TALK ON "REAL RANK OF BINARY FORMS"

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- Polynomial ring: $R = \oplus R_d$, where $R_d = \mathrm{Sym}^d(\mathbb{R}^2)$
- $f = \sum_{i=1}^{d} \binom{d}{i} a_i x^{d-i} y^i \in R_d$, binary real form of degree d

1. Complex rank

- Generic rank is $\lfloor \frac{d}{2} \rfloor + 1$
- Sylvester algorithm finds an explicit decomposition of a form f with rank r, i.e. $f = \sum_{i=1}^{r} (l_i)^d$, where $l_i \in R_1$
- Sylvester Theorem: f with generic rank has a unique decomposition if d is odd, ∞^1 decompositions if d is even.

2. Real rank

Typical rank replaces the notion of generic rank. We say that a rank is *typical* if it occurs in an open (Euclidean) subset of R_d.

Remark 2.1. The smallest typical rank of a real binary form is equal to the generic complex rank $\lfloor \frac{d}{2} \rfloor + 1$.

Remark 2.2. The rank of any binary form of degree d is $\leq d$.

2.1. **Sylvester algorithm:** The dual ring of differential operators $D = \mathbb{R}[\partial_x, \partial_y]$ acts on R and preserves the degrees:

$$R_d \otimes D_k \to R_{d-k}$$

Given $\ell = ax + by \in R_1$ the apolar operator is $\ell^\perp = -bx + ay$. We have $\ell^\perp(\ell) = 0$.

Lemma 2.3 (Apolarity lemma). *Let $f \in R_d$ and $l_i \in R_1$ distinct linear forms. There are coefficients $c_i \in \mathbb{R}$ such that $f = \sum_{i=1}^{r} c_i (l_i)^d$ if and only if*

$$(l_1^\perp \circ \ldots \circ l_r^\perp)(f) = 0$$

Apolar ideal: $f^\perp = \{h \in D : h(f) = 0\}$. A form f has rank r if r is the smallest degree k such that $(f^\perp)_k$ contains a form with real distinct roots.

$$(f^\perp)_k = \mathrm{Ker}(A_f : D_k \to R_{d-k})$$

where A_f is the *catalecticant matrix*

Proposition 2.4. *Let $f \in R^d$, then its apolar ideal $f^\perp$is a complete itersection ideal, i.e. it is generated by two real forms g_1, g_2 such that $\deg g_1 + \deg g_2 = d + 2$ and $V(g_1, g_2) = \emptyset$. Conversely, any two such forms generate an ideal $f^\perp$ for some $f \in R$ with degree $\deg g_1 + \deg g_2 - 2$.*

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Definition 2.5. We say that $f \in R_d$ is *generated in generic degrees* if $(\deg g_1, \deg g_2) = (\frac{d+2}{2}, \frac{d+2}{2})$ if d is even, $(\deg g_1, \deg g_2) = (\frac{d+1}{2}, \frac{d+3}{2})$ if d is odd. This happens if and only if the intermediate catalecticant has maximal rank.

Here we always assume that f is generated in generic degrees.

2.2. **Excursus on low degrees.**

- Degree $d = 1$: trivial
- Degree $d = 2$: R_2 = symmetric matrices 2×2. The real rank is equal to the complex rank, which is equal to the rank of the matrix. The only typical rank is 2. The rank is equal to the number of distinct roots.

Theorem 2.6 (Sylvester, 1861)**.** *Given $f \in R_d$, the number of distinct real roots of f is less than or equal to the rank of f.*

Corollary 2.7. *If $f \in R_d$ has d distinct real roots, then the rank of f is d. In particular d is a typical rank for forms of degree d.*

- Degree $d = 3$:

  Theorem 2.8. *In R_3 the typical ranks are $2, 3$. Given $f \in R_3$, generated in generic degrees and with distinct roots, then*

  $$\mathrm{rk}(f) = 2 \Leftrightarrow f \text{ has } 1 \text{ real root}$$

  $$\mathrm{rk}(f) = 3 \Leftrightarrow f \text{ has } 3 \text{ real roots}$$

- Degree $d = 4$:

  Theorem 2.9 (Comon-Ottaviani)**.** *In R_4 the typical ranks are $3, 4$. Given $f \in R_4$, generated in generic degrees and with distinct roots, then*

  $$\mathrm{rk}(f) = 3 \Leftrightarrow f \text{ has } 0 \text{ or } 2 \text{ real roots}$$

  $$\mathrm{rk}(f) = 4 \Leftrightarrow f \text{ has } 4 \text{ real roots}$$

- Degree $d = 5$:

  Theorem 2.10 (Comon-Ottaviani)**.** *In R_5 the typical ranks are $3, 4, 5$. Given $f \in R_5$, generated in generic degrees and with distinct roots, then*

  $$\mathrm{rk}(f) = 3 \Leftrightarrow \text{the generator of } (f^{\perp})_3 \text{ has } 3 \text{ real roots}$$

  $$\mathrm{rk}(f) = 4 \Leftrightarrow \text{the generator of } (f^{\perp})_3 \text{ has not } 3 \text{ real roots}$$

  $$\mathrm{rk}(f) = 5 \Leftrightarrow f \text{ has } 5 \text{ real roots}$$

3. The case of all real roots

Theorem 3.1 (Causa-Re)**.** *Given a form f of degree $d \geq 3$ with distinct roots. Then all the roots are real if and only if $\alpha f_x + \beta f_y$ has $d - 1$ real distinct roots for all $(\alpha, \beta) \neq (0, 0)$.*

Corollary 3.2. *A real binary form has all real roots if and only if it has rank 1 or d.*

4. Typical ranks

Lemma 4.1. *Let $f \in R_d$ be generated in generic degrees of rank r. Then f is typical if and only if any form in $(f^{\perp})_{r-1}$ has non-real roots.*

The previous lemma says that the only obstruction to f being a typical form of rank r is that $(f^{\perp})_{m-1}$ may contain a form with all real roots, but no form with distinct roots. Then perturbating f may result in a form of degree $r-1$.

Theorem 4.2 (Blekherman)**.** *All the integers $\lfloor \frac{d}{2} \rfloor + 1 \leq r \leq d$ are typical ranks for real binary forms of degree d.*

5. Boundaries

Let $\mathcal{R}_d^r$ be the set of forms in R_d with rank r. $\mathcal{R}_d^r$ is a semi-algebraic set in R_d.

The (algebraic) boundary of the $\mathcal{R}_d^r$ (i.e. Zariski closure of the topological boundary) are described in the two extreme cases, that is for $r = d$ and $r = \lfloor \frac{d}{2} \rfloor + 1$.

- $r = d$: The boundary of R_d^d is given by the discriminant hypersurface Δ, given by forms with at least a double root.
- $r = \lfloor \frac{d}{2} \rfloor + 1$:

  Theorem 5.1 (Lee-Sturmfels)**.** *Let $d \geq 5$. If $d = 2k+1$, the boundary of $R_d^{\lfloor \frac{d}{2} \rfloor + 1}$ is an irreducible hypersurface of degree $2k(k+1)$ which is the dual variety of the multiple root locus $\Delta_{2^{k-1},3}$. If $d = 2k$, the boundary of $R_d^{\lfloor \frac{d}{2} \rfloor + 1}$ is the union of two irreducible hypersurfaces of degrees $2k(k-1)(k-2)$ and $3k(k-1)$ which are the dual varieties of the multiple root loci $\Delta_{2^{k-3},3^2}$ and $\Delta_{2^{k-2},4}$.*

References

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