Mean Section Bodies

Wolfgang Weil (KIT)

joint work with P. Goodey

Cortona

June 2011

1. The body $M_k(K)$

For $k = 1, ..., d$, the k-th **mean section body** $M_k(K)$ of a convex body $K \subset \mathbb{R}^d$ was introduced in **Goodey-W. 1992** as the Minkowski average of all intersections of K with k-dimensional (affine) flats.

In terms of support functions

$$h(M_k(K), \cdot) = \int_{A(d,k)} h(K \cap E, \cdot)\mu_k(dE).$$

($A(d,k)$ affine Grassmannian with invariant measure μ_k)

Whereas $M_d(K) = K$ (trivial) and $M_1(K)$ is always a ball (not trivial but easy), the case $2 \leq k \leq d-1$ raises the interesting question, whether $M_k(K)$ determines K uniquely.

The mean section body arises, e.g., in the kinematic formula for (centred) support functions,

$$\int_{G_d} h^*(K \cap gM, \cdot)\mu(dg) = \sum_{k=1}^{d} h^*(M_k(K), \cdot)V_k(M).$$

The mean section body was also motivated by the (analogously defined) **mean projection body** $P_k(K)$, considered first by **Schneider 1977**.

Since $P_k : K \mapsto P_k(K)$ is an intertwining linear operator, harmonic analysis could be used to investigate the injectivity of P_k (**Spriestersbach, Goodey et al., Kiderlen...**). However, the picture is still incomplete ($3 < k < d/2$).

In our case, the operator $M_k : K \mapsto M_k(K)$ is intertwining but not linear!

For k = 2 (i.e. planar sections), uniqueness was proved in **Goodey-W. (1992)**. As it was shown, up to a linear function,

$$h(M_2(K), u) = c_d \int_{S^{d-1}} \alpha(x, u) \sin \alpha(x, u) S_{d-1}(K, du).$$

($\alpha(x, u)$ angle between x and u; $S_j(K, \cdot)$ jth area measure of K)

This shows that $h(M_2(K), \cdot) = \tilde{M}_2 S_{d-1}(K, \cdot)$, were $\tilde{M}_2$ is a continuous, linear and intertwining operator (mapping measures to functions and given by the non-symmetric sine transform). Therefore, also here spherical harmonics could be used to show uniqueness (up to translations).

The translational restriction was later removed by **Goodey (1998)**, where it was also shown that, for $3 \le k \le d-1$ and centrally symmetric bodies K, the mean section body $M_k(K)$ determines K, in case $\dim K \ge d-k+2$.

For centrally symmetric K,

$$h(M_k(K),\cdot) = c_{dk} R_{d+1-k,1} V_{d+1-k}(K|\cdot),$$

a Radon transform of a projection function, hence arguments from harmonic analysis on Grassmannians could be used.

However, the general case ($3 \le k \le d-1$ and arbitrary body K) remained open.

2. Results

Theorem 1. *For $2 \leq k \leq d$, a convex body K of dimension* $\dim K \geq d + 2 - k$ *is uniquely determined by the mean section body $M_k(K)$.*

A major ingredient of the proof is the following symmetry relation.

Theorem 2. *For convex bodies K, L and $k \in \{2, ..., d\}$, we have*

$$\int_{S^{d-1}} h(M_k(K), u) S_{d+1-k}(L, du) = \int_{S^{d-1}} h(M_k(L), u) S_{d+1-k}(K, du).$$

3. Proofs

The **proof of Theorem 2** uses the following formula for polytopes K:

$$h(M_k(K), u) = c(d, k) \sum_{F \in \mathcal{F}_{d+1-k}(K)} \gamma(F, K; -u) |\langle F, u \rangle| V_{d+1-k}(F)$$

($\mathcal{F}_j(K)$ set of j-dimensional faces, $\gamma(F, K; -u)$ common outer angle, $|\langle F, u \rangle|$ length of the projection of u onto aff (F))

as well as the representation of area measures of polytopes L:

$$S_{d+1-k}(L, A) = \sum_{G \in \mathcal{F}_{d+1-k}(L)} V_{d+1-k}(G) \int_{n(L,G)} \mathbf{1}_A(u) \omega_{k-2}^{G^\perp}(du).$$

($n(L, G)$ spherical image of L at G, $\omega_{k-2}^{G^\perp}$ spherical Lebesgue measure in $G^\perp$)

Proof of Theorem 1, for $k \geq 3$ (sketch):

- Given $M_k(K)$, we know the integrals

$$\int_{S^{d-1}} h(M_k(K), u) S_{d+1-k}(L, du) = \int_{S^{d-1}} h(M_k(L), u) S_{d+1-k}(K, du)$$

for all bodies L (by Theorem 2).

• Choose $H \in G(d, d+2-k)$ and a convex body $L \subset H$. Then almost all intersections $H \cap E, E \in A(d,k)$, are 2-dimensional and the image of μ_k under $E \mapsto H \cap E$ is (proportional to) μ_2^H. Hence

$$
\begin{aligned}
h(M_k(L), u) &= \int_{A(d,k)} h(L \cap E, u)\mu_k(dE) \\
&= c_{d,k} \int_{A(H,2)} h(L \cap E', u)\mu_2^H(dE') \\
&= c_{d,k}[\pi^*_{H,1} h(M_2^H(L), \cdot)](u), \quad u \in S^{d-1},
\end{aligned}
$$

where $M_2^H(L)$ is the mean section body of L of order 2 in H and $\pi^*_{H,1}$ is a lifting operator considered in **Goodey-Kiderlen-W. (2011)**.

• Hence, we know all integrals

$$\int_{S^{d-1}} [\pi^*_{H,1} h(M_2^H(L),\cdot)](u) S_{d+1-k}(K,du)$$
$$= \int_{S^{d-1}\cap H} h(M_2^H(L),u)[\pi_{H,1} S_{d+1-k}(K,\cdot)](du),$$

for all bodies $L \subset H$ and all $H \in G(d, d+2-j)$. Here, $\pi_{H,1}$ is the adjoint projection operator (**Goodey-Kiderlen-W. (2011)**).

• Since

$$L \mapsto M_2^H(L)$$

is injective (for bodies in H), the support functions

$$\{h(M_2^H(L), \cdot), L \subset H\}$$

span a dense subspace of $C_o(S^{d-1} \cap H)$ (the continuous functions with centroid at o).

- Since $\pi_{H,1} S_{d+1-k}(K,\cdot)$ annihilates linear functions, the integrals

$$\int_{S^{d-1}\cap H} h(M_2^H(L),u)[\pi_{H,1} S_{d+1-k}(K,\cdot)](du),$$

for all $L \subset H$ determine the measure $\pi_{H,1} S_{d+1-k}(K,\cdot)$ ($H \in G(d,d+2-j)$).

• Therefore, we know all projections $\pi_{H,1} S_{d+1-k}(K,\cdot)$, for $H \in G(d, d+2-j)$.

By Corollary 3.4 in **Goodey-Kiderlen-W. (1998)**, the measure $S_{d+1-k}(K,\cdot)$ is determined.

• Since $\dim K \geq d + 2 - k$, Minkowski's uniqueness result shows that K is determined (up to a translation t).

• Since $M_k(K) = M_k(K + t) = M_k(K) + a_{d,k} V_{d-k}(K)t$, we obtain $t = 0$.

- Goodey, P. and Weil, W., The determination of convex bodies from the mean of random sections. *Math. Proc. Camb. Phil. Soc.* **112** (1992), 419–430.

- Weil, W., Translative and kinematic integral formulae for support functions. *Geom. Dedicata* **57** (1995), 91–103.

- Goodey, P., Radon transforms of projection functions. *Math. Proc. Camb. Phil. Soc.* **123** (1998), 159–168.

- Goodey, P., Kiderlen, M. and Weil, W., Section and projection means of convex bodies. *Mh. Math.* **126** (1998), 37–54.

- Goodey, P., Kiderlen, M. and Weil, W., Spherical projections and liftings in geometric tomography. *Adv. Geom.* **11** (2011), 1–47.