REAL ALGEBRAIC GEOMETRY. A FEW BASICS. DRAFT FOR A RESEARCH SEMINAR

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Abstract. Comparisons between complex and real algebraic varieties, ideals and real ideals. Main reference is [BCR]. The case of tensors. Some directions to be studied.

Contents

1. Real affine (and projective!) algebraic varieties

We start from a complex (algebraic) variety $X \subset \mathbb{C}^{n+1}$ (affine case) or $X \subset \mathbb{P}^n_{\mathbb{C}}$ (projective case).

Proposition 1.1. *The following properties for X are equivalent:*
(i) the ideal $I(X)$ has real generators $f_1, \ldots, f_k \in \mathbb{R}[x_0, \ldots, x_n]$.
(ii) $X = \overline{X}$.

Definition 1.2. *The set of real points $V_{\mathbb{R}}(f_1, \ldots, f_k)$ where $f_i \in \mathbb{R}[x_0, \ldots, x_n]$ is called a real affine algebraic variety.*

In equivalent way, after Prop. 1.1, the conjugation operator acts on a complex variety X satisfying $X = \overline{X}$ and the real variety consists of its fixed points.

Example 1.3. *$V(y^2 - x^2(x-1))$ has a connected one dimensional component and a isolated point at the origin. It is smooth as a real manifold, with two components of different dimension, but the origin is not a regular point in the algebraic sense. We will refer to regular points in the algebraic sense, namely for a d-dimensional variety, a point P is regular when the jacobian matrix of the defining equations has maximal rank d at P.*

Theorem 1.4. *Let X be real variety which is irreducible. If $X_{\mathbb{R}}$ has a regular point in X, then $X_{\mathbb{R}}$ is Zariski dense in X. In particular $I(X_{\mathbb{R}}) = I(X)$.*

Example 1.5. *$\mathbb{P}^n(\mathbb{R})$ is an affine algebraic variety. An embedding $\mathbb{P}^n(\mathbb{R}) \to \mathrm{Sym}^2\mathbb{R}^{n+1}$ in the affine space of $(n+1) \times (n+1)$ symmetric matrices is given by the following normalized 2-Veronese map*

$$(x_0, \ldots, x_n) \mapsto \frac{x_i x_j}{\sum_{i=0}^n x_i^2}$$

The image consists of positive semidefinite matrices of rank one and unitary norm. Its equations are given by $A^2 = A$, $\mathrm{trace}(A) = 1$.

It follows from previous example that any projective real variety is affine.

2. Real ideals

In the complex setting, there is a well known 1-1 correspondence between varieties and radical ideals. This cannot be extended in a straightforward way to the real setting, as it is shown by the ideal $(1+x^2)$, which is radical, but corresponds to the empty set, like the ideal (1).

Around 1970, this difficulty was overcome with the Real NullStellenSatz, proved by Risler, relying on prevuious work by Artion and Schreyer. introduction of We recall that for any variety X, the ideal $I(X)$ is radical. This is true both in the complex or in the real setting.

Let X be a real variety. It is immediate to check that the ideal $I(X)$ satisfies the following additional property

$$a_i \in \mathbb{R}[x_1,\ldots,x_n], a_1^2+\ldots+a_p^2 \in I(X) \Longrightarrow a_i \in I(X) \text{ for } i=1,\ldots,p \tag{1}$$

A ideal I, with real generators, which satisfies the property (1) is called a *real ideal.* So for any real algebraic variety, $I(X)$ is a real ideal.

The typical examples of non real ideals are (x^2) and $(1+x^2)$. Also (x^n) for any $n\geq 2$ is not real.

Theorem 2.1. *Every real ideal in* $\mathbb{R}[x_1,\ldots,x_n]$ *is radical.*

Theorem 2.2. X *is a irreducible real variety if and only if* $I(X)$ *is a real prime ideal.*

Example 2.3. *The ideal* $(f)\subset\mathbb{R}[x,y]$ *is real if and only if* f *has all real and distinct roots, that is it is hyperbolic.*

Example 2.4. *Let* f *be an irreducible polynomial. Then* (f) *is real if and only if* $V(f)$ *has a regular real point.*

Define

$$\sum\mathbb{R}[\mathbf{x}]^2=\{\sum h_i^2|h_i\in\mathbb{R}[\mathbf{x}]\}$$

which is a monoid cone, namely it is closed under addition, multiplication and positive scalar multiplication.

Definition 2.5. *Given an ideal* $I\subset\mathbb{R}[x_1,\ldots,x_n]$, *the real radical of* I *is given by the intersection of all prime real ideals containing* I. *It is denoted as* $\sqrt[\mathbb{R}]{(I)}$.

Theorem 2.6 (Real NullStellenSatz)**.**

$$\sqrt[\mathbb{R}]{(J)}=\{f\in\mathbb{R}[x] : -f^{2m}\in\sum\mathbb{R}[\mathbf{x}]^2+I \textit{ for some } m>0\}$$

If J *is a real ideal, then* $\sqrt[\mathbb{R}]{(J)}=I(V_{\mathbb{R}}(J))=J$.

Corollary 2.7. *There is a natural bijective correspondence between affine algebraic varieties in* $\mathbb{R}^n$ *and real ideals in* $\mathbb{R}[x_1,\ldots,x_n]$.

Example 2.8. *In this example we show an interesting example of non real ideal, from spectral theory of matrices. Consider the symmetric* 2×2 *matrices* A. *We claim* $(\operatorname{tr}(A),\det(A))\subset\mathbb{R}[a_{11},a_{12},a_{22}]$ *is not a real ideal. Indeed, consider the variety* $X=V(\operatorname{tr}(A),\det(A))$. *Note that* $X_{\mathbb{R}}$ *consists of the zero matrix, by the Spectral Theorem. The complex variety* X *consists of symmetric nilpotent matrices, like* $\begin{pmatrix}1 & \sqrt{-1}\\ \sqrt{-1} & -1\end{pmatrix}$. *In this case* $I(X_{\mathbb{R}})=(a_{11},a_{12},a_{22})$ *strictly contains* $(\operatorname{tr}(A),\det(A))$. *Moreover one computes that* $a_{11}^2+2a_{12}^2+a_{22}^2=(a_{11}+a_{22})^2-2(a_{11}a_{22}-a_{12}^2)\in(\operatorname{tr}(A),\det(A))$, *while individually* $a_{ij}\notin(\operatorname{tr}(A),\det(A))$. *In other words, the real radical of* $(\operatorname{tr}(A),\det(A))$ *is* (a_{11},a_{12},a_{22}).

It is interesting to consider the map $\pi\colon\operatorname{Sym}^2\mathbb{R}^2\to\mathbb{R}^2$ *such that* $\pi(A):=(\operatorname{tr}(A),\det(A))$. *The image is given by* $(x,y)|x^2-4y\geq 0\}$. *Note that general fiber is a curve, but* $\pi^{-1}(2,1)$ *(on the* *boundary of the image) contains just the identity, by the Spectral Theorem. All other points at the boundary have the same behaviour. The complex fiber of* $(2,1)$ *is a curve containing* $I+N$ *where* N *is symmetric nilpotent.*

Example 2.9. *The above example can be generalized in several ways. Consider the symmetric* 3×3 *matrices* A *with characteristic polynomial* $\det(A-tI) = -t^3 + trace(A)t^2 - f_2(A)t + f_3(A)$, *note that* $f_3(A) = \det(A)$. *We claim that* $(f_2, f_3) \subset \mathbb{R}[a_{ij}]$ *is not a real ideal. Consider the variety* $X = V(f_2, f_3)$. *Note that* $X_{\mathbb{R}}$ *consists of symmetric matrices with two vanishing eigenvalues, so it consists of* 3×3 *matrices of rank one, which is the cone over the Veronese variety, it has codimension* 3. *But the complex variety* X *is a complete intersection of codimension* 2. *Note that* $X_{\mathbb{R}}$ *consists of singular points in* X, *it is not Zariski dense. In this case* $I(X_{\mathbb{R}})$ *is generated by the* 2*-minors of* A, *moreover* $I(X_R)$ *strictly contains* (f_2, f_3). *This can be checked in the following way. Let* A^{adj} *be the adjugate matrix of* A, *containing its* 2×2 *minors with alternate signs. Then* $\mathrm{tr}(A^{adj}A^{adj})$ *is a sum of squares of* 2*-minors. One can check the identity*

$$\mathrm{tr}(A^{adj}A^{adj}) = f_2^2 - 2\mathrm{tr}(A)f_3(A) \in (f_2, f_3). \tag{2}$$

In order to check the identity (2), one can assume that A *is diagonal. Note that, individually, no* 2*-minor of* A *belongs to the ideal* (f_2, f_3). *In conclusion,* $\sqrt[\mathbb{R}]{(f_2, f_3)} = \mathrm{minors}(2, A)$.

Remark 2.10. *The Real NullStellenSatz can be extended to a statement holding for quasi varieties, called the PositivStellenSatz.*

Example 2.11. *The variety of triangles in* $\mathbb{P}(\mathrm{Sym}^3\mathbb{C}^3) = \mathbb{P}^9$ *has codimension* 3 *and it is the closure of a* $SL(3,\mathbb{C})$*-orbit. Its real points consist of two real triangles, which is* $SL(3,\mathbb{R})\cdot(xyz)$ *and imaginary triangles, which is* $SL(3,\mathbb{R})\cdot(x(y^2+z^2))$. *Both the two euclidean closures are quasi-varieties.*

Given a real affine (algebraic) variety X, the *complexification* $X_{\mathbb{C}}$ is defined by the complex solutions to the real ideal $I(X)$. Then

Theorem 2.12 (Whitney)**.** *Let* X *be a real algebraic variety.* $\dim_{\mathbb{R}} X = \dim_{\mathbb{C}} X_{\mathbb{C}}$ *If* X_i *are the irreducible components of* X, *then* $(X_i)_{\mathbb{C}}$ *are the irreducible components of* $X_{\mathbb{C}}$.

3. Complex versus Real

Complex statement	**Real statement**
Complex tangent space	Real tangent space
A plane curve of degree d is a Riemann surface of genus g=(d-1)(d-2)/2	A plane curve of degree d has at most (d-1)(d-2)/2ovals
Bezout Theorem	Bezout inequality, parity modulo 2 is invariant
Chow Theorem, a morphism of projective varieties is closed.	False: $(a, b) \mapsto (a^2, b^2)$ but **Tarski-Seidenberg principle** shows that semialgebraicity is preserved
Zariski closure and euclidean closure coincide for constructible sets	Zariski closure and euclidean closure are often different, euclidean topology is preferred in the applications
k-secant varieties only one generic rank	k-secant semi-algebraic sets several typical ranks
Semicontinuity of fiber dimension	False: send a real symm. matrix to its char. polynomial, there is a fiber containing only the identity.
X is irreducible iff $I(X)$ is prime	X is irreducible iff $I(X)$ is real prime
NullStellenSatz	**Real NullStellenSatz**
Moduli spaces of varieties are schemes	Moduli spaces of varieties are semi-algebraic
Compact affine varieties are just points	Projective spaces and Grassmannians are affine
Topological restrictions for projective varieties given by Hodge decomposition and by Lefschetz	**Tognoli Theorem** Every smooth compact differentiable manifolds is diffeomorphic to a real affine algebraic variety
	PositivStellenSatz
	Sum Of Squares, nonnegativity
	Spectrahedra, Semidefinite programming Convex algebraic geometry[BPT]

4. Real tensors and real rank, binary forms

Let K be a field, let $X \subset \mathbb{P}(K^{N+1})$ be a projective variety and let $\hat{X} \subset K^{N+1}$ be the cone over X. We assume that X is nondegenerate, namely that X is not contained in a hyperplane. The *rank* of a point $x \in K^{n+1}$ with respect to X (X-rank for short) is the minimum k such that there exists a decomposition $x = \sum_{i=1}^{k} \lambda_i x_i$, with $x_i \in \hat{X}, \lambda_i \in K$.

The real d-Veronese variety of $\mathbb{P}(V)$ is defined by 2-minors of $V^\vee \to \mathrm{Sym}^{d-1}V$, it is the ideal of polynomials vanishing on the real Veronese variety, hence it is a real ideal.

Note that for d even, the image of the d-Veronese map $v \mapsto v^d$ fill only a semialgebraic variety.

The real Segre variety has quadratic generators in its Segre embedding, which again generate a real ideal.

Veronese variety, Segre variety and Grassmann variety are basics example where is interesting to study the rank, both in complex or real case. The rank with respect to Veronese variety is called the symmetric rank, and in the immersion $\mathrm{Sym}^d V \subset \underbrace{V \otimes \ldots \otimes V}_{d\ times}$ it is conjectured (*Comon conjecture*) to be equal to the rank. This conjecture is open both in the real or in the complex case.

In the case $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ there are exactly two typical ranks. Let $\langle e_0, e_1 \rangle = \mathbb{R}^2$. The tensor

$$2e_0e_0e_0 - 2e_0e_1e_1 - 2e_1e_0e_1 - 2e_1e_1e_0 = (e_0 + \sqrt{-1})^{\otimes 3} + (e_0 - \sqrt{-1}e_1)^{\otimes 3} = 4e_0^{\otimes 3} - (e_0 + e_1)^{\otimes 3} - (e_0 - e_1)^{\otimes 3} =$$

has complex rank equal to 2 and real rank equal to 3.

A similar example can be constructed for Waring decompositions in the symmetric seytting, namely

$$2x^3 - 6xy^2 = (x + \sqrt{-1}y)^3 + (x - \sqrt{-1}y)^3 = 4x^3 - (x+y)^3 - (x-y)^3.$$

which again has complex symmetric rank equal to 2 and real symmetric rank equal to 3.

More in general, on $\mathbb{R}^2 \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ we can call (A_0, A_1) the two $n \times n$ slices. There are two typical ranks $(n, n+1)$, according if the polynomial $\det(A_0 + tA_1)$ has all distinct real roots (in this case $\mathrm{rk} = n$) or there is at least a pair of conjugate roots (in this case $\mathrm{rk} = n+1$).

Theorem 4.1. [Fr, BBO] *Typical ranks for any irreducible variety $X \subset \mathbb{P}(\mathbb{R}^{N+1})$ make an interval $\{r_0, \ldots, r_1\}$ when r_0 is the generic complex rank.*

The space $\mathbb{P}\left(\mathrm{Sym}^d \mathbb{R}^2\right)$ contains the discriminant hypersurface Δ, given by polynomials with at least a double root. The complement $\mathbb{P}\left(\mathrm{Sym}^d \mathbb{R}^2\right) \setminus \Delta$ is divided in $\lfloor d/2 \rfloor$ chambers containing polynomials with $d - 2i$ real roots. These chambers are basic example of almost varieties. In case d even there are two extreme cases, when all roots are real (hyperbolic) and when no real root is real. This second one is the cone over a convex set.

There is a probability distribution which is orthogonally invariant. According to it, Smale and Shub prove that the expected number of real zeroes of a degree d binary form os $\sqrt{d}$.

The trace form $\mathbb{R}[x,y]/(f) \times \mathbb{R}[x,y]/(f) \to \mathbb{R}$ is defined by $B(a,b) = \mathrm{tr} M_{ab}$

Theorem 4.2 (Sylvester)**.** *f has all real roots if and only if the trace form is positive definite*

The number of distinct (complex or real) roots of f is $\mathrm{rk} B = k$

The number of distinct real roots of f is given by the signature of B, namely by the number of positive eigenvalues minus the number of negative eigenvalues.

Sylvester Theorem holds over $\mathbb{R}$ as well, namely

Theorem 4.3 (Sylvester, 1851)**.** *f has a Waring decomposition with r summands if and only if it is killed by a differential operator of degree r which has r distinct real roots.*

Theorem 4.4 (Sylvester, 1861)**.** *Let* $f \in \mathrm{Sym}^d\mathbb{R}^2$*.*

$$\#\{\textit{distinct real roots of } \ f\} \le \mathrm{rk}_{\mathbb{R}}(f)$$

Proof. By induction on d. Cases $d = 1, 2$ are obvious.

Take a derivative ∂_l (differential operator of first order) which kills one summand of f

Call $k = \#\{\text{distinct real roots of } f\}$. Then

$$k - 1 \underbrace{\le}_{\text{Rolle thm.}} \#\{\text{distinct real roots of } \partial_l f\} \underbrace{\le}_{\text{induction}} \mathrm{rk}_{\mathbb{R}}(\partial_l f) \le \mathrm{rk}_{\mathbb{R}}(f) - 1$$

□

Theorem 4.5 (Causa-Re)**.** [CR, CO] *A real binary form has all real roots if and only if it has rank* 1 *or* d*.*

The previous Theorem is proved by induction on d by relying on the following

Theorem 4.6 (Causa-Re)**.** *A binary form of degree* $d \ge 2$ $f(x, y)$ *has all real roots if and only if all the forms in the pencil* $\langle f_x, f_y\rangle$ *have all real roots.*

Finally we get, after [BBO], the following, which was proved before in [B13].

Theorem 4.7 (Blekherman)**.** *The typical ranks of a real binary form of degree* d *are all integers between* $[d/2]$ *and* d

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