The hot spot of a convex conductor

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joint work with
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Grounded conductor and hot spots

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$$\begin{cases} u_t = \Delta u & \text{in} \quad \Omega \times (0, \infty), \\ u = 1 & \text{on} \quad \Omega \times \{0\}, \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty). \end{cases}$$

Here Ω — the *heat conductor* — is a bounded domain in the Euclidean space $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary and $u = u(x, t)$ denotes the normalized temperature of the conductor at a point $x \in \Omega$ and time $t > 0$.

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A *hot spot* $x(t)$ is a point such that

$$u(x(t), t) = \max_{\overline{\Omega}} u(\cdot, t).$$

The hot spot of a convex conductor

If Ω is convex — in this case $\overline{\Omega}$ is a *convex body* that we shall denote by $\mathcal{K}$ — results of Brascamp & Lieb (1976) and Korevaar (1983) imply that

$$\log u(x,t) \text{ is concave in } x \text{ for every } t > 0.$$

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$$\log u(x,t) \text{ is concave in } x \text{ for every } t > 0.$$

Based on this result and the analyticity of u in x, we have that for every $t > 0$

$$\exists! \text{ hot spot } x(t) \in \mathcal{K} \text{ and } \nabla u(x(t),t) = 0.$$

Evolution of the hot spot

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SHORT TIMES. Since, by a result of Varadhan,

$$-4t \log\{1 - u(x,t)\} \to \operatorname{dist}(x, \partial\Omega)^2$$

uniformly for $x \in \overline{\Omega}$ as $t \to 0^+$, we can claim that

$$\operatorname{dist}(x(t), \mathcal{M}) \to 0 \ \text{ as } t \to 0^+,$$
$$\operatorname{dist}(x(t), \partial\Omega) \to r_\Omega \ \text{ as } t \to 0^+,$$

where

$$\mathcal{M} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = r_\Omega\}$$

and

$$r_\Omega = \max\{\operatorname{dist}(y, \partial\Omega) \ : \ y \in \overline{\Omega}\}$$

is the **inradius** of Ω.

The set $\mathcal{M}$

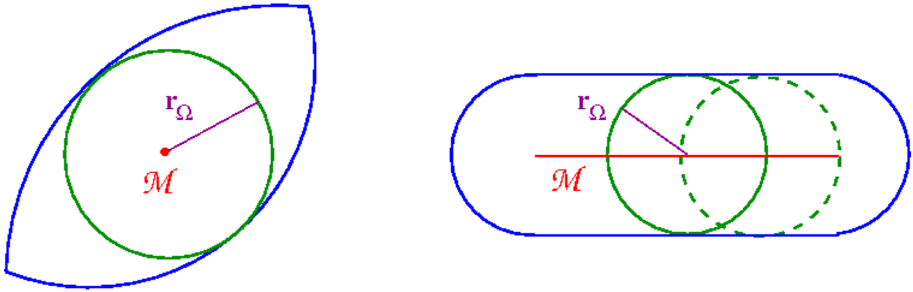

Figura: Two examples for the set $\mathcal{M}$.

Evolution of the hot spot

LARGE TIMES. Let ϕ_1 be the first Dirichlet eigenfunction of $-\Delta$ in Ω, i.e.

$$\Delta\phi_1 + \lambda_1\phi_1 = 0 \text{ and } \phi_1 > 0 \text{ in } \Omega, \quad \phi_1 = 0 \quad \text{on } \partial\Omega.$$

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Since $e^{\lambda_1 t}u(\cdot, t)$ converges to ϕ_1 locally uniformly in C^2 as $t \to \infty$, then, for a convex body $\mathcal{K}$,

$$x(t) \to x_\infty \ \text{ as } t \to \infty \,,$$

where x_∞ is the (unique) maximum point in $\mathcal{K}$ of ϕ_1.

The location of the hot spot

Remarks

1. It is relatively easy to locate the set $\mathcal{M}$ by geometrical means.
2. Saying that $x(t) \to x_\infty$ as $t \to \infty$ does not give much information: locating either $x(t)$ or x_∞ has more or less the same difficulty.

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Known result: Grieser & Jerison, JAMS 1998

In the plane they estimate:

$$|x_\infty - \overline{x}| \leq C,$$

where $\overline{x}$ is the unique maximum point of a one-dimensional eigenfunction related to $-\Delta$ and $\mathcal{K}$. The estimate is uniform w.r.t. the ratio $r_\mathcal{K}/\delta_\mathcal{K}$ (sse figure).

Methods for locating the hot spot

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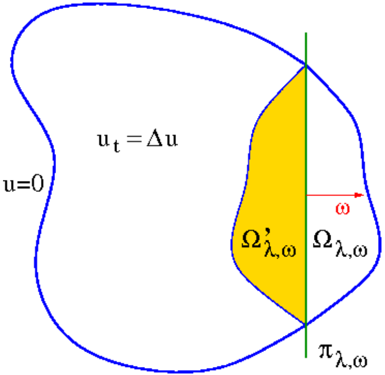
2. The latter is based on ideas related to the **Alexandrov-Bakelmann-Pucci maximum principle** and convex geometry.

Alexandrov's reflection principle

Fix a direction $\omega \in \mathbb{S}^{N-1}$ and a parameter $\lambda \in \mathbb{R}$ define the sets

$$\pi_{\lambda,\omega} = \{x \in \mathbb{R}^N : x \cdot \omega = \lambda\}, \quad \Omega_{\lambda,\omega} = \{x \in \Omega : x \cdot \omega > \lambda\},$$
$$\Omega'_{\lambda,\omega} = \text{ reflection of } \Omega_{\lambda,\omega} \text{ in the plane } \pi_{\lambda,\omega}.$$



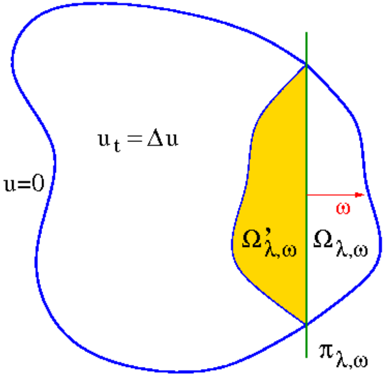
Alexandrov's reflection principle

Proposition

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$. If

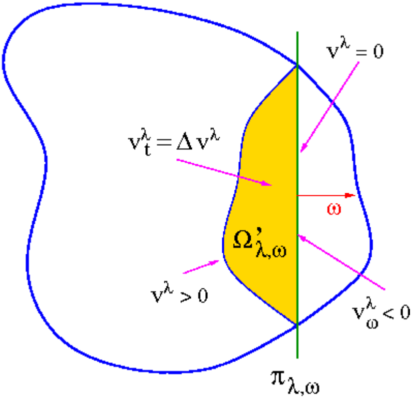
$$\overline{\Omega_{\lambda,\omega} \cup \Omega'_{\lambda,\omega}} \subset \overline{\Omega},$$

then $\Omega_{\lambda,\omega}$ cannot contain any maximum point of u.



Proof.

$$x^\lambda = \text{ reflection of } x \text{ in } \pi_{\lambda,\omega},$$
$$u^\lambda(x,t) = u(x^\lambda, t),$$
$$v^\lambda(x,t) = u(x,t) - u^\lambda(x,t),$$



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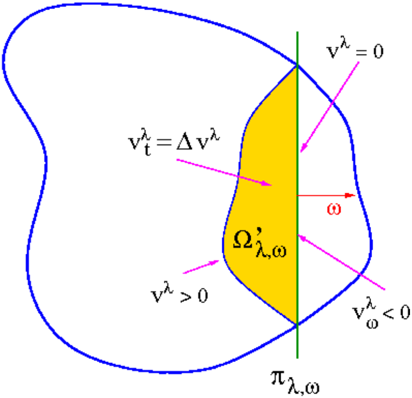

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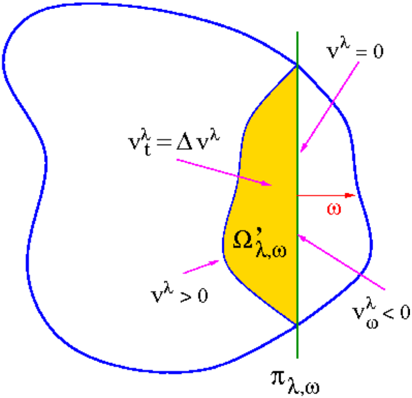
$$\Rightarrow \quad v^\lambda > 0 \text{ on } \Omega'_{\lambda,\omega} \times (0,\infty)$$



REMARK

The same result can be drawn for positive solutions of large classes of elliptic and parabolic equations, e.g.

$$F(u, Du, D^2u) = 0 \;\text{ or }\; u_t = F(u, Du, D^2u) \;\text{ in }\; \Omega,$$

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The heart of a set

These remarks motivate our interest in the convex set

$$\heartsuit(\Omega) = \bigcap \{\Omega \setminus \Omega_{\lambda,\omega} : \Omega'_{\lambda,\omega} \subset \Omega\},$$

that we call the **heart** of Ω.

The heart of a set

PROPERTIES

1. $\Omega \setminus \heartsuit(\Omega)$ does not contain any critical point of u, then $x(t) \in \heartsuit(\mathcal{K})$ for every $t > 0$ and also $x_\infty \in \heartsuit(\mathcal{K})$;

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3. $\heartsuit(\mathcal{K})$ contains the **center of mass** B of $\mathcal{K}$, the center C of the smallest ball containing $\mathcal{K}$ **(circumcenter)** and the center I of the largest ball contained in $\mathcal{K}$ **(incenter)**, if this is unique;

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5. if $\mathcal{K}$ has j independent hyperplanes of symmetry, then $\heartsuit(\mathcal{K})$ is contained in their $(N - j)$-dimensional intersection;
6. if $j = N$, then $\heartsuit(\mathcal{K})$ reduces to a single point and hence the hot spot **does not move.**

The midpoint function

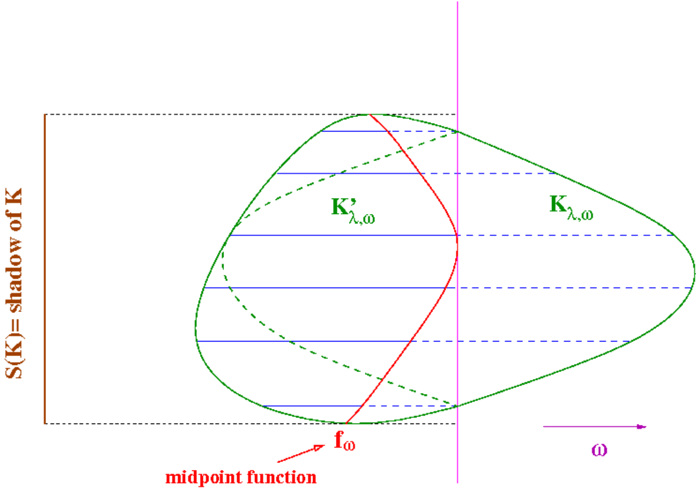


Figura: Definition of the midpoint function $f_\omega : \mathcal{S}(\mathcal{K}) \to \mathbb{R}$.

An algorithm

CONVEX POLYHEDRON

For a convex polyhedron, we prove that the maximum in the characterization can be computed **only by visiting (the projections on $\mathcal{S}(\mathcal{K})$ of) the vertices of** $\mathcal{K}$.

This fact helps us to produce an algorithm to draw $\heartsuit(\mathcal{K})$ when $\mathcal{K}$ is a convex polyhedron:

1. Fix $\omega \in \mathbb{S}^{N-1}$;
2. compute $R_{\mathcal{K}}(\omega)$ by maximizing the values $f_\omega(x_1'), \dots, f_\omega(x_m')$, where $x_1', \dots, x_m'$ are the projections on $\mathcal{S}(\mathcal{K})$ of the vertices of $\mathcal{K}$;
3. paint the halfspace $\{x \in \mathbb{R}^N : x \cdot \omega > R_{\mathcal{K}}(\omega)\}$ of yellow (kiiro);
4. iterate with a new ω.

Example 1

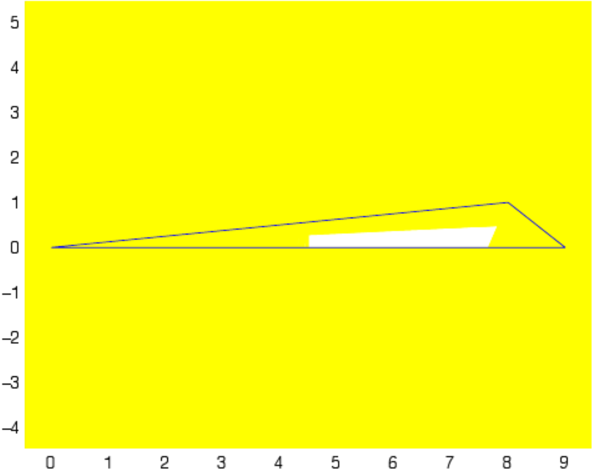
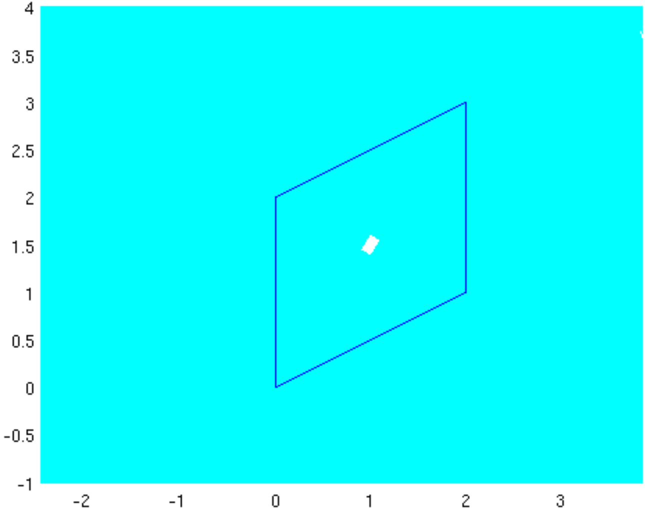
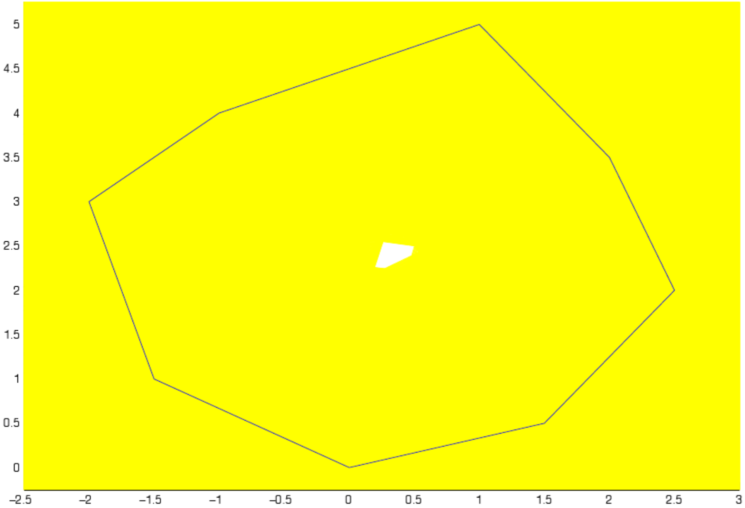


Figura: The heart of an obtuse triangle

Example 2



Example 3



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5. Does the Santalò point of $\mathcal{K}$ belong to $\heartsuit(\mathcal{K})$?

Stationary hot spot

When **8** occurs, we say that the **hot spot is stationary.**

PROBLEM (Klamkin, Siam Review 1994)

Can you characterize the convex conductors for which the hot spot does not move?

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PARTIAL ANSWERS (Magnanini - Sakaguchi, 2004, 2008)

1. triangles $\rightarrow$ equilateral;
2. quadrangles $\rightarrow$ parallelograms;
3. pentagons circumscribed to a circle $\rightarrow$ regular;
4. hexagones circumscribed to a circle $\rightarrow$ hexagons invariant w.r.t. rotations of angles $\pi/3, 2\pi/3, \pi$;
5. general formula relating the (stationary) hot spot and the curvatures of certain subsets of $\partial\mathcal{K}$.

Second method: using ABP principle

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For instance, we prove the following estimate:

$$\operatorname{dist}(x_\infty, \partial\mathcal{K}) \geq C_N\, r_{\mathcal{K}} \left(\frac{r_{\mathcal{K}}}{\delta_{\mathcal{K}}}\right)^{N^2-1},$$

where

$$C_N = \frac{(2^N N)^{N-1}}{\lambda_1(B_1)^N} \frac{\omega_{N-1}}{\omega_N} < 1,$$

$\lambda_1(B_1)$ is the first Dirichlet eigenvalue of the unit ball and

Second method: using ABP principle

The idea is condensed il the following picture.

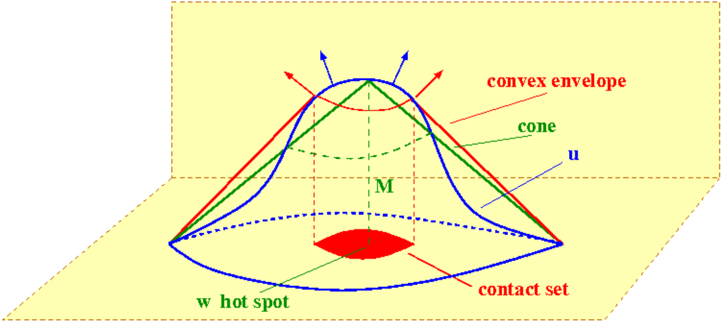

Figura: $u = u(x,t)$ or $\phi_1(x)$; $w = x(t)$ or x_∞; $M = u(w,t)$ or $\phi_1(w)$; $\mathcal{E}$ = convex envelope of u; $\mathcal{G}$ = cone with tip at the point (w, M); $\mathcal{C}$ = contact set (of points where $u = \mathcal{E}$.)

Estimating the volume of the polar set

Using the definition of the polar set, it is easy to see that $|\mathcal{K}_w^*|$ goes to ∞ as the point w approaches $\partial\mathcal{K}$. The following estimate gives a quantitative version of this fact and helps us to prove explicit estimates of the position of x_∞.

$$|\mathcal{K}_w^*| \geq |E_w^*| \geq \frac{\omega_{N-1}/N}{R^{N-1}\, d}.$$

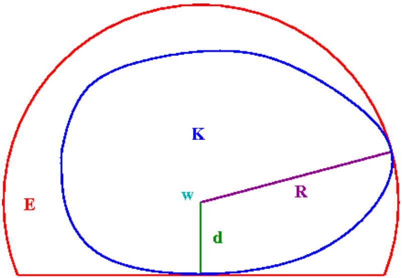

Figura:

Conclusion

Thus,

$$\frac{\omega_{N-1}/N}{R^{N-1}\, d} \le \left[\frac{\lambda_1}{N}\right]^N |\mathcal{K}|$$

and the bound

$$\mathrm{dist}(x_\infty, \partial\mathcal{K}) \ge C_N\, r_{\mathcal{K}} \left(\frac{r_{\mathcal{K}}}{\delta_{\mathcal{K}}}\right)^{N^2-1}$$

follows by choosing $w = x_\infty$ and by using a standard inequality to bound λ_1 from above.

Concluding Remark

The two methods for locating the hot spot can be coupled. For example, in the case of the obtuse triangle, we know that its heart extends to part of the boundary; however, by the estimate we have just proved, we can quantitatively say how far x_∞ must be from the boundary.