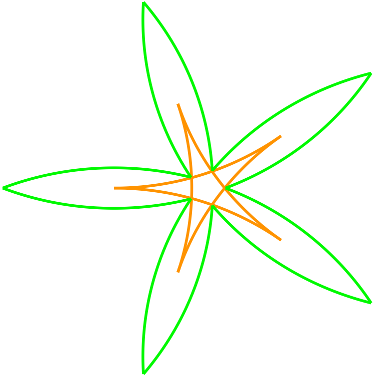
Some non-convex examples

Here is a non-convex object of area 2.54



Some non-convex examples

Here is a non-convex object of area 2.45



Some non-convex examples

Here is a non-convex object of area 2.41