Vanishing Hessian and Wild Polynomials

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Outline

1. Wild Polynomials
   - Different notions of ranks
   - Known results about wild polynomials
   - An example of wild cubic
2. Vanishing Hessian Implies Wild
   - Apolarity and border apolarity
   - Concise polynomials of minimal border rank
     - Wild cubic = cubic with vanishing Hessian
   - A wild polynomial with non-vanishing Hessian
3. Two Infinite Series of Wild Polynomials and Their border VSP

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- $\sigma_r(X) = \overline{\bigcup \{\langle R \rangle : R = \{x_1, \dots, x_r\}, x_i \in X\}} \subset \mathbb{P}^N$.

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    $\underline{\mathbf{r}}_X(F) = 3$ while $\mathrm{sr}_X(F) = \mathrm{cr}_X(F) = 4$.
- Higher border rank examples when $\underline{\mathbf{r}}_X(F) < \mathrm{sr}_X(F)$?

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Polynomials: Tame and Wild

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Theorem (Buczyńska, Buczyński,2014)

For cubic polynomials, F is tame if $n \leq 3$.

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First Example of a Wild Cubic

Example(Buczyńska, Buczyński,2014)

$$F = x_0 x_1^2 - x_2(x_1 + x_4)^2 + x_3 x_4^2.$$

F is a wild cubic with $\underline{\mathbf{r}}(F) = 5$ and $\mathrm{cr}(F) = \mathrm{sr}(F) = 6$.

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$$F = \lim_{t\to 0}\Big(\frac{1}{3}(x_1 + tx_0)^3 - \frac{1}{3}((x_1 + x_4) + tx_2)^3 + \frac{1}{12}(2x_4 - tx_2)^3 - \frac{1}{9}(x_1 - x_4)^3 + \frac{1}{9}(x_1 + 2x_4)^3\Big).$$

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Question(Ottaviani)

F is a concise polynomial with vanishing Hessian.
Is there a relation between wild polynomials and concise polynomials with vanishing Hessian?

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Theorem

Let $d \geq 3$ and $F \in S^d V$ be a concise polynomial of minimal border rank. Then:

$$\mathrm{Hess}(F) = 0 \implies F \text{ is wild.}$$

Further, for $d = 3$, one has the following equivalences:

$$\mathrm{Hess}(F) = 0 \iff \mathrm{cr}(F) > \underline{\mathbf{r}}(F) \iff \mathrm{sr}(F) > \underline{\mathbf{r}}(F) \iff F \text{ is wild.}$$

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Border Apolarity(Border Rank)(Buczyńska, Buczyński,2019)

$\underline{\mathbf{r}}(F) \leq r \iff \exists$ a homogeneous ideal $\mathcal{I} \subset \mathrm{Ann}(F)$ with $\mathrm{HF}(T/\mathcal{I}, d) = \min\{r, \dim S^d V^*\}$ s.t. $\mathcal{I}$ is a flat limit of saturated ideals defining r distinct points.

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$\mathrm{sr}(F) \leq r \iff \exists$ a saturated homogeneous ideal $\mathcal{I} \subset \mathrm{Ann}(F)$ with $\mathrm{HF}(T/\mathcal{I}, d) = r$ for $d \gg 0$ s.t. $\mathcal{I}$ is a flat limit of saturated ideals defining r distinct points.

Border Apolarity(Border Rank)(Buczyńska, Buczyński,2019)

$\underline{\mathbf{r}}(F) \leq r \iff \exists$ a homogeneous ideal $\mathcal{I} \subset \mathrm{Ann}(F)$ with $\mathrm{HF}(T/\mathcal{I}, d) = \min\{r, \dim S^d V^*\}$ s.t. $\mathcal{I}$ is a flat limit of saturated ideals defining r distinct points.

Wild $\iff$ All ideals realizing border rank are not saturated.

Outline

Concise Polynomials of Minimal Border Rank

Wild Example: $F = x_0x_1^2 - x_2(x_1 + x_4)^2 + x_3x_4^2$.

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  $\langle (y_0, y_2, y_3)^2, y_0 y_4, y_1 y_3, -y_1 y_2 + y_2 y_4, y_0 y_1 + y_1 y_2 + y_3 y_4 \rangle$.

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    - $\mathcal{J}_2 = \mathrm{Ann}(F)_2 \Rightarrow \mathcal{I} \subset \mathcal{J} \Rightarrow \mathcal{I}^{\mathrm{sat}} \subset \mathcal{J}^{\mathrm{sat}} = \mathcal{J} \Rightarrow \mathcal{J}$ contains $y_0 \notin \mathrm{Ann}(F)$ while $\mathrm{Ann}(F)_1 = \emptyset$.

Concise Polynomials of Minimal Border Rank

Theorem (Part I)

Let $d \geq 3$ and $F \in S^d V$ be a concise polynomial of minimal border rank. Then:

$$\mathrm{Hess}(F) = 0 \implies F \text{ is wild.}$$

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Lemma

$R \subset \mathbb{P}^n$ is a scheme defined by the saturated ideal $\mathcal{J}$ such that $\langle R \rangle = \mathbb{P}^n$. Then $(\mathcal{J}_d)^{\perp}$ is spanned by $(n+1)$ algebraically independent forms.

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$$(\mathcal{I}^{\mathrm{sat}}_{d-1})^\perp \subset (\mathcal{I}_{d-1})^\perp = \langle \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n} \rangle.$$

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$\{\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n}\}$ are algebraically independent $\Rightarrow \mathrm{Hess}(F) \neq 0$. □

Outline

Vanishing Hessian and Wild Cubics

Theorem (Part II)

Let $F \in S^3\mathbb{C}^{n+1}$ be a concise cubic of minimal border rank. Then:

$$\mathrm{Hess}(F) \neq 0 \implies \mathrm{cr}(F) \leq n+1 \iff F \text{ is not wild.}$$

In particular:

$$\mathrm{Hess}(F) = 0 \iff \mathrm{cr}(F) > \underline{\mathbf{r}}(F) \iff \mathrm{sr}(F) > \underline{\mathbf{r}}(F) \iff F \text{ is wild.}$$

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$\mathrm{Hess}(F) \neq 0 \Leftrightarrow T_F$ is the structure tensor of a $(n+1)$-dimensional smoothable algebra A.
We can use A to construct a scheme of length $(n+1)$ that spans F. □

Outline

1. Wild Polynomials
   - Different notions of ranks
   - Known results about wild polynomials
   - An example of wild cubic
2. Vanishing Hessian Implies Wild
   - Apolarity and border apolarity
   - Concise polynomials of minimal border rank
     - Wild cubic = cubic with vanishing Hessian
   - A wild polynomial with non-vanishing Hessian
3. Two Infinite Series of Wild Polynomials and Their border VSP

A Wild Polynomial with Non-vanishing Hessian

$$F = v_0 u_0^3 u_1 + v_1 u_0 u_1^3 + v_0^3 v_1^2 \in S^5\mathbb{C}^4$$

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- F is a wild polynomial with non-vanishing Hessian not of minimal border rank.

Outline

VSP and Border VSP

- $F \in S^d V$, $V \cong \mathbb{C}^{n+1}$, $T = \mathrm{Sym}^\bullet V^*$.

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Variety of Sums of Powers

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Border Variety of Sums of Powers(Buczyńska, Buczyński,2019)

$$\underline{\mathrm{VSP}}(F, r) = \left\{\mathcal{I} \in \mathrm{Slip}_{r,\mathbb{P}^n} \mid \mathcal{I} \subset \mathrm{Ann}(F) \subset T\right\}.$$

Wild Polynomials of Higher Degree and Their VSP

$$G_d = v_0 u_1^{d-1} + v_1 u_0 u_1^{d-2} + \dots v_{d-1} u_0^{d-1} = \sum_{i=0}^{d-1} v_i u_0^i u_1^{d-1-i}.$$

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- G_d are concise polynomials and $\underline{\mathbf{r}}(G_d) = d + 2$.
- $\mathrm{Hess}(G_d) = 0$.
- G_d is an infinite series of wild polynomials of degree d.

Wild Polynomials of Higher Degree and Their $\underline{\mathrm{VSP}}$

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Proposition

$\underline{\mathrm{VSP}}(G_d, d+2)$ is isomorphic to the projective space $\mathbb{P}^{d+2} \cong \mathbb{P}(S^{d+2}\mathbb{C}^2) \cong S^{d+2}\mathbb{P}^1$.

An Infinite Series of Wild Cubics and Their VSP

$$F_n = x_0x_1^2 + x_1x_2x_4 + x_3x_4^2 + x_4x_5x_7 + x_6x_7^2 + x_8x_7x_{10} + x_9x_{10}^2 + \cdots + x_{n-4}x_{n-3}^2 + x_{n-3}x_{n-2}x_n + x_{n-1}x_n^2.$$

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- $n = 3k + 1$.

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An Infinite Series of Wild Cubics and Their VSP

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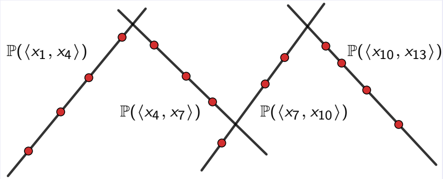

Figure: $\mathcal{J}(R) \in \underline{\mathrm{VSP}}(F_{13}, 14)$ and $R \subseteq C_4$

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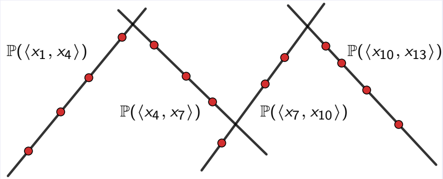

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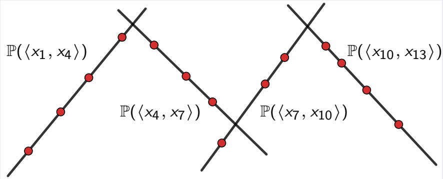

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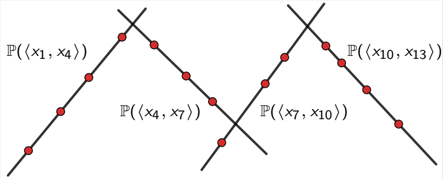

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