How often is a random quantum state k–entangled?

Stanislaw Szarek
Elisabeth Werner
Karol Zyczkowski

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Determine the "size" of certain convex sets that appear naturally in quantum information theory.

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Some notation

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- Φ is positive if $\Phi(\rho) \geq 0$ for all $\rho \in \mathcal{M}_d$, $\rho \geq 0$, i.e. positive semidifinite

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$\mathcal{P}_k(\mathcal{M}_d)$= set of k-positive maps on $\mathcal{M}_d$

This set is a convex cone.

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f.i. via Jamiolkowski-Choi isomorphism

$$\mathcal{P}_k(\mathcal{M}_d) \longleftrightarrow \mathcal{BP}_k(\mathbb{C}^d \otimes \mathbb{C}^d),$$

the space of k-block positive $d^2 \times d^2$ matrices.

The set of k- entangled operators on $\mathbb{C}^d \otimes \mathbb{C}^d$ is

$$Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d) =$$

$$\text{conv}\left(\left\{|\xi\rangle\langle\xi| : \xi = \sum_{j=1}^{k} u_j \otimes v_j, u_j, v_j \in \mathbb{C}^d, j = 1, \ldots, k\right\}\right)$$

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$\xi = \sum_{j=1}^k u_j \otimes v_j \in \mathbb{C}^d \otimes \mathbb{C}^d$ is called a k- entangled vector, i.e.

k-entangled states have rank $\leq k$: $Ent_k^v(\mathbb{C}^d \otimes \mathbb{C}^d)$

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- For all k: $Ent_k(\mathbb{C}^d \otimes \mathbb{C}^d) \subset Ent_{k+1}(\mathbb{C}^d \otimes \mathbb{C}^d)$

Via Jamiolkowski-Choi isomorphism k-entangled states on $\mathbb{C}^d \otimes \mathbb{C}^d = \mathbb{C}^{d^2}$ are in correspondence with maps on $\mathcal{M}_d$

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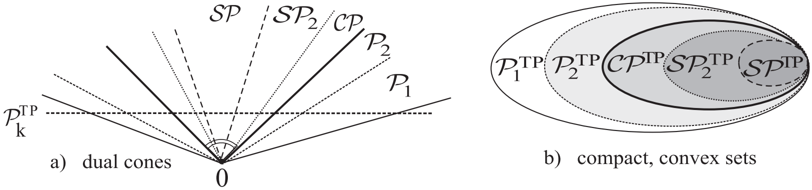
$\mathcal{SP}_k(\mathcal{M}_d)$ is the convex cone of k- superpositive operators Φ on $\mathcal{M}_d$:

$$\Phi(\rho) = \sum A_i^\dagger \rho A_i$$

such that each A_i has rank $\leq k$.

For $d = 3$



a) dual cones

b) compact, convex sets

Normalizations

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- maps $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ get normalized such that Φ is trace preserving: for all states ρ

  $$Tr(\Phi(\rho)) = Tr(\rho)$$

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$$\mathcal{SP}_k^{TR}(\mathcal{M}_d) =$$
$$\mathcal{SP}_k(\mathcal{M}_d) \cap \{\Phi : \mathcal{M}_d \to \mathcal{M}_d : Tr(\Phi(\rho)) = Tr(\rho), \forall \rho\}$$

Geometrically, this means that we intersect the cones cones with hyperplanes and get convex sets.

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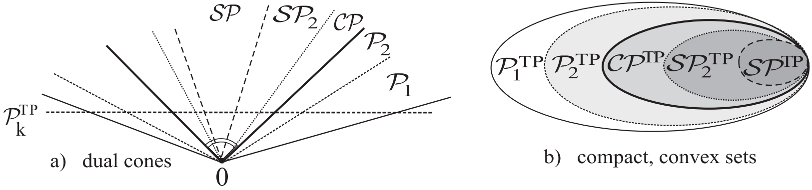

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Measure "size" of a convex body K via **volume radius**

$$\mathrm{vrad}(\mathrm{K}) = \left(\frac{|\mathrm{K}|}{|\mathrm{B}_2^{\mathrm{n}}|} \right)^{\frac{1}{\mathrm{n}}}$$

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$$\mathrm{vrad}\left(\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) = ?$$

Lower Bound

SKETCH

- Step 1: We show that

$$B_{HS}\left(\frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}\right) \subset \mathrm{conv}\left(-Ent^{1}_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d) \cup Ent^{1}_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d)\right)$$

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- Step 2:

$$\text{Rogers Shepard} \implies \mathrm{vrad}\left(\mathrm{Ent}^1_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d)\right) \geq \frac{1}{2}\ \frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}$$

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- **Upper Bound**

$$\mathrm{vrad}\left(\mathrm{Ent}_k^1(\mathbb{C}^d \otimes \mathbb{C}^d)\right) \leq C\frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}$$

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$$B_{HS}\left(0, \frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}\right) \subseteq \mathrm{conv}\left(-Ent^{1}_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d) \cup Ent^{1}_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d)\right)$$

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$$\Longleftrightarrow$$

$$h = h_{-Ent^1_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d) \cup Ent^1_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d)} \geq \frac{k^{\frac{1}{2}}}{d^{\frac{3}{2}}}$$

Let A be a hermitian operator on $\mathbb{C}^d \otimes \mathbb{C}^d$, $\|A\|_{HS} = 1$.

$$\begin{aligned} h(A) &= \max_{M \in Ent_{2k}^{1}(\mathbb{C}^d \otimes \mathbb{C}^d)} \left| Tr\ AM \right| = \max_{\eta \in Ent_{2k}^{v}(\mathbb{C}^d \otimes \mathbb{C}^d)} \left| Tr\ A|\eta\rangle\langle\eta| \right| \\ &= \max_{\eta\ \in Ent_{2k}^{v}(\mathbb{C}^d \otimes \mathbb{C}^d)} \left| \langle\eta|A|\eta\rangle \right| \end{aligned}$$

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Claim 1

$$\geq \max_{\phi,\psi\ \in Ent^v_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d)} \mathrm{Re}\left(\langle\phi|A|\psi\rangle\right)$$

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&= \max_{\psi\ \in Ent^v_{2k}(\mathbb{C}^d \otimes \mathbb{C}^d)} \left\| A|\psi\rangle \right\|^{(k)}
\end{aligned}
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For $k \in \{1, \dots d\}$, $\xi \in \mathbb{C}^d \otimes \mathbb{C}^d$

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Claim 2 For all $\xi \in \mathbb{C}^d \otimes \mathbb{C}^d$

$$\|\xi\|^{(k)} \geq \sqrt{\frac{k}{d}} \ \|\xi\|_{HS}$$

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