A problem of Klee on inner section functions of convex bodies

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(joint work with R. J. Gardner, D. Ryabogin, and A. Zvavitch)

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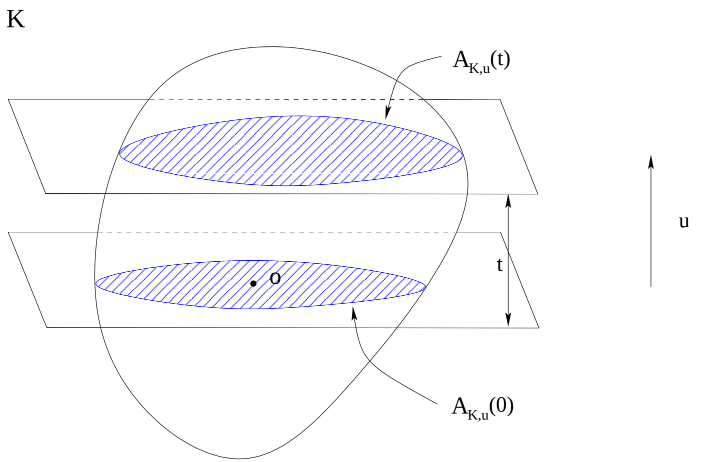
Let K be a convex body in $\mathbb{R}^n$. $u \in S^{n-1}$.
The parallel section function $A_{K,u}(t)$ is defined by

$$A_{K,u}(t) = \mathrm{Vol}_{n-1}(K \cap (u^{\perp} + tu)),$$

for $t \in \mathbb{R}$.

Here, $u^{\perp} = \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\}$.

Introduction



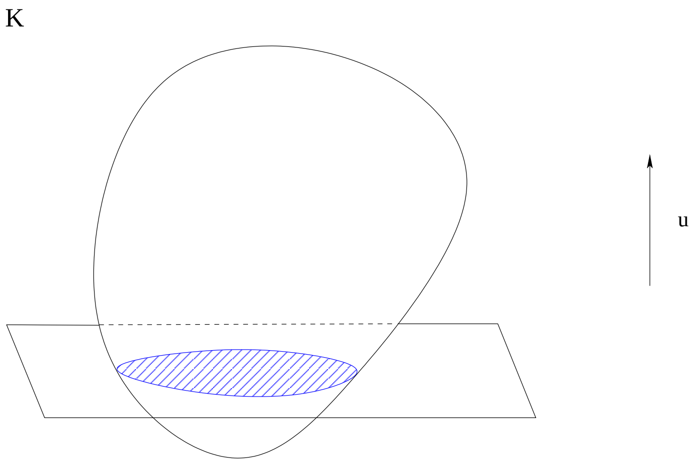
Introduction

The inner section function m_K is defined by

$$m_K(u) = \max_{t \in \mathbb{R}} A_{K,u}(t),$$

for $u \in S^{n-1}$.

Proof



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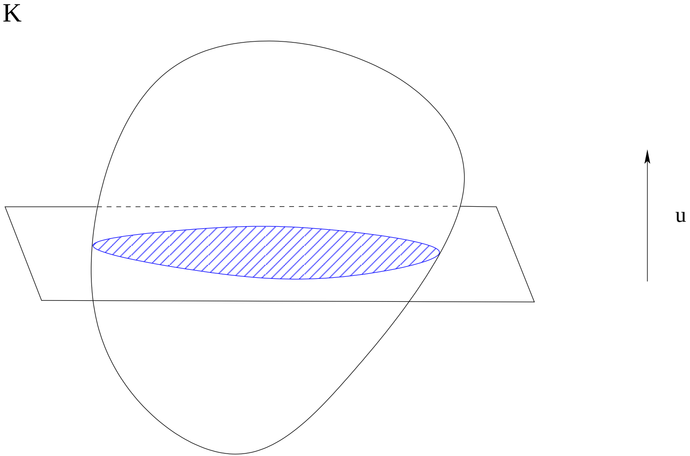

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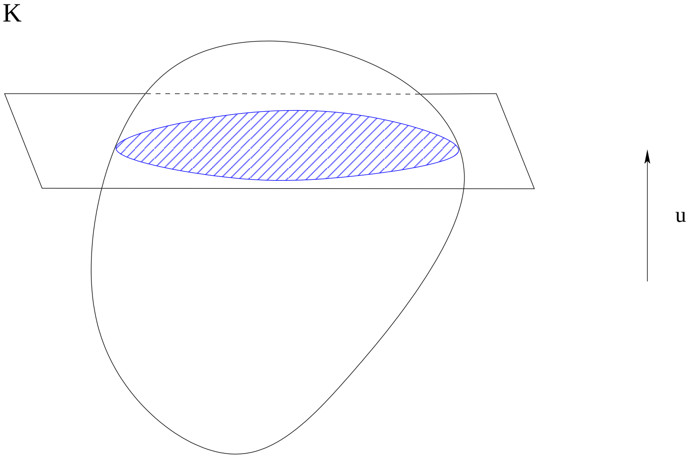

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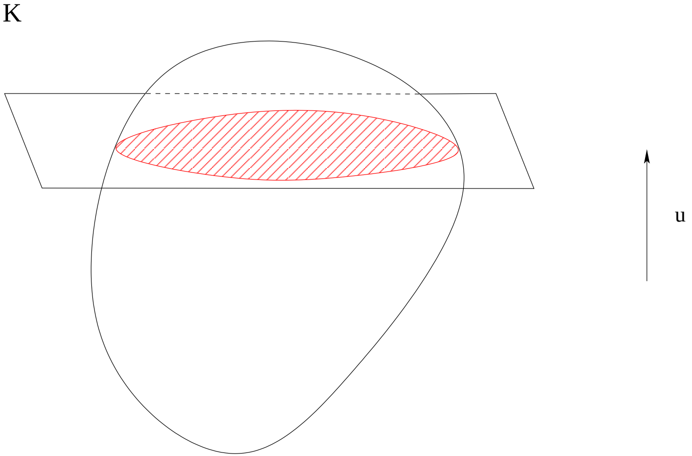

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The inner section function and cross-section bodies were studied by Meyer, Makai, Martini, Ódor, Brehm,...

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That is, given two convex bodies K and L such that

$$m_K(u) = m_L(u), \qquad \forall u \in S^{n-1},$$

is it true that $K = \pm L + a$?

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Thus, $\forall u \in S^{n-1}$,

$$\int_{S^{n-1} \cap u^{\perp}} \rho_K^{n-1}(\theta)\, d\theta = \int_{S^{n-1} \cap u^{\perp}} \rho_L^{n-1}(\theta)\, d\theta.$$

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$$R\rho_K^{n-1} = R\rho_L^{n-1} \quad \Rightarrow \quad \rho_K^{n-1} = \rho_L^{n-1} \quad \Rightarrow \quad K = L.$$

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Main result

Theorem

There exist convex bodies K and L in R^n, $n \geq 3$, such that K is not centrally symmetric, L is origin-symmetric, and $m_K = m_L$.

Proof

Define K by its radial function:

$$\rho_K(x) = (|x|^3 + \epsilon x_n^3)^{-1/3}, \qquad x \in \mathbb{R}^n \setminus \{0\},$$

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One can check that

- K is convex for small enough $\epsilon > 0$,
- K is not centrally symmetric.

Proof

Since K is a body of revolution about the x_n-axis, for any fixed $t \in \mathbb{R}$, the function $A_{K,u}(t)$, $u \in S^{n-1}$ is rotationally symmetric.

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Therefore we can write $A_{K,u}(t) = A_{K,\phi}(t)$, $\phi \in S^1$.

Proof

Lemma

(i) For each $\phi \in S^1$, the parallel section function $A_{K,\phi}(t)$ has a maximum at a unique point $t = t_\varepsilon(\phi)$.

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Thus, we have

$$m_K(\phi) = A_{K,\phi}(t_\varepsilon(\phi)).$$

Proof

(ii) For all ε small enough we have

$$t_\varepsilon(\phi) = T_\varepsilon(\phi)\varepsilon,$$

where $T_\varepsilon(\phi) \in C^\infty(S^1)$.
(iii) For $k = 0, 1, \dots,$ there is a constant $c_1(k, n)$ such that

$$\left| \frac{d^k T_\varepsilon(\phi)}{d\phi^k} \right| \leq c_1(k, n),$$

for all $\phi \in S^1$.

Proof

Lemma

$m_K \in C_e^\infty(S^{n-1})$ and there is a rotationally symmetric function $g_\varepsilon \in C_e^\infty(S^{n-1})$ such that

$$m_K(u) = \kappa_{n-1} + g_\varepsilon(u)\varepsilon/(n-1),$$

for all $u \in S^{n-1}$. Here, $\kappa_{n-1} = |B_2^{n-1}|$.

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for all $u \in S^{n-1}$. Here, $\kappa_{n-1} = |B_2^{n-1}|$.
Moreover, there is a constant $c_2(k,n)$ such that

$$\left| \frac{d^k g_\varepsilon(\phi)}{d\phi^k} \right| \le c_2(k,n),$$

for all $\phi \in S^1$.

Proof

We have

$$(n-1)(R^{-1}m_K)(u) = 1 + (R^{-1}g_\varepsilon)(u)\varepsilon,$$

for $u \in S^{n-1}$, where R is the spherical Radon transform.

Proof

Define a new body L by

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If we show that L is an origin-symmetric convex body (for small enough ε), then we are done, since

$$\begin{aligned} m_L(u) = \mathrm{Vol}_{n-1}(L \cap u^\perp) &= \frac{1}{n-1}(R\rho_L^{n-1})(u) \\ &= (R(R^{-1}m_K))(u) = m_K(u), \end{aligned}$$

for all $u \in S^{n-1}$.

Proof

By Lemma,

$$\rho_L(u) = \left(1 + (R^{-1}g_\varepsilon)(u)\varepsilon\right)^{1/(n-1)}.$$

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We will use Fourier transform techniques of Koldobsky. Extend g_ε to a homogeneous of degree -1 function on $\mathbb{R}^n \setminus \{o\}$. Then,

$$(R^{-1}g_\varepsilon)(u) = \frac{\pi}{(2\pi)^n}\widehat{g_\varepsilon}(u),$$

for $u \in S^{n-1}$.
Hence,

$$\rho_L(u) = \left(1 + \frac{\pi}{(2\pi)^n}\widehat{g_\varepsilon}(u)\varepsilon\right)^{1/(n-1)},$$

for all $u \in S^{n-1}$.

Proof

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Define

$$G_{\varepsilon,u}(z) = (1 - z^2)^{(n-3)/2} \int_{S^{n-1}\cap u^\perp} g_\varepsilon(zu + \sqrt{1 - z^2}\, v)\, dv,$$

for $-1 \le z \le 1$.
Then, for $k = 0, 1, \dots,$ there are constants $c_3(k, n)$ such that

$$|G^{(k)}_{\varepsilon,u}(0)| \le c_3(k, n),$$

for $u \in S^{n-1}$.

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If n is even, then $\widehat{g_\varepsilon}(u) = (-1)^{(n-2)/2}\pi G_{\varepsilon,u}^{(n-2)}(0)$.

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If n is odd, then $\widehat{g_\varepsilon}(u) = (-1)^{(n-1)/2}(n-2)!(I_\varepsilon(u) + \Sigma_\varepsilon(u))$, where

$$I_\varepsilon(u) = \int_{-1}^{1} |z|^{-n+1} \left(G_{\varepsilon,u}(z) - \sum_{k=0}^{n-2} G_{\varepsilon,u}^{(k)}(0) \frac{z^k}{k!} \right) dz$$

and

$$\Sigma_\varepsilon(u) = 2 \sum \left\{ \frac{G_{\varepsilon,u}^{(k)}(0)}{k!(2+k-n)} : k = 0, \dots, n-3, k \text{ even} \right\}.$$

Proof

For both n even and odd, $\widehat{g_\varepsilon}$ is bounded on S^{n-1} uniformly in ε.

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For both n even and odd, $\widehat{g_\varepsilon}$ is bounded on S^{n-1} uniformly in ε. We see that

$$\rho_{L_\varepsilon}(u) = \left(1 + \frac{\pi}{(2\pi)^n}\widehat{g_\varepsilon}(u)\varepsilon\right)^{1/(n-1)} > 0,$$

for all $u \in S^{n-1}$, and hence L_ε is an origin-symmetric star body.

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Q.E.D.

Other open problems

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Bonnesen's question: does a convex body in $\mathbb{R}^n$, $n \geq 3$, have to be a ball if both its inner section function and its brightness function are constant?

THANK YOU!!!