Mahler's Conjecture and Curvature

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A convex body K in $\mathbb{R}^n$ is a convex, compact subset with nonempty interior.

The dual body to K w.r.t. $z \in K$ is

$$K^z = \{y | \forall x \in K :< y, x - z >\leq 1\}$$

We consider the volume product

$$vol_n(K) \cdot vol_n(K^z)$$

There is a unique point $s(K)$ for which the volume product is minimal: *Santaló point*.

Blaschke-Santaló:

$$vol_n(K) \cdot vol_n(K^{s(K)}) \leq (vol_n(B_2^n))^2$$

Mahler's conjecture:

The cube and the crosspolytope are minimizers in the class of centrally symmetric convex bodies.

The simplex is a minimizer in the class of all convex bodies.

Mahler's conjecture has been verified for certain classes of convex bodies: *Reisner*, *Gordon*, *StRaymond*

It is not known whether a minimizer for the volume product is a polytope

Bourgain-Milman:

$$\exists C > 0 \forall n \in \mathbb{N} \forall K \in \mathcal{K}^n :$$

$$\left(\frac{C}{n}\right)^n \leq vol_n(K) \cdot vol_n(K^{s(K)})$$

Kuperberg, Nazarov: new proofs, improved constants

Local minima:

Nazarov, Petrov, Ryabogin, Zvavitch:

The cube is a local minimizer.

Reisner, Kim:

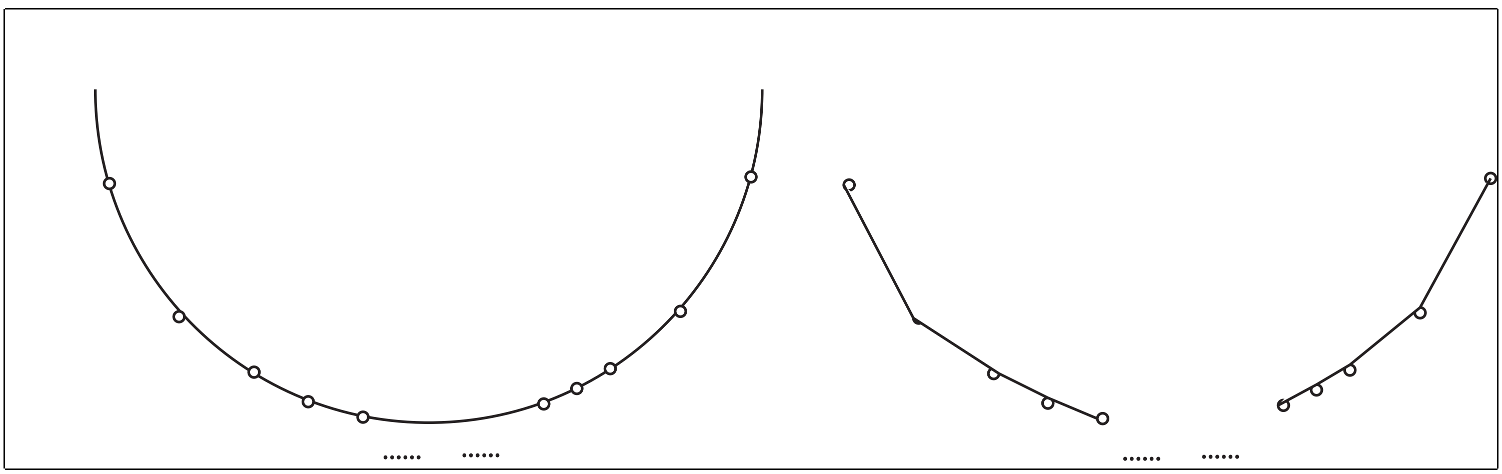
The simplex is a local minimizer.

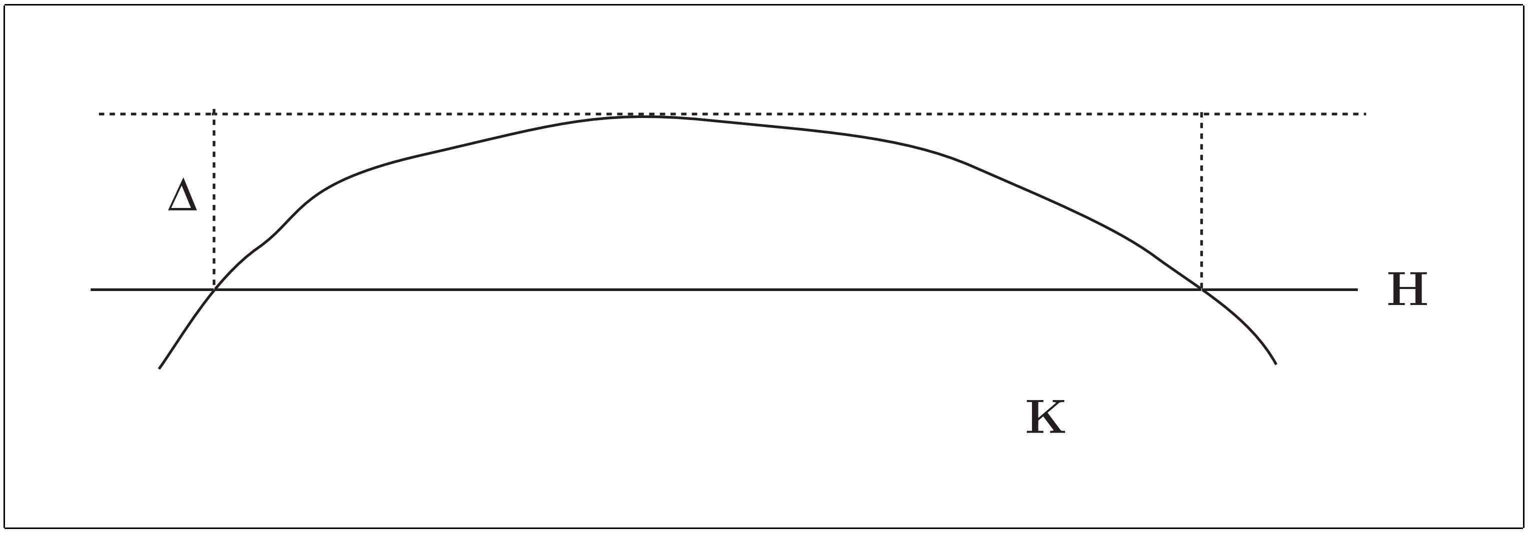
Stancu:

A convex body with everywhere strictly positive Gauß curvature is not a local minimizer.

Theorem 1 *Let K be a convex body in $\mathbb{R}^n$. Suppose that there is a point in the boundary of K with strictly positive generalized Gauß curvature, then the volume product is not a local minimum.*

Generalized Gauß curvature





Intersect K with a hyperplane H with distance Δ from the boundary point x.

Take the orthogonal projection of $K \cap H$ onto the support hyperplane at x. Blow up the projected image by $\frac{1}{\sqrt{2\Delta}}$.

If the convex bodies in the support hyperplane converge for $\Delta \to 0$ to an ellipsoid, then the product of the lengths of the principal axes is

$$\frac{1}{curvature(x)} = \frac{1}{\kappa(x)}$$

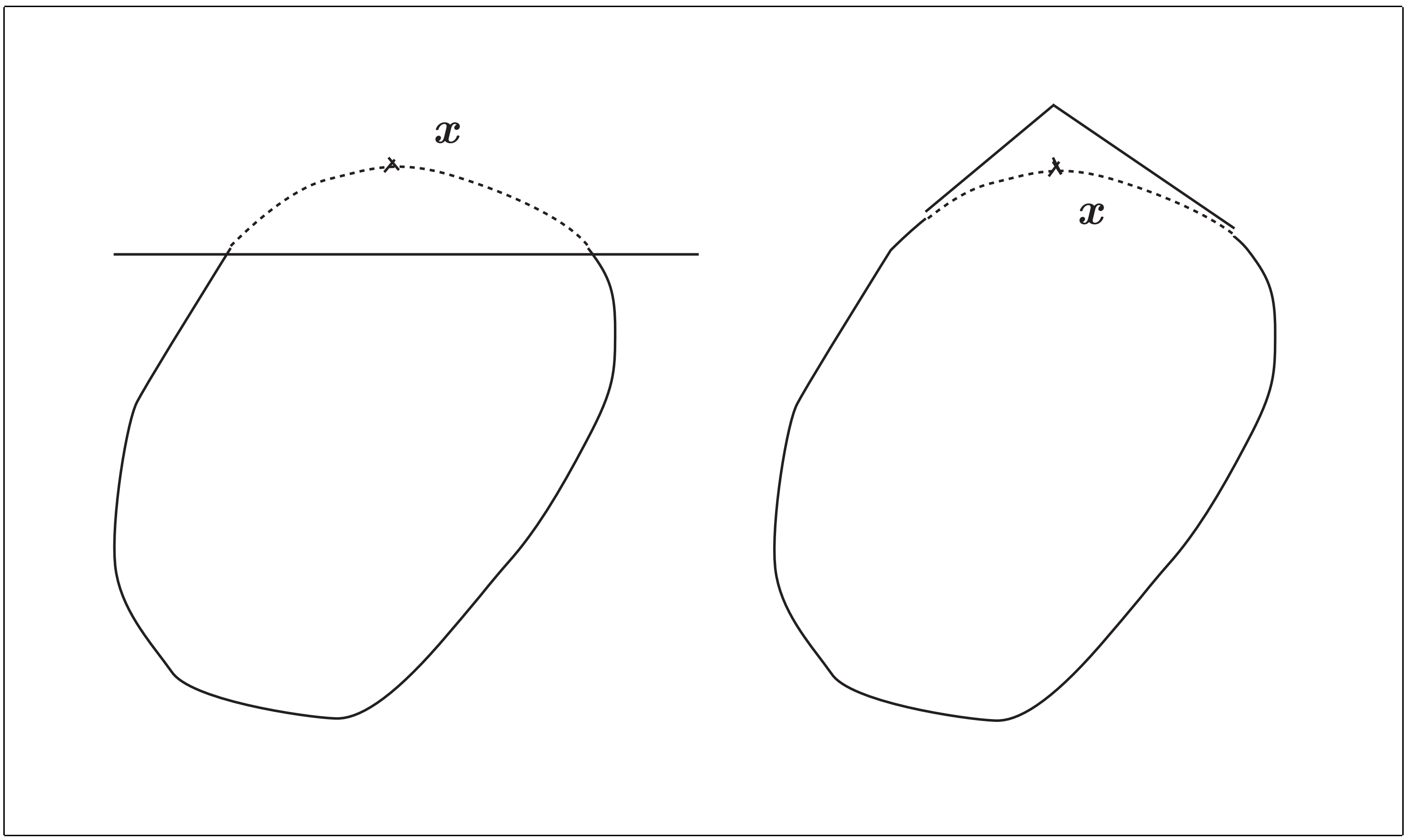
Aleksandrov-Busemann-Feller:

The boundary of a convex body has a.e. generalized Gauß curvature.

Therefore, a convex body that is a minimizer for the volume product has a.e. curvature equal to 0.

This suggests that a minimizer is a polytope.

Proof.



$K_x(\Delta)$ $K^x(\Delta)$

K has strictly positive Gauß curvature at x.

Standard position of K:

$$x = e_n \quad N_K(x) = e_n \quad s(K) = 0$$

Indicatrix of Dupin at x is a Euclidean ball

We shall prove: Either

$$vol_n(K_x(\Delta))vol_n((K_x(\Delta))^\circ) < vol_n(K)vol_n(K^\circ)$$

or

$$vol_n(K^x(\Delta))vol_n((K^x(\Delta))^\circ) < vol_n(K)vol_n(K^\circ)$$

Observations:

$$(K_x(\Delta))^\circ = (K^\circ)^x\left(\frac{\Delta}{1-\Delta}\right)$$

$$\kappa_K(x) = \frac{1}{\kappa_{K^\circ}(x)}$$

Lemma 2 *Let K be a convex body in $\mathbb{R}^n$ and suppose that the indicatrix of Dupin at $x \in \partial K$ exists and is a Euclidean ball of radius $r > 0$. Let $C(r, \Delta)$ be the cap at x of height Δ. Then*

$$|C(r,\Delta)| = g\left(\frac{\Delta}{r}\right)^{\frac{n+1}{2}} \frac{1}{n+1} 2^{\frac{n+1}{2}} vol_{n-1}(B_2^{n-1}) \Delta^{\frac{n+1}{2}} r^{\frac{n-1}{2}}$$

where $\lim_{t\to 0} g(t) = 1$.

Lemma 3 *[?] Let K be a convex body in $\mathbb{R}^n$ and suppose that the indicatrix of Dupin at $x \in \partial K$ exists and is a Euclidean ball of radius $\rho > 0$. Then for all $\epsilon > 0$ there is a Δ_ϵ such that for all Δ with $0 < \Delta \leq \Delta_\epsilon$*

$$\frac{2^{\frac{n+1}{2}}}{n(n+1)}|B_2^{n-1}|\Delta^{\frac{n+1}{2}}\rho^{\frac{n-1}{2}} \times \left\{(n+1)\left(1-\frac{\epsilon}{\rho}\right)^{\frac{n-1}{2}}(1-c\ \epsilon) - n\left(1+\frac{\epsilon}{\rho}\right)^{\frac{n-1}{2}}(1+c\ \epsilon)g\left(\frac{\Delta}{\rho+\epsilon}\right)^{n+1}\right\}$$
$$\leq |K^x(\Delta)\setminus K|$$
$$\leq \frac{2^{\frac{n+1}{2}}}{n(n+1)}|B_2^{n-1}|\Delta^{\frac{n+1}{2}}\rho^{\frac{n-1}{2}} \times \left\{(n+1)\left(1+\frac{\epsilon}{\rho}\right)^{\frac{n-1}{2}}(1+c\ \epsilon) - n\left(1-\frac{\epsilon}{\rho}\right)^{\frac{n-1}{2}}(1-c\ \epsilon)g\left(\frac{\Delta}{\rho+\epsilon}\right)^{n+1}\right\}.$$

$$
\begin{aligned}
&|K_x(\Delta)||K_x(\Delta)^\circ| \\
&\leq |K||K^\circ| + |K|\frac{2^{\frac{n+1}{2}}}{n(n+1)}|B_2^{n-1}|\Delta^{\frac{n+1}{2}} r^{-\frac{n-1}{2}} \\
&\times \Bigg\{ (n+1)\,(1+r\epsilon)^{\frac{n-1}{2}}\,(1+c\ \epsilon) \\
&-n\,(1-r\epsilon)^{\frac{n-1}{2}}\,(1-c\ \epsilon) g\left(\frac{\Delta}{\frac{1}{r}+\epsilon}\right)^{n+1} \Bigg\} \\
&-|K^\circ| g\left(\frac{\Delta}{r}\right)^{\frac{n+1}{2}} \frac{1}{n+1} 2^{\frac{n+1}{2}} vol_{n-1}(B_2^{n-1})\Delta^{\frac{n+1}{2}} (r-\epsilon)^{\frac{n-1}{2}}.
\end{aligned}
$$

$$
\begin{aligned}
&|K| \Bigg\{ (n+1)\,(1+r\epsilon)^{\frac{n-1}{2}}\,(1+c\ \epsilon) \\
&-n\,(1-r\epsilon)^{\frac{n-1}{2}}\,(1-c\ \epsilon) g\left(\frac{\Delta}{\frac{1}{r}+\epsilon}\right)^{n+1} \Bigg\} \\
&< n|K^{\circ}| g\left(\frac{\Delta}{r}\right)^{\frac{n+1}{2}} r^{n-1}\left(1-\frac{\epsilon}{r}\right)^{\frac{n-1}{2}} .
\end{aligned}
$$