Apolarity, border rank, and multigraded Hilbert scheme

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Joint work with **Weronika Buczyńska** $\rightarrow\rightarrow$

Toric variety

Assume X is a smooth projective toric variety over $\mathbb{C}$, and L is a very ample line bundle on X.

The Cox ring of X is the multigraded polynomial ring

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Three main examples:

- $X = \mathbb{P}^{n-1}$, $L = \mathcal{O}_{\mathbb{P}^{n-1}}(d)$, $S = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$ graded by $\mathbb{Z}$.

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- $X = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots$ (one factor $\mathbb{P}^2$ and $k-1$ factors $\mathbb{P}^1$),
  $L = \mathcal{O}_X(d_1, d_2, d_3, \dots, d_k)$,
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Dual polynomials: $$\widetilde{S} := \bigoplus_{D \in \mathrm{Pic}(X)} H^0(D)^* = \mathbb{C}[x_1, \ldots, x_n].$$

S acts on $\widetilde{S}$ via $\lrcorner$, the apolarity action:

$$\alpha_i \lrcorner \left(x_1^{(a_1)} \cdot x_2^{(a_2)} \cdots x_n^{(a_n)}\right) = \begin{cases} x_1^{(a_1)} \cdots x_i^{(a_i-1)} \cdots x_n^{(a_n)} & \text{if } a_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Saturated ideals and apolarity lemma for linear spans

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$$X \subset \mathbb{P}\left(H^0(L)^*\right) = \mathbb{P}\left(\widetilde{S}_L\right)$$

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$$R \subset X \subset \mathbb{P}\left(H^0(L)^*\right) = \mathbb{P}\left(\widetilde{S}_L\right)$$

$$I_R \subset \bigoplus\nolimits_{D \in \mathrm{Pic}(X)} H^0(D) = S$$

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A homogeneous ideal $I \subset S$ is saturated $\iff$
$I = I_R$ for some subscheme $R \subset X$ or $I = \left(I : (H^0(L))^\infty\right)$.

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Rank

$I_R = (\Theta \mid \Theta_{|_R} = 0)$, $\mathrm{Ann}(F) = (\Theta \mid \Theta \lrcorner F = 0)$, $I_R \subset \mathrm{Ann}(F) \Leftrightarrow \langle R \rangle \ni F$

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Definition

The X-rank of $F \in \mathbb{P}\left(H^0(L)^*\right)$ is the minimal integer $r(F)$ such that

$$F \in \langle p_1, \ldots, p_{r(F)} \rangle \text{ for some } p_i \in X.$$

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- $X = \mathbb{P}^{n-1}, L = \mathcal{O}_X(d) \Rightarrow$ Waring rank,
- $X = \mathbb{P}^{a-1} \times \mathbb{P}^{b-1} \times \mathbb{P}^{c-1}, L = \mathcal{O}_X(1,1,1) \Rightarrow$ tensor rank.
- $X = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \cdots, L = \mathcal{O}_X(d_1, d_2, d_3, \dots) \Rightarrow$ partially symmetric rank.

Apolarity for rank

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Lemma (Apolarity for rank)

Fix $F \in \mathbb{P}\left(\widetilde{S}_L\right)$ and $r \in \mathbb{N}$. Then

$r(F) \leq r \iff \exists$ *saturated radical homogeneous $I \subset S$, such that $I \subset \mathrm{Ann}(F)$ and $\dim(S/I)_D = r$ for "very large" D.*

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- Tensor in $\mathbb{P}(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3)$ with $R(F) = 5$, $F =$ $[x_1 \otimes y_1 \otimes z_1 + x_2 \otimes y_1 \otimes z_2 + x_2 \otimes y_2 \otimes z_1 + x_3 \otimes y_1 \otimes z_3 + x_3 \otimes y_3 \otimes z_1]$

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$$\lim_{t \to 0} [(tx_1 - x_2 - x_3) y_1 z_1 + x_2 (y_1 + ty_2)(z_1 + tz_2) + x_3 (y_1 + ty_3)(z_1 + tz_3)].$$

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- Monomial in $\mathbb{P}\left(S^d\ \mathbb{C}^3\right)$

$$F = \left[x_0^{(a_0)} x_1^{(a_1)} x_2^{(a_2)}\right]$$

where $d \ = a_0 + a_1 + a_2$

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[Landsberg-Teitler 2010]: $br(F) \leq (a_1+1)(a_2+1)$

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Example:

- Partially symmetric monomial tensor in $\mathbb{P}\left(S^{d_1}\mathbb{C}^3 \otimes S^{d_2}\mathbb{C}^2 \otimes S^{d_3}\mathbb{C}^2 \otimes \cdots\right)$

$$F = \left[x_0^{(a_0)} x_1^{(a_1)} x_2^{(a_2)} \otimes y_0^{(b_0)} y_1^{(b_1)} \otimes z_0^{(c_0)} z_1^{(c_1)} \otimes \cdots\right]$$
$$\in \lim_{t\to 0}\left\langle \left(x_0 + \sqrt[a_1+1]{1}\, t x_1 + \sqrt[a_2+1]{1}\, t x_2\right)^{d_1} \otimes \left(y_0 + \sqrt[b_1+1]{1}\, t y_1\right)^{d_2} \otimes \cdots \right\rangle,$$

where $d_1 = a_0 + a_1 + a_2, d_2 = b_0 + b_1, \ldots$

[Landsberg-Teitler 2010]: $br(F) \leq (a_1+1)(a_2+1)(b_1+1)(c_1+1)\cdots$.

Weak apolarity for border rank

$$br(F) = \min\{r \mid F = \lim F_t \text{ with } r(F_t) \leq r\}.$$

Lemma (Apolarity for rank)

$r(F) \leq r \iff \exists$ saturated radical homogeneous $I \subset S$, such that $I \subset \mathrm{Ann}(F)$ and $\dim(S/I)_D = r$ for "very large" D.

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Theorem (Weak apolarity for border rank, Buczyńska-B.)

Fix $F \in \mathbb{P}\left(H^0(L)^*\right)$ and $r \in \mathbb{N}$. Then

$br(F) \leq r \Longrightarrow \exists$ homogeneous ideal $I \subset S$, such that $I \subset \operatorname{Ann}(F)$ and $\dim(S/I)_D = \quad r \quad$ for "many" $D \in \operatorname{Pic}(X)$.

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Theorem (Weak apolarity for border rank, Buczyńska-B.)

Fix $F \in \mathbb{P}\left(H^0(L)^*\right)$ *and* $r \in \mathbb{N}$*. Then*

$br(F) \leq r \Longrightarrow \exists$ *homogeneous ideal* $I \subset S$*, such that* $I \subset \operatorname{Ann}(F)$ *and* $\dim(S/I)_D = \min\{r, \dim S_D\}$ *for all* $D \in \operatorname{Pic}(X)$.

Examples

$$br(F) \leq r \Rightarrow \exists I \subset S, I \subset \operatorname{Ann}(F), \dim(S/I)_D = \min\{r, \dim S_D\}$$

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Let $F = [x_1^{d-1}x_2]$. Since $br(F) = 2$, the following ideal $I = (\alpha_2^2, \alpha_3, \ldots, \alpha_n)$ satisfies $I \subset \mathrm{Ann}(F) = (\alpha_1^d, \alpha_2^2, \alpha_3, \ldots, \alpha_n)$ and

$$\dim(S/I)_d = \min\{2, \dim S_d\} = \begin{cases} 0 & \text{if } d < 0, \\ 1 & \text{if } d = 0, \\ 2 & \text{if } d > 0. \end{cases}$$

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If $F = [x_0^{(2)} x_1^{(2)} x_2] \in S^5\mathbb{C}^3$, then $br(F) = 6$, and $I = (\alpha_1^3, \alpha_0\alpha_2^2, \alpha_1\alpha_2^2, \alpha_2^3)$.

Examples, continued

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If $F = [x_1 y_1 z_1 + x_2 y_1 z_2 + x_2 y_2 z_1 + x_3 y_1 z_3 + x_3 y_3 z_1]$, then $br(F) = 3$,

$$I = \begin{pmatrix} \{\alpha_1\alpha_i\}_{i=1,2,3}, & \begin{array}{c}\{\alpha_i\beta_i - \alpha_3\beta_3\}_{i=1,2}, \\ \{\alpha_1\beta_i\}_{i=2,3}, \\ \alpha_2\beta_3, \\ \alpha_3\beta_2,\end{array} & \begin{array}{c}\{\alpha_i\gamma_i - \alpha_3\gamma_3\}_{i=1,2}, \\ \{\alpha_1\gamma_i\}_{i=2,3}, \\ \alpha_2\gamma_3, \\ \alpha_3\gamma_2,\end{array} \\ \alpha_2\alpha_3(\alpha_2 - \alpha_3), & & \\ \{\beta_i\beta_j\}_{i,j=2,3}, & \begin{array}{c}\{\beta_1\gamma_i - \beta_i\gamma_1\}_{i=2,3}, \\ \{\beta_i\gamma_j\}_{i,j=2,3},\end{array} & \{\gamma_i\gamma_j\}_{i,j=2,3} \end{pmatrix}.$$

Multigraded Hilbert scheme

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Let $h\colon \operatorname{Pic} X \to \mathbb{N}$, and consider the multigraded Hilbert scheme:

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- The closure of $\mathcal{I}_r$ is the set of limits of ideals of points:

$$\mathcal{I}_r \subset \overline{\mathcal{I}_r} = \mathrm{Slip}_r \subset \mathrm{Hilb}_S^{h_r}.$$

Apolarity for border rank

$$\mathrm{Slip}_r = \overline{\{I_R \subset S\colon R = \{p_1, \ldots, p_r\} \text{ and } \dim(S/I_R)_D = \min\{r, \dim S_D\}\}}.$$

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Fix $F \in \mathbb{P}\left(H^0(L)^*\right)$, $r \in \mathbb{N}$. Then

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For F and r we set

$$\begin{aligned} VSP^0(F,r) &= \{R \subset X \colon R = \{p_1, \dots, p_r\} \text{ and } F \in \langle R \rangle\} \\ &= \{R \subset X \colon R = \{p_1, \dots, p_r\} \text{ and } I_R \subset \mathrm{Ann}(F)\}. \end{aligned}$$

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Theorem (Fixed Ideal Theorem, or *fit*)

Let $B \subset G$ be a Borel subgroup.

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Matrix multiplication

Suppose $F \in \mathbb{C}^{nm} \otimes \mathbb{C}^{mp} \otimes \mathbb{C}^{pn}$ is the matrix multiplication tensor. Its border rank is a measure of the computational complexity of multiplying two (typically large) matrices.

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(stamp from `clipart.email`)

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[Connor–Harper–Landsberg 2019] use border apolarity and fit to get new lower bounds for the border rank of matrix multiplication.
See the talk by Landsberg next week!

Tensors of minimal border rank

Proposition (Buczyńska–B.–Kleppe–Teitler 2015)

Suppose $F \in \mathbb{P}(S^d\mathbb{C}^n)$ *is such that* $br(F) = n$

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Suppose $F \in \mathbb{P}(\mathbb{C}^a \otimes \mathbb{C}^a \otimes \mathbb{C}^a)$ *is concise and* $br(F) = a$, *then* $\mathrm{Ann}(F)$ *has at least* $a-1$ *minimal generators in (multi)-degree* $(1,1,1)$.

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Example: if $F = [x_1y_1z_1 + x_2y_1z_2 + x_2y_2z_1 + x_3y_1z_3 + x_3y_3z_1]$, then $\alpha_2\beta_1\gamma_1$ and $\alpha_3\beta_1\gamma_1$ are among minimial generators of $\operatorname{Ann}(F)$.

Border rank of monomials

Theorem (Fixed Ideal Theorem, or *fit*)

F is preserved by G, and $B \subset G$ is a Borel subgroup.

- $br(F) \leq r \iff bVSP(F, r)^B \neq \emptyset$ *(there is a BT-fixed point).*
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Border rank of monomials

Theorem (Monomial Version of Fixed Ideal Theorem, or *move-fit*)

F is a monomial (presereved by $T = (\mathbb{C}^*)^n$),

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Theorem (Border rank of monomials in few variables)

Let $X = \mathbb{P}^2 \subset \mathbb{P}(S^d\, \mathbb{C}^3)$. If $F = \left[x_0^{(a_0)} x_1^{(a_1)} x_2^{(a_2)}\right]$ is a monomial, then

$$br(F) = (a_1 + 1)(a_2 + 1)$$

(assuming $a_0 \geq a_1 \geq a_2$).

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Theorem (Border rank of monomials in few variables)

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$$br(F) = (a_1 + 1)(a_2 + 1)(b_1 + 1)(c_1 + 1)\cdots$$

(assuming $a_0 \geq a_1 \geq a_2$, and $b_0 \geq b_1$, and $c_0 \geq c_1, \ldots$).

Secant and cactus varieties

$$X \subset \qquad \mathbb{P}\left(H^0(L)^*\right).$$

Secant and cactus varieties

The r^{th} **secant variety** of X is

$$\sigma_r(X) = \{F \mid br(F) \leq r\}$$

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For $\dim X \geq 6$, $r \geq 14$, and embeddings of sufficiently large degrees

$$\sigma_r(X) \neq \mathfrak{K}_r(X).$$

Equations

$$\sigma_r(X) = \{F \mid br(F) \leq r\}, \ \mathfrak{K}_r(X) = \overline{\bigcup \{\langle R \rangle \mid R \text{ is finite of degree } r\}}.$$

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Does the border apolarity technique help?

Border cactus apolarity

$F \in \mathbb{P}\left(H^0(L)^*\right),\ r \in \mathbb{N},\ h_r(D) = \min\{r, \dim S_D\}.$

For an ideal I set $H_I(D) := \dim((S/I)_D)$, the **Hilbert function**.

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Theorem (Apolarity for border rank)

$br(F) \leq r \iff \exists$ *a homogeneous ideal* $I \subset \mathrm{Ann}(F)$
with $H_I = h_r$, *and such that*
I is a limit of saturated ideals of points.

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with $H_I \in \left\{h_r, h_r', h_r'', \ldots, h_r^{(k)}\right\}$, and such that

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Limits of saturated ideals and smoothable schemes

$I \subset S$, an ideal with $H_I(D) = h_r(D) = \min\{\dim S_D, r\}$.
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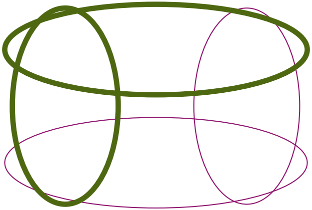

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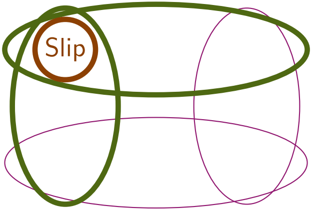

Distinguishing secant and cactus

Example of [Gałązka, Mańdziuk, Rupniewski]:

Suppose $X = \mathbb{P}^6 \subset \mathbb{P}\left(S^d\mathbb{C}^7\right)$, $d \geq 6$.

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- If $F \in \mathfrak{K}_{14}(X)$, then they propose a simple and effective algorithm to determine if $F \in \sigma_{14}(X)$.
- In the proof they use border apolarity (for both secant and cactus varieties) and the description of finite smoothable Gorenstein schemes in degree 14 [Jelisiejew 2016].

Thank you!