FAMOUS PROBLEMS on TENSORS
A GEOMETRIC POINT of VIEW

FIRENZE 19-12-19

- <u>STRASSEN'S PROBLEM</u> Find conditions under which

  a block tensor $T = T_1 \oplus T_2$ satisfies $\mathrm{rank}(T) = \mathrm{rank}(T_1) + \mathrm{rank}(T_2)$

- <u>COMON'S PROBLEM</u> Find conditions under which

  a symmetric tensor T satisfies $\mathrm{rank}(T) < \text{symmetric rank}(T)$

- <u>MULTIPLICATIVE PROBLEM</u> Find conditions under which

  a tensor product $T = T_1 \otimes T_2$ satisfies $\mathrm{rank}(T) < \mathrm{rank}(T_1) \cdot \mathrm{rank}(T_2)$

I) Notation and Problems

T tensor over $\mathbb{C}$

Elementary expression

(e.g. Waring expression) in the symmetric case

$$T = \sum^{r} a_i T_i$$

at this stage coefficients could be avoided $/\mathbb{C}$

T_i = elementary (rank 1)

r = length

Geometric version

T corresponds to a point in a projective space of tensors $\mathbb{P}^N$.

T_i corresponds to a point in a multi-projective space $P_i \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ which maps to $\mathbb{P}^N$ via the Segre map σ

elementary expression $\Leftrightarrow$ $T \in \langle \sigma(P_1), \dots, \sigma(P_r) \rangle = \langle T_1, \dots, T_r \rangle$

If we fix the T_i's we must maintain the coefficients a_i in order to distinguish points in $\langle T_1, \dots, T_r \rangle$.

<u>N.B.</u> In the symmetric case $P_i \in \mathbb{P}^n$ and σ corresponds to the Veronese map (of degree k) which is also denoted as v_k.

<u>Def</u> The expression $T = \sum a_i T_i$ is

- <u>non-redundant</u> if T does not belong to the span of a <u>proper</u> subset of the T_i's.
  It corresponds to say that the T_i's are linearly independent and no a_i is 0.
  (STANDARD ASSUMPTION)
- <u>minimal</u> If there are no expressions of T with length $< r$
- <u>unique</u> If given another expression of the same length $T = \sum^r b_i T_i'$, after rearranging, we have $\forall i \quad T_i' = \alpha_i T_i$ and $b_i = \frac{a_i}{\alpha_i}$ for some scalars α_i's.

<u>Def</u> Given an elementary expression $T = \sum^{r} a_i T_i$ with $T_i = \sigma(P_i)$ the set $\{P_i\} = A$ is a (GEOMETRIC) DECOMPOSITION of $T \in \mathbb{P}^N$

Notice that all points in the span $\langle \sigma(P_1), \dots, \sigma(P_r) \rangle$ share the same decomposition (another reason to maintain the a_i's)

The notation "non-redundant", "minimal", "unique" applies also to decompositions.

<u>PROBLEM</u> (certification) Find a method which guarantees that a given decomposition A of T is minimal or unique.

(N.B. For non-redundant the problem is elementary)

Target of the talk

1) discuss some solutions to the certification problem

2) discuss applications to some problems on tensors:

2a) STRASSEN PROBLEM

Find conditions that imply that one (every) minimal decomposition of a block tensor has the same block structure

2b) COMON PROBLEM

Find conditions that imply that one (every) minimal decomposition of a symmetric tensor is symmetric (or find many counterexamples)

2c) MULTIPLICATIVE PROBLEM

Find examples of tensors T_1, T_2 such that the rank of $T_1 \otimes T_2$ is (much) smaller than $\mathrm{rank}(T_1) \cdot \mathrm{rank}(T_2)$.

1) THE CERTIFICATION PROBLEM

Old results

KRUSKAL'S CRITERION . $\{P_i\} = A$ decomposition

$P_i \in \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k}$ $\qquad$ $\pi_j : \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_k} \to \mathbb{P}^{n_j}$ projection

compute $\rho_i :=$ kruskal's rank of $\pi_j(A)$

If $r \leq \dfrac{\sum^k \rho_i - (k-1)}{2}$ then the decomposition is unique and minimal

Symmetric version $\qquad P_i \in \mathbb{P}^n$ $\qquad \rho =$ kruskal's rank of A

If $r \leq \dfrac{k\rho - (k-1)}{2}$ then the decomposition is unique and minimal

Reshaped kruskal's criterion (symmetric case)

Write $k = k_1 + k_2 + k_3$ $\qquad \rho_i :=$ kruskal's rank of $V_{k_i}(A)$

if $r \leq \dfrac{\rho_1 + \rho_2 + \rho_3 - 2}{2}$ then the decomposition is unique and minimal

FLATTENINGS

If a flattening of T has rank r then the minimality holds

CATALECTICANT MAPS (Symmetric case)

Let L be the space generated by the m-derivatives of T and compute $W = L \cap v_{k-m}(\mathbb{P}^n)$.
If the intersection consists of r points, then minimality and uniqueness hold.

Notice that L depends on $A = \{P_i\}$ only.

GENERAL WEAKNESS

All the methods cannot work when r is big

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All the methods cannot work when r is big

- RANGE FOR r

The rank of a generic tensor is known
- in the symmetric case (Alexander-Hirschowitz)
- in the $2 \times \dots \times 2$ case (Catalisano-Geramita-Simigliano)

In other cases we have a guess

When $r >$ generic rank r_g then uniqueness cannot hold.

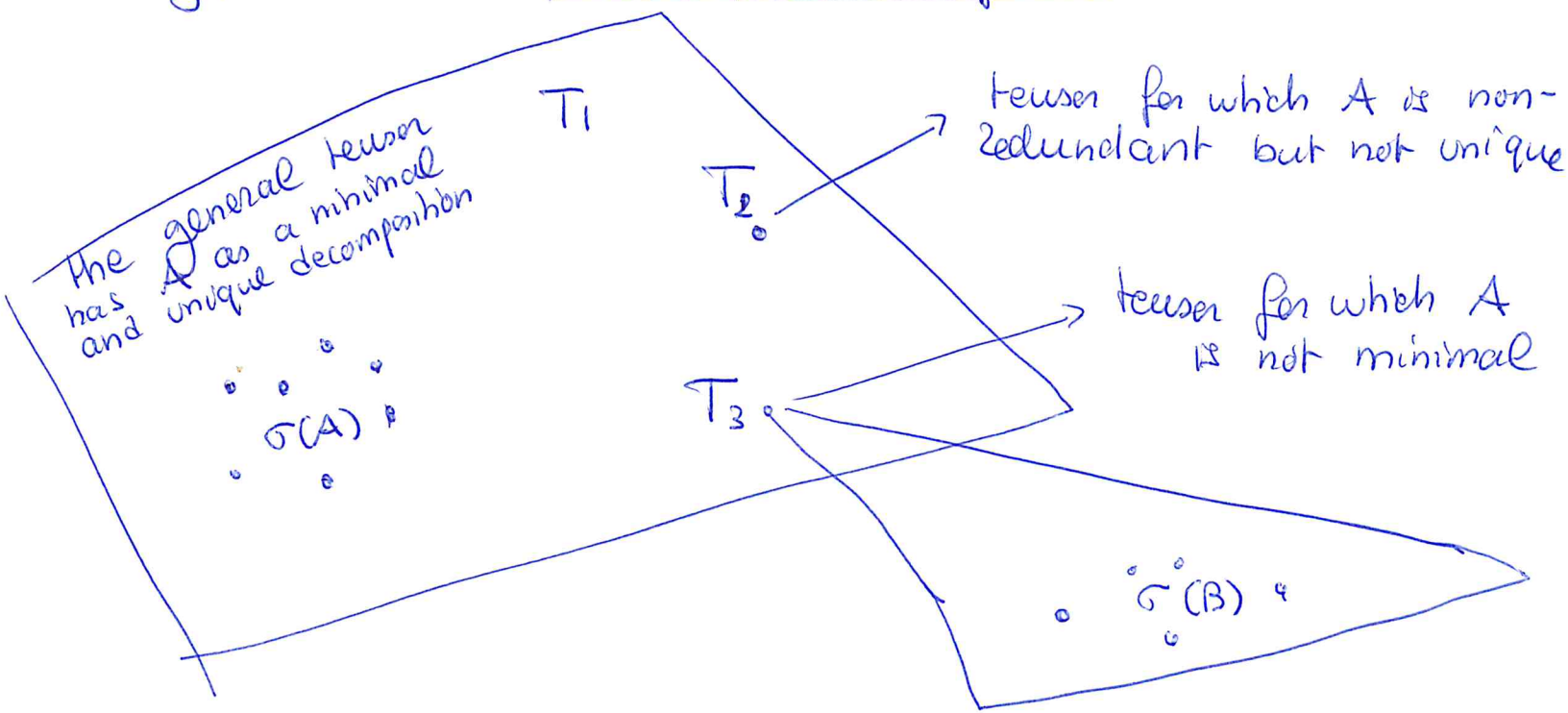
- YET

All the previous methods cannot work over a bound r_0 for r which is <u>much smaller</u> than r_g.

Geometric reason why Kruskal's criterion and the catalecticant criterion cannot work when r is big

The Kruskal's and the catalecticant criteria only look at geometric properties of A

If there are two tensors T_1, T_2 in the span of $\sigma(A)$ with different behavior with respect to uniqueness or minimality, then the <u>criteria must fail.</u>



Need a more accurate analysis when r is close to r_g

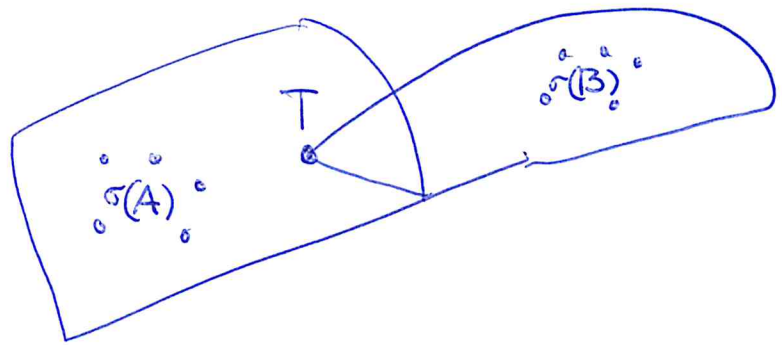
- CASE of SYMMETRIC TERNARY TENSORS (CUBIC FORMS)

  A method for the accurate analysis exists in terms of Hilbert functions and Hilbert-Burch matrices.

<u>Ex</u> When the higher Kruskal's ranks (computable) of A are general, then there exists a <u>sharp</u> bound for the range in which uniqueness holds uniformly in the span $\langle \sigma(A) \rangle$ (Angelini, –) 2019 preprint

<u>N.B.</u> <u>Generic</u> uniqueness for symmetric tensors is understood (– Oeding Vannieuwenhoven $r < r_g$ } 2017
Galuppi - Mella $r = r_g$)

Example of the method



Given the decomposition A assume there exists B with $\ell(B) \leq r$

$Z = A \cup B$ $\qquad$ h_Z = Hilbert function $\qquad$ Dh_Z = 1st difference of h_Z

Restrictions on Dh_Z

- If $h_Z(k) = r$ i.e. $Dh_Z(k+1) = 0 \Rightarrow \langle\sigma(A)\rangle \cap \langle\sigma(B)\rangle = \emptyset$

  Thus $Dh_Z(k+1) > 0$

  If $A \cap B = \emptyset$ then $r - h_Z(k) = \sum_{i=k+1}^{\infty} Dh_Z(i) = \dim(\langle\sigma(A)\rangle \cap \langle\sigma(B)\rangle) + 1$

- (CAYLEY-BACHARACH)

  If $A \cap B = \emptyset$ then $CB(k)$ holds for Z hence

  $$\forall j \quad Dh_Z(0) + \dots + Dh_Z(j) \leq Dh_Z(k+1) + \dots + Dh_Z(k+1-j)$$

- <u>no-flat</u>

  Since A, B non-redundant

  $$0 < Dh_Z(i) = Dh_Z(i+1) < \max Dh_Z(j)$$

  cannot hold.

  Moreover if $Dh_Z(i) \leq i \Rightarrow Dh_Z(i+1) \leq Dh_Z(i)$

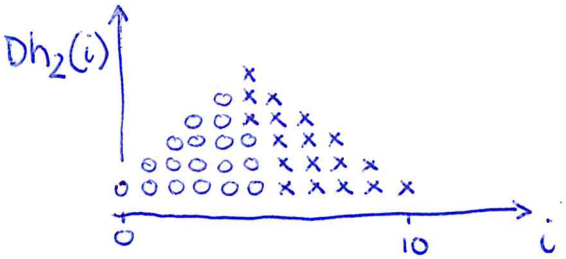
- $Dh_2(k+1) > 0$
- Cayley - Bacharach
- no - flat

"When you eliminated the impossible, what remains, though improbable, must be the truth" (SHERLOCK HOLMES)

<u>EXAMPLE</u> 3 variables $(r_g = 19)$ degree $k = 9$ case of length $r = 18 = e(A)$

$Z = A \cup B$

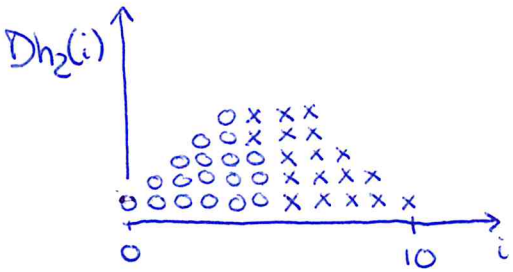
$A \cap B = \emptyset$



Case I

$e(B) = 18$

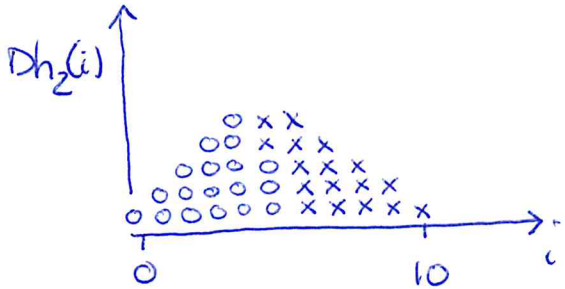
$\langle V_g(A) \rangle \cap \langle V_g(B) \rangle = \{T\}$



Case II

$e(B) = 18$

$\langle V_g(A) \rangle \cap \langle V_g(B) \rangle = \{T\}$



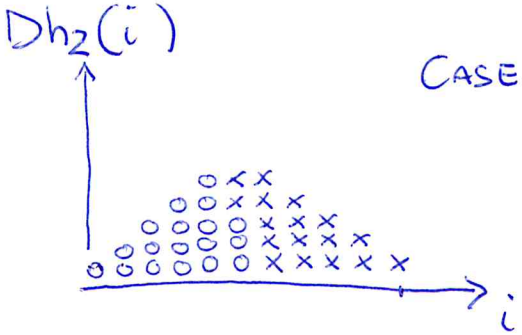
Case III

$e(B) = 17$

$\langle V_g(A) \rangle \cap \langle V_g(B) \rangle = \{T\}$

$\Downarrow$

CASE of NON-MINIMALITY

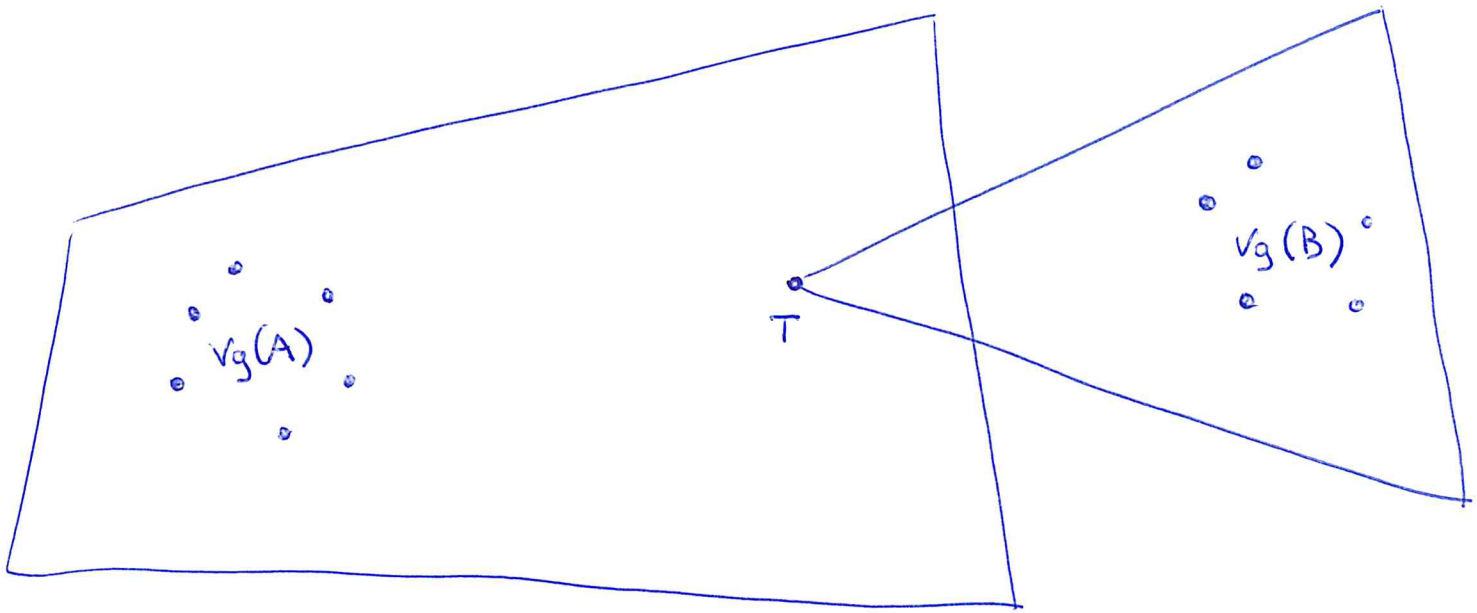


CASE III exists for a general set of 18 points in $\mathbb{P}^2$

2 is complete intersection of type 5, 7

CONVERSELY

If Z is a (smooth) complete intersection of type 5, 7 containing A, then $B = Z - A$ is a second decomposition of length 17 for some tensor in $\langle V_g(A) \rangle$



Case IV

$A \cap B \neq \emptyset$

$$T = a_1 T_1 + \cdots + a_{18} T_{18}$$

$$= b_1 T_1 + \cdots + b_i T_i + b_{i+1} T'_{i+1} + \cdots + b_{r'} T_{r'}$$

rank $T_i, T'_j = 1$

$r' \leq 18$

$T'_j = V_g(P'_j)$

$B' = \{P'_{i+1}, \ldots, P'_{r'}\}$

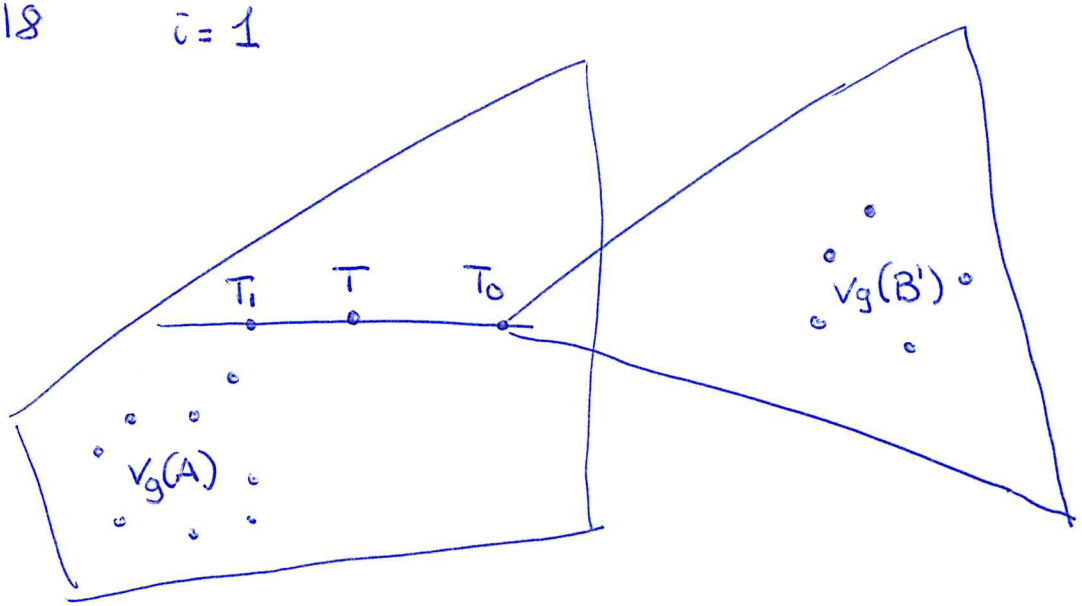
$$T_0 = (a_1 - b_1) T_1 + \cdots + (a_i - b_i) T_i + a_{i+1} T_{i+1} + \cdots + a_{18} T_{18}$$

$$= b_{i+1} T'_{i+1} + \cdots + b_{r'} T_{r'}$$

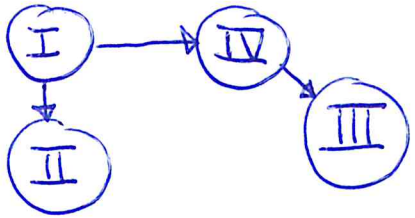
Applying the previous results to B' and T_0

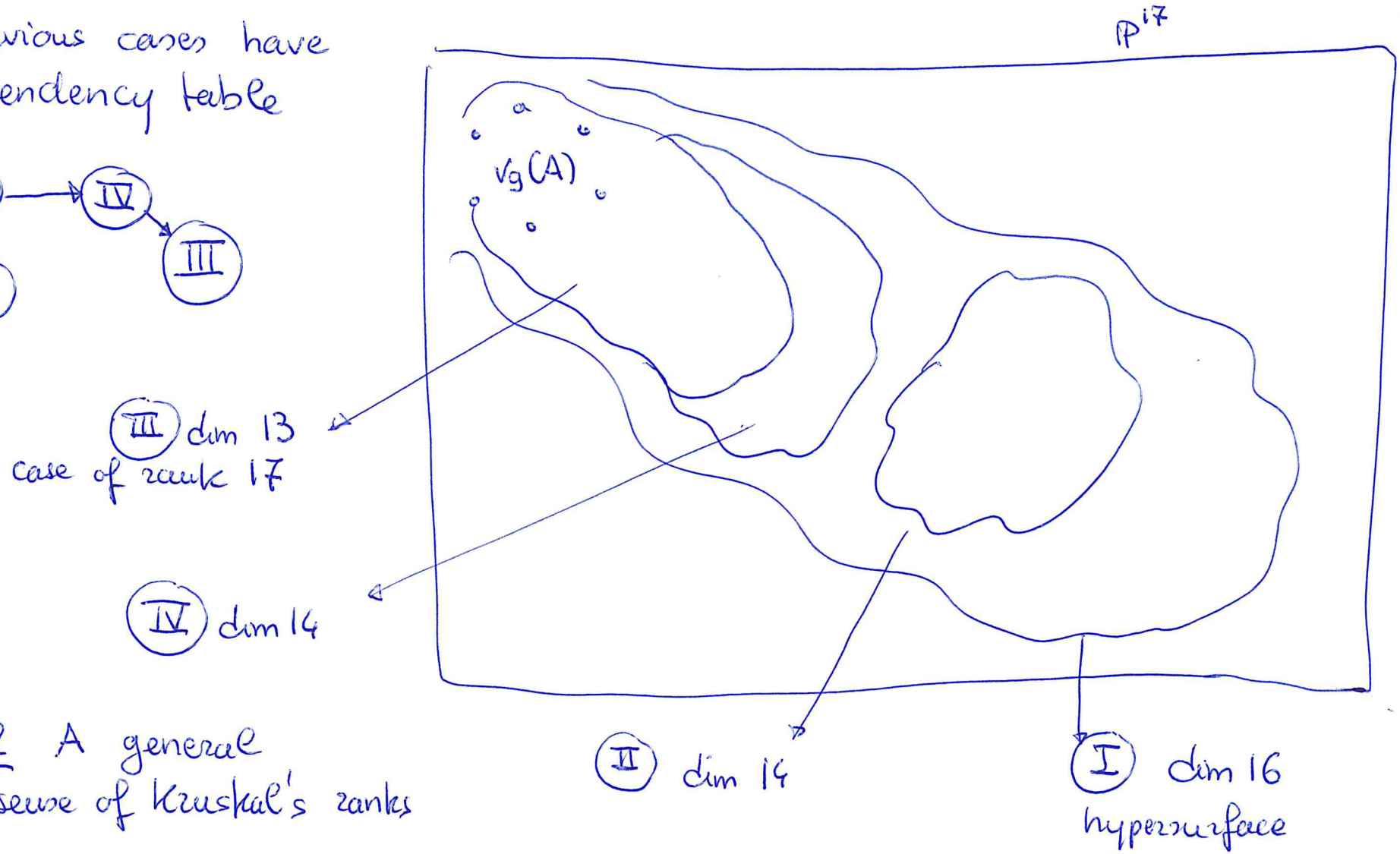
$\Rightarrow \quad r' = 18 \qquad i = 1$

$e(B') = 17$



The previous cases have a dependency table





For <u>all</u> A general in the sense of Kruskal's ranks

<u>CLAIM</u> In 3 variables, for all degrees k one can find, with the <u>same procedure</u>, a similar table

- When k grows, the number of cases blows.

GOING FURTHER

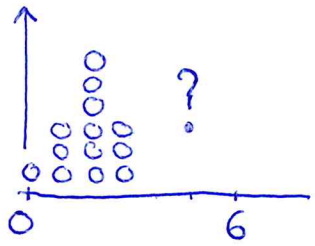
Case of 4 variables

One cannot repeat the same analysis because our knowledge of conditions on h_2 and Dh_2 are <u>too</u> <u>faint</u>.

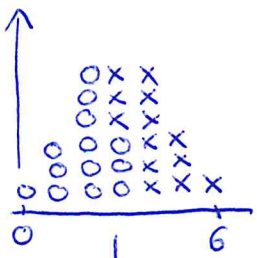
(The previous three conditions hold, but they are no longer sufficient to characterize Dh_2).

<u>Example</u> degree $k=5$ $(r_g = 14)$ $r = 13$

Dh_A for 13 general points in $\mathbb{P}^3$



Dh_2



proposal ?

$\downarrow$

28 points on an ACM curve of degree 6 C

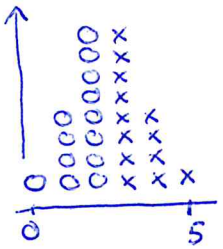
<u>QUESTION</u> It is easy to see that 13 general points cannot stay on a curve like C.
Find explicit conditions to test when $A \subseteq C$

GOING FURTHER

- Case of 5 variables $k=4$ $r_g = 15$

A complete description for A krukal-general is work in progress (we hope we have it)

The most complicate case is the case of rank 13 (of course)



geometric description

- Case of non-krustkal-general A

A. MAZZON's PhD thesis

- "easy" in 3 variables
- unknown in general

STRASSEN'S PROBLEM (symmetric version)

Divide the set of $n+1$ variables in two subsets of n_1+1 and n_2+1 variables respectively.

Take tensors (forms) T_1 in the first set of variables
T_2 " second " "

Put $T = T_1 + T_2$

QUESTION Is $\operatorname{rank} T = \operatorname{rank} T_1 + \operatorname{rank} T_2$?
(symmetric rank)

COUNTEREXAMPLE (SHITOV 2017)

T cubic in a huge number of variables

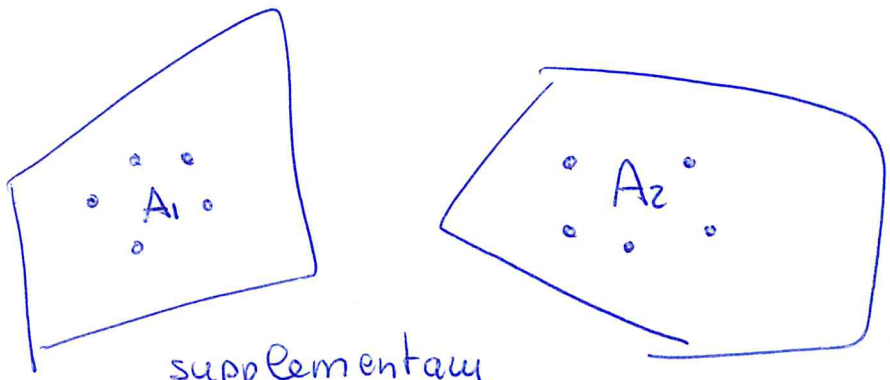
$\operatorname{rank}(T) \gg r_g$

Geometric setting of Strassen's problem

A_1 = decomposition of T_1 $\qquad$ A_2 = decomposition of T_2

$A_1 \cup A_2$ = decomposition of T

MINIMAL?



supplementary spaces

<u>Prop</u> Except for the first step

$$Dh_{A_1 \cup A_2}(i) = Dh_{A_1} + Dh_{A_2}$$

Thus $A_1 \cup A_2 = A$ not general

<u>Known cases</u> When r is close to the flattening maximal rank (FRIEDLAND)

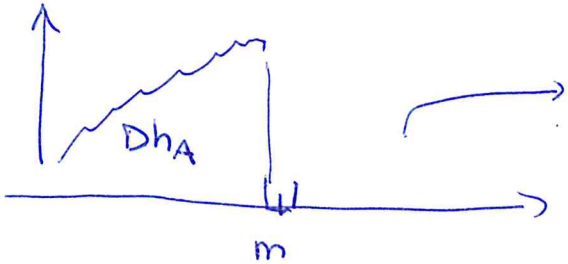
When $n_1 = n_2 = 1$

(CARLINI - CATALISANO - C, WOO)

Next case $n_1 = 1$ $n_2 = 2$ (but the procedure can be generalized)

Prop $r_1 < \frac{k+2}{2}$ and $Dh_{A_2}(k/2+1) = 0$ $\Rightarrow$ Strassen's additivity holds

pf degree $k = 2m$. The condition implies

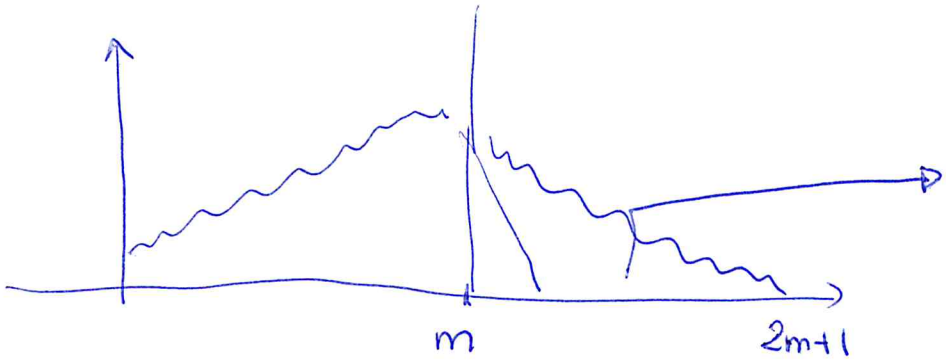


$\longrightarrow$ by Cayley-Bacharach $\mathbb{B}$ cannot exist.

Remark on the assumptions $r_1 < \frac{k+2}{2}$ mild because $r_1 \leq$ generic rank

$Dh_{A_2}(k/2+1) = 0$ should be improved in order to arrive to the generic rank for A_2

If $r_2 =$ generic rank $\sum_{i=m+1}^{\infty} Dh_A(i)$ is about $\frac{m^2}{6} - \frac{m}{2} - \frac{2}{3}$



If k is big enough,
one can use Cayley-Bacharach
and no-flat to prove
that B yields a contradiction
(purely numerical computation)

<u>Problem</u> the computation holds only if
no points of B belong either
to the line of A_1 or to the
plane of A_2

<u>WORK IN PROGRESS</u>

<u>Target</u> Find that for $r_1, r_2 <$ respective generic ranks
and A_1, A_2 are Kruskal-general $\Rightarrow$ Strassen's additivity holds

COMON'S PROBLEM

T = symmetric tensor = form of degree k in $n+1$ variables

r = symmetric rank of T

r' = rank of T

PROBLEM Is $r = r'$?

COUNTEREXAMPLE (Shitov 2017)

T cubic in a huge number of variables

$r, r' \gg r_g$

The general feeling is that it could be possible to produce examples for lower r (w.r.t. r_g)

MAYBE In the range $r \leq r_g$

Refinement (Gesmundo - Oneto - Ventura)

fix $k_1 + k_2 = k$

A (k_1, k_2) - tensor of rank 1 is $L_1^{k_1} \otimes L_2^{k_2}$

(non - symmetric)

(k_1, k_2) - rank = minimum r s.t.

$$T = T_1 + \dots + T_r$$

$r(k_1, k_2)$ with T_i (k_1, k_2) - tensor of rank 1.

Problem Find examples where symmetric rank > (k_1, k_2) - rank

~~Thm~~

known positive results

- small k and n (e.g. binary forms)
- monomials
- elementary symmetric forms $\sum x_{i_1} \cdots x_{i_k}$ k odd

  symmetric rank $= (1, k-1)$-rank [GOV]

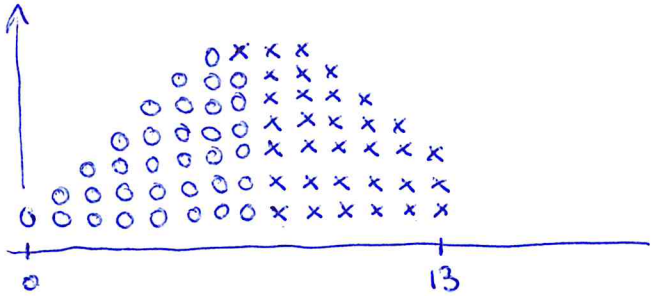
Proposal to find counterexamples
(or even (k_1, k_2)-counterexamples)
to the Comon's problem
via a geometric construction.

toy construction

3 variables $\quad k=13 \quad (r_g = 35)$

tensors with a non-redundant decomposition A formed by 34 general points (the general tensor of this type has rank 34)

Dh_Z



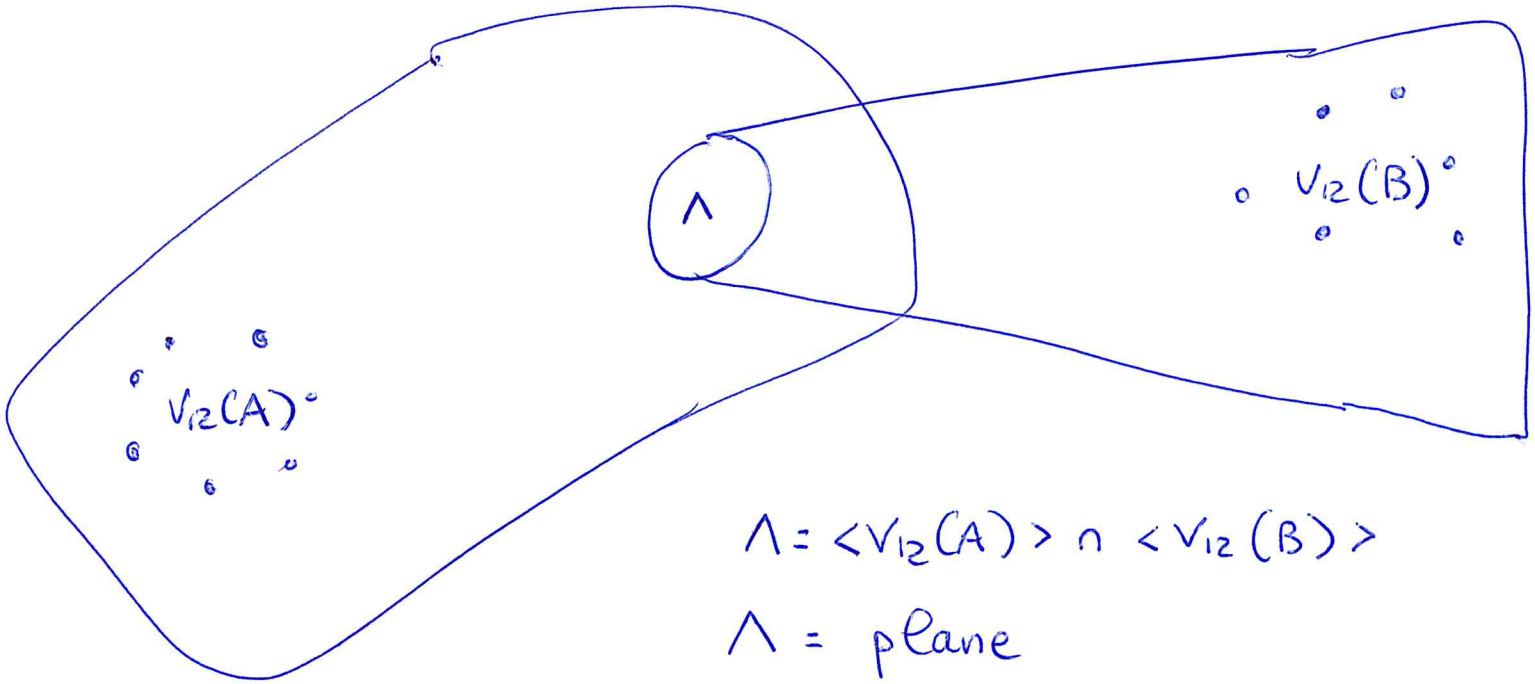
$Z = A \cup B \qquad e(A) = 34 \quad e(B) = 33$

Z linked 7, 10 to 3 general points

$$e(Z) - h_Z(12) = 3$$

$$\Downarrow$$

$$\langle \underline{V_{12}}(A) \rangle \cap \langle \underline{V_{12}}(B) \rangle \text{ has dimension } 3$$



$$\Lambda = \langle V_{12}(A) \rangle \cap \langle V_{12}(B) \rangle$$

Λ = plane

<u>Fact</u> (TEITLER) Assume that Λ is the plane spanned by the 3 derivatives of a form T of degree 13.

Then if $B = \{L_1, \dots, L_{33}\}$ there are linear forms $M_1 \dots M_{33}$ such that

$$T = L_1^{12} \otimes M_1 + \dots + L_{33}^{12} \otimes M_{33}$$

<u>Construction</u> Find $T \in \langle V_{13}(A) \rangle$ such that

- symmetric rank $T = 34$ (T general)

- $\Lambda = \langle \frac{\partial T}{\partial x_0}, \frac{\partial T}{\partial x_1}, \frac{\partial T}{\partial x_2} \rangle$

  is the intersection of $\langle V_{12}(A) \rangle$ with $\langle V_{12}(B) \rangle$ for some $B = \{L_1, \dots, L_{33}\}$ of lenght 33 (hard!)

This proves that

$$T = L_1^{12} \otimes M_1 + \dots + L_{33}^{12} \otimes M_{33}$$

i.e. the $(1,12)$-rank of T is ≤ 33.

<u>WORK IN PROGRESS</u>

MULTIPLICATIVE PROBLEM

- Find examples T_1, T_2 such that
$$\operatorname{rank}(T_1 \otimes T_2) < \operatorname{rank}(T_1) \cdot \operatorname{rank}(T_2)$$

- Examples such that $\operatorname{rank}(T \otimes T) < \operatorname{rank}(T)^2$

<u>Prop</u> Assume $T \in V_k(A)$ $T \notin V_k(A')$ for all $A' \subsetneq A$
(i.e. A is a non-redundant decomposition of T)
If the rank of T is $s < r = \ell(A)$
then there exists T_0 with $\operatorname{rank}(T_0 \otimes T_0) < \operatorname{rank}(T_0)^2 - (r-s)^2$

<u>pf</u> Divide $A = A_1 \cup A_2$ where $\ell(A_1) = s$ $\ell(A_2) = r - s$
Fix a general $S \in V_k(A_1)$ $S' \in V_k(A_2)$ so that
$T \in$ line $\overline{SS'}$ and $T = S - S'$. Put $T_0 = S + S' = T + 2S'$
Then $T_0 \otimes T_0 = T \otimes T + T \otimes 2S' + 2S' \otimes T + 2S' \otimes 2S' = T \otimes T + S \otimes 2S' + 2S' \otimes S$
$\operatorname{rank}(T_0 \otimes T_0) \leq s^2 + s(r-s) + (r-s)s = 2rs - s^2 = r^2 - (r-s)^2$

Easy to find, $\forall s$, a ternary form $T \in V_k(A)$ $k \gg 0$ such that $\underset{\overset{\shortparallel}{s}}{\mathrm{rank}}(T) \leq r - q$ $\forall q > 0$

E.g. for $q = 1$ we can use the example in degree 9 for ternary forms

$$\mathrm{rank}(T) = 18 - 1 \qquad T \in V_9(A) \qquad e(A) = 18$$

Find T_0 such that $\mathrm{rank}\, T_0 = 18$

$$\mathrm{rank}(T_0 \otimes T_0) = 18^2 - 1.$$

<u>Example</u> (Ballico - Bernardi - Christiandl - Gesmundo)
2018

with $\mathrm{rank}(T_0 \otimes T_0) = \mathrm{rank}(T_0)^2 - 1$.

Problem (open) Find examples where

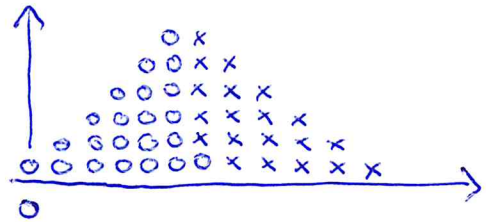
$$s = \mathrm{rank}(T) \leq r - q \qquad q > 1$$

Construction for $q = 2$

ternary forms of degree 10 $\qquad r_g = 22$

$\overline{T}$ general $\qquad$ A = decomposition of T with 22 points

= general set of 22 points in $\mathbb{P}^2$

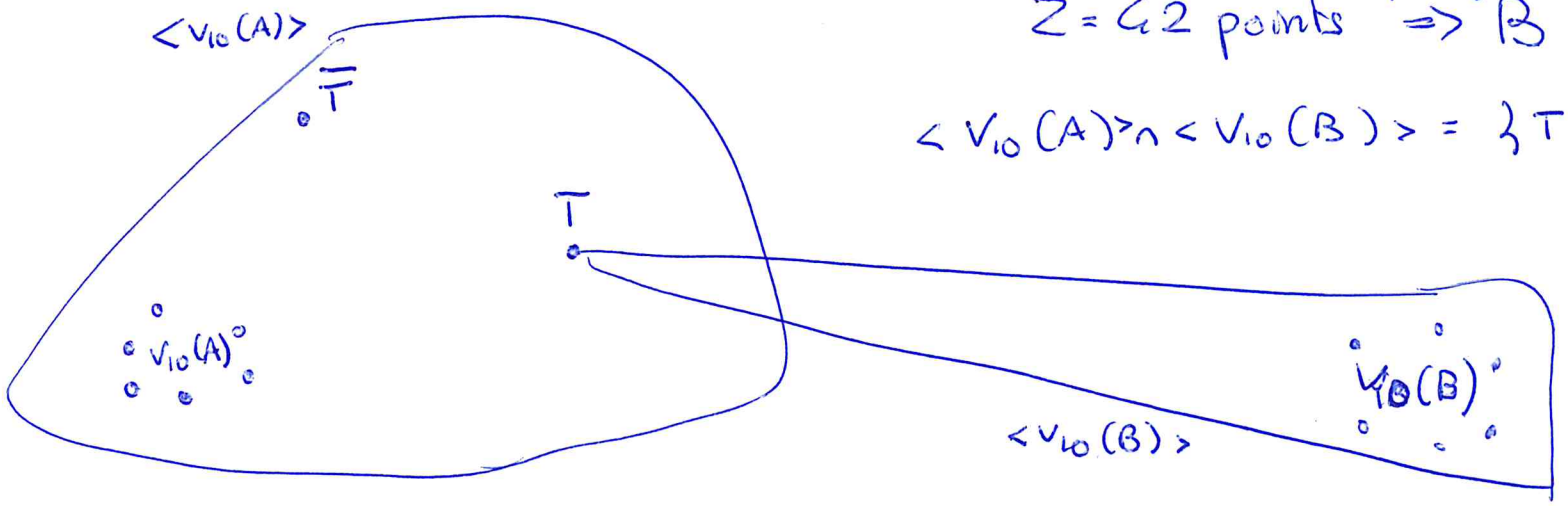


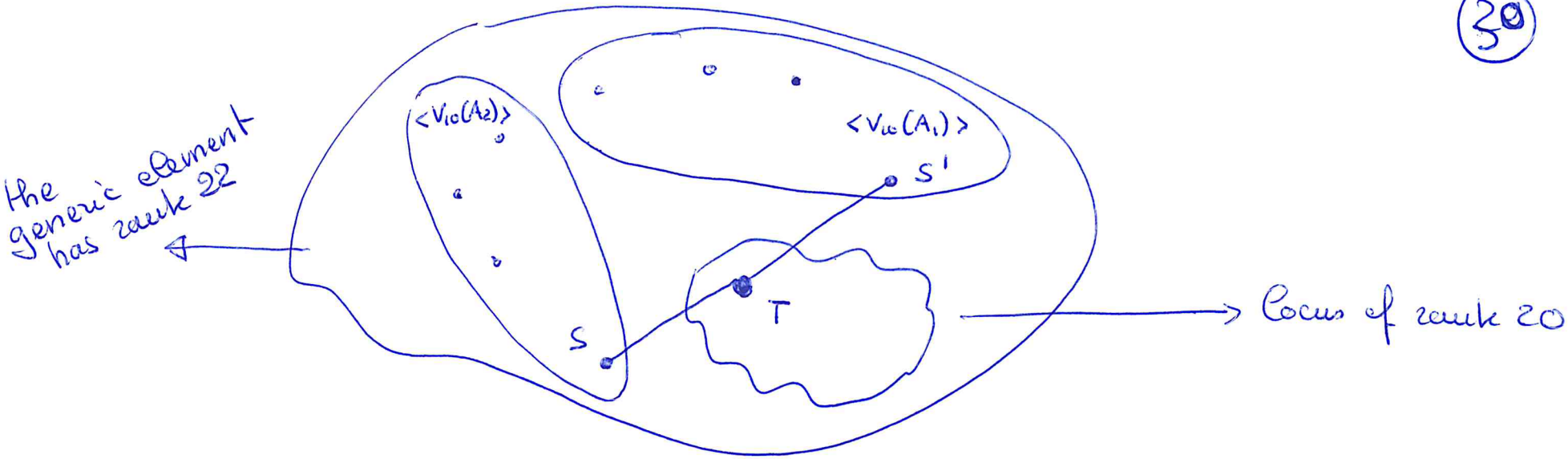
$\Rightarrow$ B such that $Z = A \cup B$

is c.i. of type 6, 7

Z = 42 points $\Rightarrow$ B = 20 points

$$\langle V_{10}(A) \rangle \cap \langle V_{10}(B) \rangle = \{T\}$$





so done

<u>PROVIDED</u> one can prove that $\operatorname{rank}(T_0) = \operatorname{rank}(T + 2S') = 22$

By increasing the degree one finds similar <u>proposals</u> for examples with increasing q

PROVIDED .... $\Rightarrow$?

<u>WORK IN PROGRESS</u>

BIBLIOGRAPHY

Hilbert functions and decompositions

[C] Hilbert functions and tensor analysis
QUANTUM PHYSICS AND GEOMETRY.
Lecture Notes UMI 25

[ACV] Angelini - C - Vannieuwenhoven
Identifiability beyond the Kruskal's bound for symmetric tensors of degree 4 Rend. Lincei 2018

Tensors strata

[COV] C - Ottaviani - Vannieuwenhoven
On generic identifiability of symmetric tensors of subgeneric rank

[ACM] Angelini - C - Mattson
Identifiability for a class of symmetric tensors
Mediterr. J. Math. 16 2019

[AC] Angelini - C
On the identifiability of ternary forms ArXiv 1901.01796

[M] Mattson
On a geometric method for the identifiability of forms
Boll. UMI to appear