Order-isomorphisms for cones and ellipsoids

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Order-isomorphisms - Motivating results

- In recent years, various characterizations of dualities and identities have been established, mainly for classes of convex bodies and functions. We call a bijective mapping $T : S \to S$
  - Order reversing isomorphism if : $x \leq y \iff Tx \geq Ty$
  - Order preserving isomorphism if : $x \leq y \iff Tx \leq Ty$
- Examples:
  - Theorem (Böröczky-Schneider): On the class of convex bodies (compact, containing the origin in the interior) $K \mapsto K^\circ$ is essentially the only inclusion reversing map.
  - Theorem (Artstein-Milman): The Legendre transform is, up to obvious linear modifications, the only involution on the class of l.s.c convex functions which reverses the (point-wise) order.
  - An earlier result: Gruber ; classification of the endomorphisms of lattices of convex bodies. More similar results by Schneider, Artstein-Milman,...

Some standard definitions

Cone: A nonempty subset K of a vector space which satisfies:

- $K + K \subset K$,
- $\alpha K \subset K$ for all $\alpha \geq 0$,
- $K \cap (-K) = \{0\}$.

Clearly, every cone is a convex set.

Partially ordered vector space: A vector space X equipped with an order relation compatible with the algebraic structure. Namely, if $x \leq y$ then

- $x + z \leq y + z$ for each $z \in X$
- $\alpha x \leq \alpha y$ for all $\alpha \in \mathbb{R}_+$

Given $(X, \leq)$, $X_+ = \{x \in X : x \geq 0\}$ is a cone

Given a cone K it induces a vector ordering: $x \leq y$ whenever $y - x \in K$

Cone order-isomorphisms in $\mathbb{R}^n$

- Let $K \subset \mathbb{R}^n$ be a closed non-degenerate cone.

  What is the general form of the following order-isomorphisms:

  - $T : (\mathbb{R}^n, \leq_K) \to (\mathbb{R}^n, \leq_K)$
  - $T : (K, \leq_K) \to (K, \leq_K)$
  - $T : (int(K), \leq_K) \to (int(K), \leq_K)$

- For which cones must such transformations be affine (linear) ?

Some History

- A.D Alexandrov-Ovchinnikova ('53), Zeeman ('64); if K is a right-circular cone in $\mathbb{R}^4$, then all order isomorphisms are affine, with a Lorentz transformation being the linear part.

- Rothaus ('66) has shown that for any non-angular closed non-degenerate cone, all order-isomorphisms are affine (holds both for a map defined on $\mathbb{R}^n$ or on the interior of K).
  - A cone K is non-angular if it has a compact base which does not have isolated extreme points.

Basic examples

- Every linear transformation B for which $BK = K$ is an order-isomorphism
- Compositions and translations are allowed
- $K = \mathbb{R}^n_+, \ T(x) = (x_1^3, x_2^5, ..., x_n^7)$
- Our result: essentially there are no other options.

Main results

For any cone K, the general form of an order-isomorphism is described as follows. The set $A \subset \mathbb{R}^n$ shall stand for either K, $int(K)$ or $\mathbb{R}^n$.

Theorem

Let $n \geq 2$. Let $K \subset \mathbb{R}^n$ be a closed non-degenerate cone. Let $T : (A, \leq_K) \to (A, \leq_K)$ be an order-isomorphism. Then, $\exists$ bijective increasing functions $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}$, and linearly independent vectors $v_1, ..., v_n \in K$ and $w_1, ..., w_n \in K$ such that

$$T(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} f_i(\alpha_i) w_i$$

for $\alpha_1 v_1 + \cdots + \alpha_n v_n \in A$. If $A = K$ or $A = \mathbb{R}^n$, $w_i = T(v_i)$.

Main results - continued

An additional condition on the order inducing cone, forces the order-isomorphism to be affine-linear;
We say that a set of $m > n$ vectors in $\mathbb{R}^n$ is n-independent if each n of them are linearly independent.

Theorem

Let $n > 2$. Let $K \subset \mathbb{R}^n$ be a closed cone. Assume K has at least $n+1$ n-independent extremal vectors. Let $T : (A, \leq_K) \to (A, \leq_K)$ be an order-isomorphism. Then, T is an affine transformation, i.e., $\exists v_0 \in \mathbb{R}^n \;\; \exists B \in GL_n$ s.t

$$T(x) = v_0 + Bx \;\; \text{and} \;\; BK = K$$

- Recall: a vector $e \in K$ is said to be an extremal vector of K if $e = x + y$ with $x, y \in K$ implies that x and y are linearly dependent.

Remarks

- An order isomorphism is continuous
- Any three extremal vectors of a cone (that generate three distinct extremal rays) are linearly independent. Hence, in $\mathbb{R}^3$, for a cone with more than 3 extremal rays, an order-isomorphism must be affine-linear.
- The same holds if we set two different orderings : $T : (A, \leq_{K_1}) \to (A, \leq_{K_2})$.

An application

Let us go back to the context of convex bodies; Let E^n denote the class of all compact ellipsoids centered at the origin and let E_0^n denote its subclass consisting of all non-degenerate ellipsoids.

Theorem

Let $T : E_0^n \to E_0^n$ or $T : E^n \to E^n$ be an order-isomorphism (with respect to inclusion). Then, T is induced by a linear point-map on $\mathbb{R}^n$.

Corollary

Let $T : E_0^n \to E_0^n$ be an order-reversing isomorphism. Then, there exists a linear transformation $G \in GL_n$ such that $T(\mathcal{E}) = G\mathcal{E}^\circ$, for all $\mathcal{E} \in E_0^n$.

Application - continued

- Trying the usual method from previous (mentioned) works, one encounters difficulties. For example: the class E_0^n, ordered by inclusion, does not satisfy lattice requirements. Namely there is no maximal object which is the "smallest" greater than some two given ellipsoids (and similarly minimum does not exist).

- Observation: Let D_n denote the standard Euclidean unit ball in $\mathbb{R}^n$. For each ellipsoid $\mathcal{E}$, let A be the unique symmetric positive definite matrix for which $\mathcal{E} = AD_n$ and denote $\mathcal{E} = \mathcal{E}_A$. Then, $\mathcal{E}_A \subset \mathcal{E}_B \iff A \leq B$. Proof:

$$\mathcal{E}_A \subset \mathcal{E}_B \iff ||B^{-1}A||_{op} \leq 1 \iff ||(B^{-1}A)^*||_{op} \leq 1$$
$$\iff ||AB^{-1}||_{op} \leq 1 \iff A \leq B$$

Application - continued

- By setting $T(\mathcal{E}_A) = \mathcal{E}_{F(A)}$, we get a bijective map F on
  - the **cone** of symmetric positive semi-definite, denoted by $\mathcal{C}^n$ $(T : E^n \to E^n)$,
  - all symmetric positive definite matrices, $\mathcal{A} = int(\mathcal{C}^n)$ $(T : E_0^n \to E_0^n)$.
- So F is an order-isomorphism with the ordering induced by $\mathcal{C}^n$, which is a good cone (its extreml vectors are of the form $v \otimes v$). By our main results: F is linear. But, we are not done yet!
- Theorem (Hue Geometry Group, Vietnam 2010): a bijective linear map $F : \mathcal{A} \to \mathcal{A}$ is of the form $A \mapsto UAU^*$.

  So, If I is fixed (T fixes D_n, w.l.o.g), then $I = UU^*$ and: $T(\mathcal{E}_A) = \mathcal{E}_{F(A)} = \mathcal{E}_{UAU^*} = UAU^* D_n = UAD_n = U\mathcal{E}_A$.

Ingredients of the proof

- Extremal lines and their translates are mapped to translations of extremal lines. The reason is the following characterization:

  $v \in K$ is extremal $\iff$ $0 \leq x, y \leq v$ implies that x, y are comparable.

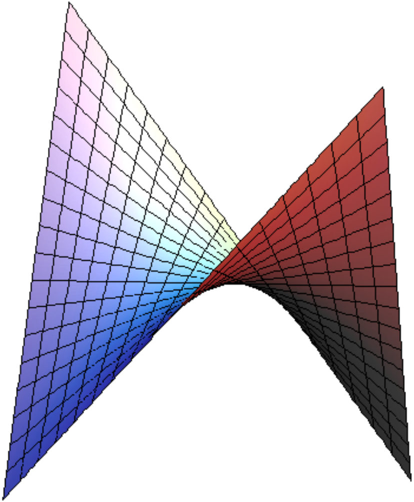
- Further investigation yields: parallel lines (rays) in extremal directions are mapped to parallel lines (rays). This is due to a characterization of doubly ruled surfaces;

Doubly ruled mappings

Proposition

Let $n \geq 3$. Let $F : \mathbb{R}^2_+ \to \mathbb{R}^n$ be injective. Assume that F maps each ray parallel either to the X-axis or the Y-axis onto a ray, and that the end-point is mapped to the endpoint. That is, for any $x_0 \in \mathbb{R}_+$, $F(x_0, \mathbb{R}_+)$ is a ray emanating from $F(x_0, 0)$ and for any $y_0 \in \mathbb{R}_+$, $F(\mathbb{R}_+, y_0)$ is a ray emanating from $F(0, y_0)$. Then each three rays parallel either to the X-axis or to the Y-axis are mapped to three rays which are parallel to one plane.

- In general, the image of such F is contained in a doubly ruled surface, which is known to be a subset of either a plane, a one-sheeted hyperboloid or a hyperbolic paraboloid. In our case (injective map; full rays) the second possibility is ruled out.
- For $F : \mathbb{R}^2 \to \mathbb{R}^n$ - the general form is given; up to linear modifications: $F(x, y) = (f(x), g(y), c \cdot f(x)g(y))$, $c \in \{0, 1\}$.



$$F(x, y) = (x, y, xy)$$

A fundamental theorem of affine geometry

- The fundamental theorem of affine geometry (many versions): a bijective mapping which maps lines to lines must be affine.
- A need for a F.T.A.G for finite number of directions, with a parallelism condition (for a map on a cone and on $\mathbb{R}^n$).

Theorem

Let $n \geq 2$. Let K_1 and K_2 be two closed non-degenerate cones in $\mathbb{R}^n$ and let $v_1, ..., v_n \in K_1$ be linearly independent vectors. Let $F : K_1 \to K_2$ be injective. Assume that for all $x \in K_1$ and $i \leq n$, $F((x + \mathrm{sp} v_i) \cap K_1) = (F(x) + \mathrm{sp} F(v_i)) \cap K_2$. Then, $\exists$ bijective functions $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}$ so that for $x = a_1 v_1 + \cdots + a_n v_n \in K_1$,

$$F(x) = \sum_{i=1}^{n} f_i(a_i) F(v_i)$$

- Adding another generic direction, for which F maps parallel lines to parallel lines yields that F is affine-additive.