A characterization of some mixed volumes via the Brunn-Minkowski inequality

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ϕ is homogeneous of degree i if for $\lambda > 0$ it follows that

$$\phi(\lambda K) = \lambda^i \phi(K).$$

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$$h_K : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$$

$$h_K(u) = \sup\{\langle x, u\rangle : x \in K\}$$

Mixed volumes

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Let $K_1, K_2, \dots, K_m \in \mathcal{K}^n$ and let $\lambda_i \geq 0$ for $i = 1, \dots, m$. The volume of the linear combination $\sum_{i=1}^m \lambda_i K_i$ can be expressed as

$$\mathrm{V}\left(\sum_{i=1}^{m} \lambda_i K_i\right) = \sum_{i_1=1}^{m} \cdots \sum_{i_n=1}^{m} \lambda_{i_1} \dots \lambda_{i_n} \mathrm{V}(K_{i_1}, \dots, K_{i_n}).$$

The coefficients $\mathrm{V}(K_{i_1}, \dots, K_{i_n})$, so defined, are called mixed volumes and they are symmetric in every index for any permutation.

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The coefficients $\mathrm{V}(K_{i_1}, \ldots, K_{i_n})$, so defined, are called mixed volumes and they are symmetric in every index for any permutation.

- In particular, for any $L \in \mathcal{K}^n$, $\phi(K) = \mathrm{V}(L, K[n-1])$ is a continuous, translation invariant, homogeneous of degree $n-1$ valuation.

Brunn-Minkowski inequality

The classical Brunn-Minkowski inequality assures the concavity of the n-th root of the volume on $\mathcal{K}^n$:

Classical Brunn-Minkowski inequality

$\mathrm{V}^{\frac{1}{n}} : \mathcal{K}^n \longrightarrow \mathbb{R}$ is concave, i.e., for $t \in [0,1]$ and $K, L \in \mathcal{K}^n$,

$$\mathrm{V}\big((1-t)K + tL\big)^{\frac{1}{n}} \geq (1-t)\mathrm{V}(K)^{\frac{1}{n}} + t\mathrm{V}(L)^{\frac{1}{n}}.$$

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Mixed volumes satisfy a Brunn-Minkowski inequality too:

Brunn-Minkowski inequality for mixed volumes

Let $K_0, K_1, \cdots, K_n \in \mathcal{K}^n$, and $1 \leq i \leq n$. For $0 \leq t \leq 1$ the function

$$f(t) := \mathrm{V}((1-t)K_0 + tK_1[i], K_{i+1}, \ldots, K_n)^{\frac{1}{i}}$$

is concave on $[0,1]$.

Surface area measures

Mixed surface area measure

Let $K_2, \ldots, K_n \in \mathcal{K}^n$ be $n-1$ convex bodies. There exists a unique non-negative Borel measure $\mathrm{S}(K_2, \ldots, K_n; \cdot)$, the *mixed surface area measure*, so that, for every convex body K_1,

$$\mathrm{V}(K_1, K_2, \ldots, K_n) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_{K_1}(u)\, d\mathrm{S}(K_2, \ldots, K_n; u).$$

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Surface area measure of order $n-1$

The surface area measure of order $n-1$ of K is the mixed surface area measure

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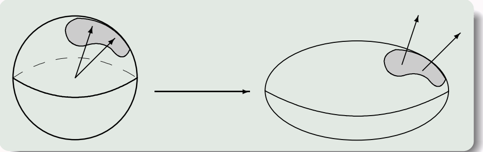

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In particular for $L, K \in \mathcal{K}^n$:

$$\mathrm{V}(L, K[n-1]) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u)\, d\mathrm{S}_{n-1}(K; u)$$

The functional $\mathcal{F}(K) = \int_{\mathbb{S}^{n-1}} f(u) d\mathrm{S}_{n-1}(K; u)$

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McMullen's theorem

McMullen 1980

$\phi : \mathcal{K}^n \mapsto \mathbb{R}$ is a continuous, translation invariant and $(n-1)$-homogeneous valuation if and only if there is $f \in \mathcal{C}(\mathbb{S}^{n-1})$ with

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- In particular, does it follow that f is a support function and in consequence that $\mathcal{F}$ is the mixed volume $\mathrm{V}(nL, K[n-1])$?

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Case $n = 2$

For every $f \in \mathcal{C}(\mathbb{S}^1)$ the functional

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is linear and in this case the Brunn-Minkowski inequality for $\mathcal{F}$ becomes an equality.

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Problem: $n \geq 3$

$f \in \mathcal{C}(\mathbb{S}^{n-1})$, $\mathcal{F}(K) = \int_{\mathbb{S}^{n-1}} f(u) d\mathrm{S}_{n-1}(K; u)$ and $\mathcal{F} \geq 0$. If

$$\mathcal{F}((1-t)K + tL)^{\frac{1}{n-1}} \geq (1-t)\mathcal{F}(K)^{\frac{1}{n-1}} + t\mathcal{F}(L)^{\frac{1}{n-1}}$$

for every $K, L \in \mathcal{K}^n$: *Is f the support function of a convex body?*

The symmetric case

Positive answer

Let $n \geq 3$ and $f \in \mathcal{C}(\mathbb{S}^{n-1})$ **even**.

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Brunn-Minkowski type inequality for $\mathcal{F}$

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is non-negative and satisfies a Brunn-Minkowski type inequality, then there exists a convex body L, whose support function is f, i.e., $f = h_L$ and

$$\mathcal{F}(K) = \int_{\mathbb{S}^{n-1}} h_L(u) d\mathrm{S}_{n-1}(K; u) = n\mathrm{V}(L, K[n-1]).$$

Remarks to the symmetric case

- If instead of the standard form of the Brunn-Minkowski inequality, the weaker form

$$\mathcal{F}((1-t)K + tL) \geq \min\{\mathcal{F}(K), \mathcal{F}(L)\}$$

for all $K, L \in \mathcal{K}^n$ and $t \in [0, 1]$ is used, then

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where $\overline{f}$ is even and Λ is the restriction to $\mathbb{S}^{n-1}$ of a linear function. Since

$$\int_{\mathbb{S}^{n-1}} \Lambda(u) d\mathrm{S}_{n-1}(K; u) = 0, \text{ for every } K \in \mathcal{K}^n,$$

if $\mathcal{F}$ is symmetric, f may be assumed to be even.

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Corollary

Let $n \geq 3$ and $\phi : \mathcal{K}^n \mapsto \mathbb{R}$ be a continuous, translation invariant, $(n-1)$-homogeneous and symmetric valuation.

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$$\phi(K) = n\mathrm{V}(L, K[n-1]).$$

The non-symmetric case

Positive answer for regular f

Let $n \geq 3$ and $f \in \mathcal{C}(\mathbb{S}^{n-1}) \cap W^{2,2}(\mathbb{S}^{n-1})$.

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for $K, L \in \mathcal{K}^n$ and $t \in [0,1]$, then there exists a convex body $L \in \mathcal{K}^n$ such that f is the support function of L.

Ideas of the proof

If K is of class C^2_+, then

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if ∂K is of class C^2 and the Gauß curvature is positive.

Ideas of the proof

If K is of class C^2_+, then

- its support function h_K is of class C^2 and $((h_K)_{ij} + \delta_{ij} h_K) > 0$.

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- $h_K + \epsilon\varphi$ is the support function of some $K_\epsilon \in \mathcal{K}^n$ of class C^2_+.
- $\mathcal{F}$ can be defined on $(-\epsilon, \epsilon)$ as:

$$g(\lambda) := \mathcal{F}(K_\lambda) \text{ for every } \lambda \in (-\epsilon, \epsilon).$$

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- This functional inequality holds for all h_K and φ satisfying the mentioned conditions.
- This provides a strong condition on f from which it follows that it is the support function of a convex body $L \in \mathcal{K}^n$.

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- This implies that f is a support function.

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