ON VARIETIES WITH ONE APPARENT DOUBLE POINT

FRANCESCO RUSSO

Contents

Introduction

In these notes we would like to give an introduction to varieties with one apparent double point, a special class of projective varieties having nice geometric properties firstly studied by Severi in [Severi01], to present most of the known results in the area and to discuss some opens problems coming from classical papers and from the study of the known examples.

This is done gradually beginning with some general remarks on projections of smooth varieties, section 1, and then by defining varieties with one apparent double point, briefly VOADP, as smooth irreducible n-dimensional varieties $X_n \subset \mathbb{P}^{2n+1}$ such that through the general point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to X_n. This condition imposes many restrictions to the intrinsic and extrinsic geometry of OADP varieties, see propositions 2.1, 2.2 and 2.3. In section 3 we construct three different series of VOADP's which were studied classically by Babbage and Edge and some "isolated" examples coming essentially from the classification of homogenous varieties. In section 4 we show a relation between special subhomaloidal systems of

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quadrics and VOADP's which was not noticed before. This tool was useful to verify that some variety is a VOADP so constructing some new example, see section 4 for definitions, examples and results; moreover the examples presented in sections 3 and 4 seem to be the only known examples of VOADP's. In section 5 we present the classification of smooth OADP surfaces and 3-folds following the ideas of [CMR01], theorems 5.1 and 5.4. While the classification of OADP surfaces was stated by Severi in [Severi01] and completed in [Russo00] following the original ideas of Severi, the classification of OADP 3-folds is new and revealed further geometrical properties of VOADP's. In section 6 we present the classification of VOADP's of dimension n and degree $d \leq 2n+4$ after [AR02], see theorem 6.3. In this section it is also presented, corollary 6.4, the positive answer, given in [AR02], to the following questions posed by F.L. Zak: *is a linearly normal variety $X_n \subset \mathbb{P}^{2n+1}$ with $Sec(X_n) = \mathbb{P}^{2n+1}$ and of degree $d \leq 2n+1$ regular, i.e. $q(X_n) = 0$? Or more generally is such a variety a VOADP?* We will present the origins and motivations for these questions and the relationship with VOADP's.

In section 7 we deal with the geometry of the projection of a variety $X_n \subset \mathbb{P}^N$, $N \geq (k+1)n+k$ from the linear space $L = < T_{x_0}(X_n), \dots, T_{x_k}(X_n) >$, $x_0, \dots, x_k$ general, onto a disjoint $\mathbb{P}^{N-k(n+1)}$, $\pi_L : X \dashrightarrow \mathbb{P}^{N-k(n+1)}$. Terracini' s lemma immediately implies that this projection is dominant if and only if $\dim(S^kX_n) = (k+1)n+k$, where S^kX_n is the variety of $(k+1)$-secant k-planes to X_n (see proposition 7.1). In [Bronowski33] Bronowski claimed that if $S^kX_n = \mathbb{P}^{(k+1)n+k}$, then through the general point of $\mathbb{P}^{(k+1)n+k}$ there passes a unique $(k+1)$-secant $\mathbb{P}^k$ to X_n if and only if π_L is a birational map. This would be a nice geometric characterization of this class of varieties and would clearly imply that such varieties are rational. The implication VOADP $\Rightarrow \pi_L$ birational was obtained in [CMR01] as an application of the method of degeneration of projections developed there giving the important consequence that every VOADP is rational. Unfortunately we were unable to understand Bronowski' s proof and this beautiful characterization remains open for arbitrary n and k. A special case of the above implication was given in [AR02] for OADP varieties having quadratic entry locus for every $p \in \mathbb{P}^{2n+1} \setminus X_n$ (X_n is then a VOADP of special kind), 7.4. On the other hand, it should be noticed that not all the known examples of VOADP's have the property that the entry locus is quadratic for every $p \in \mathbb{P}^{2n+1} \setminus X_n$ (in fact all but one!). The relation which seems to exist between VOADP's and varieties $X_n \subset \mathbb{P}^{2n+1}$ which projects birationally from a general tangent space onto a disjoint $\mathbb{P}^n$ naturally poses also the question of classifying this last class of varieties. For $n \leq 3$ these classifications coincide and for the moment this seems to be the greater evidence for the truth of Bronowski' s claim in full generality for $k = 1$. Section 8 contains some applications of properties of VOADP's to two interesting problems: the existence of a rational point on a quintic Del Pezzo surface over any field and to the existence of rational smooth cubic hypersurfaces in spaces of odd dimension. In section 9 we pose some questions raising from the analysis of the known examples and from the strong restrictions of the geometry of VOADP's. In fact, probably, it could be the case that a variety with one apparent double point naturally carries a special subhomaloidal system of quadrics, the linear system of quadrics through it, for which the general fiber of the associated map is a secant line, that it is an arithmetically Cohen-Macaulay variety. This is motivated also by the empirical observation that "essentially" the unique way of realizing geometrically a *congruence* of secant lines to a smooth variety with one apparent double point is to organize the secant lines as fibers of the rational map given by quadrics through it. The answer to some of the above questions could eventually lead to the classification of all smooth varieties with one apparent double point, which also in the range $d > 2n+4$, $n \geq 4$, seem to be isolated and sporadic.

In the appendix we give a proof of a theorem, proved by Severi in [Severi01], to the effect that the Veronese surface in $\mathbb{P}^5$ is the unique irreducible non-degenerate surface of $\mathbb{P}^N$, $N \geq 5$, not a cone, such that $\dim(SX) = 4$ (or of another classical theorem characterizing it as the only surface as above containing a two dimensional family of irreducible conics).

Ringraziamenti. The problem of classification of VOADP's was posed some years ago by Edoardo Sernesi and it was through him that I knew that the paper of Severi contained a small gap. Many discussions with him clarified my ideas in the beginning and he informed me about classical references on the subject.

I am indebted to Fyodor Zak for many discussions and e-mail correspondences, which helped and stimulated me very much and for informing me about the paper of Bronowski. His deep insight, his useful explanations and some problems posed by him were and are of fundamental importance.

Last but not least, a special thank to my friends Alberto Alzati, Ciro Ciliberto and Massimiliano Mella for many stimulating discussions on these and other topics along last years and for having explained me a lot of things. Most of the material included here comes from our joint work and their interest was very stimulating for me. Alberto, Antonio Lanteri and Massimiliano invited me many times to Milano and Ferrara during last years and I wish to express my gratitude to them. The visits to Recife of Ciro and Massimiliano were very important for me not only from the mathematical point of view. Unforgettable our discussions on Recife's beaches or eating a picanha ("una bella invenzione!"). Moreover, the frequent correspondence with Alberto and Massimiliano, especially about less serious (but not less important) subjects, "me permitiram de matar a saudade". Finally I acknowledge Giorgio Ottaviani for organizing this Workshop in Firenze and for his interest in our work along last years.

1. Generalities on projections of smooth algebraic varieties

Let $X_n \subset \mathbb{P}^N$ be a non-degenerate smooth projective variety of dimension $n \geq 1$. In this section we want to show that, if $N = 2n+1$ ($N \geq 2n+1$), we can project X_n from a general point (linear subspace of codimension $2n+1$) of $\mathbb{P}^N$ such that the image of X_n in $\mathbb{P}^{2n}$ has at most a finite number of *improper nodes*, i.e. singular points arising from collapsing the finite number of couples of distinct points of X_n collinear with the center of projection and such that the projections of the tangent r-planes in the points of contact of the secant lines intersect transversally only in the nodes; they are usually called improper nodes of the first kind (see [Severi01], pg. 37). So for $n = 1$ an improper node is a point of multiplicity 2 with distinct tangent lines; for $n = 2$ is a point of multiplicity 2 having tangent cone breaking into two planes intersecting in the singular point and so on. In particular these singularities are non-normal, since (for example) they arise from the contraction of isolated points.

To this aim we recall below a number of well known results and definitions.

The *secant variety of* X_n, $Sec(X_n) \subset \mathbb{P}^N$, is the closure of the union of all the lines with at least two points in common with X_n, called secant lines to X_n. It is constructed as follows. Let $G := G(1, N)$ be the Grassmannian of lines in $\mathbb{P}^N$. Denote by

$$\Gamma(X_n) \subset X_n \times X_n \times G$$

the closure of the graph of the morphism $\alpha : X_n \times X_n \setminus \Delta_{X_n} \to G$ which takes (x, y) to the point representing the line $< x, y >$. Since $\alpha_1 : \Gamma(X_n) \to X_n \times X_n$ is the blowing-up of $X_n \times X_n$ along the diagonal Δ_{X_n}, $\Gamma(X_n)$ is an irreducible variety of dimension $2n$. If α_2 is the projection to G, then put

$$S(X_n) = \alpha_2(\Gamma(X_n)) \subset G.$$

The morphism $\alpha_2 : \Gamma(X_n) \to S(X_n)$ is generically finite because X is not linear and hence $S(X_n)$ is of dimension $2n$ and irreducible. The lines of $\mathbb{P}^N$ parametrized by $S(X_n)$ are called secants to X_n.

Let

$$I = \{(x, l)|\ \ x \in l\} \subset \mathbb{P}^N \times G$$

be the incidence correspondence and let $\beta_1 : I \to \mathbb{P}^N$ and $\beta_2 : I \to G$ be the two projections; then put

$$SB(X_n) = \beta_2^{-1}(S(X_n)) \subset I.$$

Since $\beta_2 : I \to G$ is a $\mathbb{P}^1-$bundle, $SB(X_n)$ will be an irreducible variety of dimension $2n + 1$.

Finally we have

$$Sec(X_n) = \beta_1(SB(X_n)).$$

It follows that $Sec(X_n)$ is an irreducible variety of dimension at most $2n+1$.

This fact has the immediate consequence that if $N > 2n + 1$, then we can project X_n isomorphically to $\mathbb{P}^{N-1}$, because the projection will be a one-to-one unramified morphism onto the image of X_n. Remember that the tangent variety to X_n is contained in the secant variety under our smoothness hypothesis.

Let $l \in G$ be a line not contained in X. Then $l \cap X = Spec(A)$, with A an artinian k-algebra, and we put length$(l \cap X)$=length(A). If l is a secant line to X, then length$(l \cap X) \geq 2$. Let

$$m = min\{\text{length}(l \cap X)|\ \ l \in S(X)\}.$$

We then say that the general secant line of X is a m-secant. When $m > 2$, we say that the general secant of X is a multisecant or trisecant.

If X_n is a hypersurface of degree d in $\mathbb{P}^{n+1}$, it is clear that the general secant line is a d-secant. On the other hand it is well known that if X_n has codimension at least two, a general secant line is a 2-secant (see for example [23]).

In the same way one constructs the abstract secant variety $S^k_{X_n}$, $k \geq 1$, as the closure in $X_n \times \ldots \times X_n \times \mathbb{P}^N$ of the locus of points $(x_0, \ldots, x_k, z) \in X_n \times \ldots \times X_n \times \mathbb{P}^N$ such that dim$< x_0, \ldots, x_k >= k$ and such that $z \in < x_0, \ldots, x_k >$. Finally the *$k$-secant variety to X_n*, $S^k X_n$, is defined as the image of $S^k_{X_n}$ by the restriction of the projection to the last factor $\mathbb{P}^N$. In particular we clearly have that $S^k X_n$ is an irreducible variety of dimension less or equal to the minimum between N and the dimension of $S^k_{X_n}$, which is $(k+1)n + k$.

An important tool in studying the geometry of tangent and secant lines to an algebraic variety is the following interesting result due to Terrcacini, universally known as Terracini's Lemma.

Theorem 1.1. *([Terracini11]) Let $X_n \subset \mathbb{P}^N$ be a non-degenerate irreducible variety. Let $x_0, \ldots, x_k \in X_n$, $k \geq 1$ be distinct points and let $z \in < x_0, \ldots, x_k >$ be an arbitrary point. Then $< T_{x_0}(X_n), \ldots, T_{x_k}(X_n) > \subseteq T_z(S^k X_n)$. If moreover $char(K) = 0$, $x_0, \ldots, x_k \in X_n$ and $z \in < x_0, \ldots, x_k >$ are general, then $< T_{x_0}(X_n), \ldots, T_{x_k}(X_n) >= T_z(S^k X_n)$.*

From now on let us suppose $N = 2n + 1$, char(K)=0 and that the secant variety of X_n fills the whole space, i.e. $Sec(X_n) = \mathbb{P}^{2n+1}$; or in other words that X_n cannot be projected isomorphically to $\mathbb{P}^{2n}$. Then $\beta_1 : SB(X_n) \to \mathbb{P}^{2n+1} = Sec(X_n)$ is a generically finite morphism of degree $\delta(X_n)$ and hence there exists an open set $V \subseteq \mathbb{P}^{2n+1}$ such that for every $z \in V$ there pass a finite number $\delta(X_n) \geq 1$ of secants to X_n; for a general choice of z every secant through z is a 2-secant. A general choice of z also assures by Terracini's Lemma that the projections from z to $\mathbb{P}^{2n}$ of the tangent n-planes in the points of contact of the secant lines through z intersect transversally only in the nodes.

We have thus shown that every smooth projective variety $X_n \subset \mathbb{P}^{2n+1}$ can be projected from a general point to $\mathbb{P}^{2n}$ such that the projection of X_n has exactly $\delta(X_n)$ improper nodes as singularities (we put $\delta(X_n) = 0$ if dim$(Sec(X_n))$=$2n$, the usual convention for the degree of a morphism).

We will also say that $\delta(X_n)$ is *the number of apparent double points of* X_n.

Remark 1. We can define in the same manner the number of apparent double points of $X_n \subset \mathbb{P}^N$, $N > 2n+1$, as the number of improper nodes obtained by projecting from a general $(N-2n-1)$-plane of $\mathbb{P}^N$. In this case we have that the number of apparent double points $\delta(X_n)$ is equal to the number of secants going through the general $(N-2n-1)$-plane of $\mathbb{P}^N$ and hence is equal to $deg(\beta_1) \cdot deg(Sec(X_n))$, where as above $\beta_1 : SB(X) \to Sec(X_n) \subset \mathbb{P}^N$.

Let $X_n \subset \mathbb{P}^N$ be a non-degenerate variety and let $p \in Sec(X_n) \setminus X_n$. One defines the *cone of secant lines through* p, $C_p(X_n)$, as the union of the secant lines to X_n passing through p. The *entry locus* Σ_p *of* p is defined scheme-theoretically as the intersection of C_p with X_n, i.e.

$$\Sigma_p(X_n) = C_p(X_n) \cap X_n.$$

If there is at least a proper secant line through p (for example if $p \in Sec(X_n) \setminus X_n$ is general), then we have

$$(\Sigma_p(X_n))_{red} = \overline{\{x \in X_n \mid \exists y \neq x, y \in X_n, p \in \langle x, y \rangle\}}.$$

For a general point $p \in Sec(X_n)$ the dimension of Σ_p is a projective invariant of X_n, called the *secant deficiency* of X_n, which was studied classically and more recently in [Zak93] and [Zak99]. A variety $X_n \subset \mathbb{P}^r$ is said a *variety with quadratic entry locus* if for the general $p \in Sec(X_n) \setminus X_n$, the entry locus Σ_p is a non-degenerated quadric hypersurface in the linear space $C_p(\Sigma_p)$, the cone over Σ_p of vertex p. Varieties with quadratic entry locus have a nice geometry and a lot of interesting properties.

Varieties with one apparent double point, which will be introduced in the next section, are a notable class of varieties for which the quadratic entry locus for a general point consists of a 0-dimensional quadric, i.e. 2 points on a secant line.

2. Varieties with one apparent double point

Let us recall the following definition introduced by Severi in [Severi01].

Definition 1. A smooth irreducible n-dimensional variety $X_n \subset \mathbb{P}^{2n+1}$ is said a *variety with one apparent double point*, briefly VOADP, if through a general point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to X_n; or equivalently, if the projection of X_n from a general point of $\mathbb{P}^{2n+1}$ is a variety having only one *improper double point* as singularities; or equivalently if and only if $\delta(X_n) = 1$.

We begin by showing some general properties of smooth VOADP's and we shall discuss some examples. More examples will be "constructed" in the next two sections.

Proposition 2.1. *(*[Severi01]*) If X_n is a VOADP, then X_n is linearly normal in $\mathbb{P}^{2n+1}$.*

Proof. If X_n is not linearly normal, then there exists a non-degenerate $Y_n \subset \mathbb{P}^N$, $N > 2n+1$, which projects isomorphically to X_n from a $(N-2n)$-plane. Since Y_n is non-degenerated, we have $deg(Sec(Y_n)) \geq 2$. Finally we obtain that either

$$\delta(X_n) = \delta(Y_n) = 0 \cdot deg(Sec(Y_n)) = 0$$

or

$$\delta(X_n) = \delta(Y_n) = deg(\beta_1(Y_n)) \cdot deg(Sec(Y_n)) \geq 2. \quad \square$$

Remark 2. It is quite easy to see that if C is an irreducible non-degenerate curve of $\mathbb{P}^N$, $N \geq 4$, then $deg(\beta_1) = 1$, i.e. β_1 is always birational; if C is a smooth curve, then the degree of $Sec(C)$ is equal to the number of nodes of the plane model of C obtained by projecting from a general $(N-3)$-plane.

One immediately sees that the symmetric product $X_n^{(2)}$ is birational to $S(X_n)$ and hence is unirational because $\beta_2 \circ \beta_1^{-1}$ gives a dominant rational map from $\mathbb{P}^{2n+1}$ to $S(X_n)$ (see section 1 for the definition of the β_i's and of $S(X_n)$).

Let φ be the restriction of $\beta_2 \circ \beta_1^{-1}$ to a general hyperplane H in $\mathbb{P}^{2n+1}$. The general secant line to X_n is not contained in H and so intersects H in one point; on the other hand through a general point of H there passes only one secant line to X, i.e. $\varphi : H \dashrightarrow S(X_n)$ is birational and hence $S(X_n)$ (or equivalently $X_n^{(2)}$) is a rational variety.

We collect this remark in the following:

Proposition 2.2. *Let $X_n \subset \mathbb{P}^{2n+1}$ be a VOADP. Then the symmetric product $X_n^{(2)}$ of X_n is a rational variety , $p_m(X_n) = h^0(\omega_{X_n}^{\otimes m}) = 0$ for every $m \geq 1$ and $q(X_n) = h^1(\mathcal{O}_{X_n}) = 0$. In particular every OADP surface is rational.*

Proof. The first part was proved above, while the second one is a consequence of the first, see [Mattuck69]. Let us sketch how to prove $q(X_n) = 0$. From general properties of the Albanese variety, we have $Alb(X_n) = Alb(X_n^{(2)})$ and from the surjective morphism $0 = Alb(\mathbb{P}^{2n}) \to Alb(X_n)$, we deduce that $q(X) = h^1(\mathcal{O}_{X_n}) = 0$, see [Mattuck69]. The fact that a OADP surface is rational follows from Castelnuovo' s theorem, since $p_2(X_2) = q(X_2) = 0$. $\square$

Remark 3. In [CMR01] it was recently proved the very strong result that every OADP variety is rational since it projects birationally from the tangent space at a general point onto a disjoint $\mathbb{P}^n$. The previous proposition gives another proof of the fact that a smooth curve $C \subset \mathbb{P}^3$ having one apparent double point is rational; since it is linearly normal, it is the twisted cubic, which clearly has one apparent double point.

Let us prove a further fundamental geometric property of VOADP's.

It is clear that if $C \subset \mathbb{P}^3$ is the twisted cubic and that if $l \subset \mathbb{P}^3$ is a secant line to C cutting it in two distinct points p_1 and p_2, then through every point of $l \setminus \{p_1, p_2\}$ there pass no secant or tangent lines to C. In this way we show the well known fact that C is a VOADP.

Let us take the Del Pezzo surface of degree 5 in $\mathbb{P}^5$, $X_2^5 \subset \mathbb{P}^5$. It is well known X_2^5 is isomorphic to the blow-up of $\mathbb{P}^2$ in four general points $p_1, \dots, p_4$, $Bl_{p_1,\dots,p_4}(\mathbb{P}^2)$, embedded by the linear system of cubics through the p_i's. If $l \subset \mathbb{P}^5$ is a general secant line cutting X_2^5 in the two distinct points p_5 and p_6, then since the linear system of cubics passing through 6 general points of the plane is very ample, we deduce that the projection of X_2^5 from the secant line l into $\mathbb{P}^3$ is an isomorphism on $X_2^5 \setminus \{p_5, p_6\}$; in particular through every point of $l \setminus \{p_5, p_6\}$ there passes no secant line to X_2^5. Hence X_2^5 has one apparent double point. This idea is due to Severi, see [Severi01], who tried to show that also the converse is true, i.e. that if X_n is a VOADP, then the projection from a general secant line l is an isomorphism on $X_n \setminus (l \cap X_n)$. This would be true only if the variety had not too many trisecant lines, see [Russo00] for a discussion.

Let us recall that if $X_n \subset \mathbb{P}^{2n+1}$ is a VOADP, the morphism $\beta_1 : SB(X_n) \to Sec(X_n) = \mathbb{P}^{2n+1}$ from the abstract to the embedded secant variety is birational by definition. Moreover, by Zariski's Main theorem β_1^{-1} is not defined along a closed set $Z(X_n)$ of codimension al least 2. If $p \in \mathbb{P}^{2n+1} \setminus Z(X_n)$ is a general point and if l is a general secant line through p, then from the birationality of β_1 we can deduce that through the general point of l there pass no secant line to X_n different from l. An application of Zariski's Main Theorem and of Terracini' s lemma gives the following stronger geometric property which we show holds for the twisted cubic and the Del Pezzo quintic surface. For a different proof using the classical theory of focal loci due to C. Segre, originally given by Severi, one can consult [Severi01] or [Russo00]. The following proof was communicated to me by F. L. Zak.

Proposition 2.3. *([Severi01]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a VOADP. If l is a general secant line to X_n and if $l \cap X_n = \{p_1, p_2\}$, then through every point p of l, $p \neq p_i$, $i = 1, 2$, there passes no secant line different from l.*

Proof. If through a point $p \in \mathbb{P}^{2n+1}$ there passes more than one secant line, then p is contained in infinitely many secant lines (this follows from Zariski's Main Theorem since β_1 is birational). Suppose that for a general pair of points $p_1, p_2 \in X_n$ there exists a point $p \in \langle p_1, p_2 \rangle$ distinct from p_1, p_2 and through which there passes a secant line different from l. Let

$$\Sigma_p = (\Sigma_p(X_n))_{red} = \overline{\{x \in X_n \mid \exists y \neq x, y \in X_n, p \in \langle x, y \rangle\}}$$

be the entry locus of X_n with respect to p. Then $\dim \Sigma_p > 0$ by the above remark (in fact by Zariski's Main Theorem, all components of Σ_p have positive dimension). Let C_p be the cone with vertex p and base Σ_p. If $p \in T_{p_i}(\Sigma_p)$ for some $i = 1, 2$, then $\langle p, p_i \rangle = \langle p_1, p_2 \rangle \subset T_{p_i}(X_n)$, but this contradicts the generality of the pair (p_1, p_2). So we may assume that $p \notin T_{p_i}(\Sigma_p)$, $i = 1, 2$. Then $T_{p_1}(C_p) = T_{p_2}(C_p) = \langle p, T_{p_i}(\Sigma_p)\rangle$, and $T_{p_i}(\Sigma_p)$ is a hyperplane in $T_{p_i}(C_p)$. Thus $T_{p_1}(\Sigma_p) \cap T_{p_2}(\Sigma_p) \neq \emptyset$ so that $T_{p_1}(X_n) \cap T_{p_2}(X_n) \neq \emptyset$. We obtain a contradiction with Terracini's Lemma which in our setting says: if p_1 and p_2 are general points of X_n, then $dim(T_{p_1}(X_n) \cap T_{p_2}(X_n)) = 2n - dim(Sec(X_n)) = -1$, i.e. $T_{p_1}(X_n) \cap T_{p_2}(X_n) = \emptyset$. □

Remark 4. The lines which are locus of foci, i.e. such that, roughly speaking, intersect another secant line, are contained in the fundamental locus of the proper surjective birational morphism $\beta_1 : SB(X_n) \to \mathbb{P}^{2n+1}$ and describe a locus of codimension at least two by Zariski' s Main Theorem.

This is a reformulation of the following result of C. Segre: "un sistema ∞^{n-1} di raggi del 1^o ordine (di $\mathbb{P}^n$) (dicendo *ordine* di un sistema il numero di raggi uscenti da un punto qualunque di $\mathbb{P}^n$) non puo' avere una M_{n-1} focale (giacchè ogni punto che sia fuoco è singolare per il sistema)" (see [16], pg. 149).

3. Examples

Clearly everyone's example of a VOADP is the twisted cubic curve $C = \nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$, whose general projection is a plane cubic curve with a node as singularity. In fact, it is easy to show, for example by using the Castelnuovo' s bound on the arithmetic genus of a non-degenerate curve $C \subset \mathbb{P}^3$, that the twisted cubic curve is the only irreducible curve with one apparent double point. We proved it in remark 2 as a consequence of the properties of VOADP's. On the other hand, there is one reducible smooth curve with one apparent double point, the union of two disjoint lines, $\mathbb{P}^1 \amalg \mathbb{P}^1 \subset \mathbb{P}^3$. Besides these two examples, if we consider reducible reduced curves, then there is only one more example: the union of a smooth conic and a line intersecting in one point. So in the one dimensional case everything is known and easy to prove. Let us remark that the twisted cubic (and the union of a conic and a line intersecting in one point) can be realized as a divisor of type $(1, 2)$ (or $(2, 1)$) on the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, while the union of two skew lines is a divisor of type $(0, 2)$ (or $(2, 0)$) on the quadric surface.

This has been generalized by Edge and Babbage to construct two series of VOADP's, the smooth divisors of type $(0, 2)$ and $(1, 2)$ on the Segre variety $Y_{n+1} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$.

Example 1. (*Edge varieties have one apparent double point*). Let $Y_{n+1} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$ be the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$. Then Y_{n+1} is an $(n+1)$-dimensional subvariety of degree $n+1$. Let π be one of the $\mathbb{P}^n$'s of the ruling of Y_{n+1} and let Q_1 be a general hyperquadric of $\mathbb{P}^{2n+1}$ containing π. Then the residual intersection of Q_1 with Y_{n+1} is an n-dimensional smooth variety X_n^{2n+1} of degree $2n + 1$ in $\mathbb{P}^{2n+1}$ and it is a divisor of type $(1, 2)$ on Y_{n+1}. The variety X_n^{2n+1} is isomorphic to the blow-up of $\mathbb{P}^n$ along the complete intersection Z of two quadric hypersurfaces of $\mathbb{P}^n$, i.e. $X_n^{2n+1} \simeq Bl_Z(\mathbb{P}^n)$. The isomorphism is given by the (strict transform of) linear system $|3H - Z|$

of cubic hypersurfaces through Z. For $n = 2$ one clearly recovers the Del Pezzo surface of degree 5.

Let now π_α and π_β be two fixed n-planes of the ruling of Y_{n+1}. If Q_2 is a general hyperquadric through π_α and π_β, then the residual intersection of Q_2 with Y_{n+1} is an n-dimensional smooth variety X_n^{2n} of degree $2n$ and it is a divisor of type $(0,2)$ on Y_{n+1}, i.e. $X_n^{2n} \simeq \mathbb{P}^1 \times Q^{n-1}$.

Edge shows geometrically that these varieties have one apparent double point (see [Edge32], I.2–I.5). We will reproduce Edge's argument in lemma 3.1. The above construction gives, for $n = 1$, as X_1^3 the twisted cubic and as X_1^2 a pair of lines in a ruling of the quadric Y_1. This last one is not irreducible. For $n = 2$, X_2^5 is a Del Pezzo surface of degree 5 in $\mathbb{P}^5$, while the surface X_2^4 is a rational normal scroll $S(2,2)$ of degree 4 in $\mathbb{P}^5$.

Let us now describe a third series of VOADP's, the smooth divisors of type $(2,1)$ on the Segre variety $Y_{n+1} = \mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$. Even if Edge' s argument given below to show that divisors of type $(0,2)$ and $(1,2)$ are VOADP's applies also to divisors of type $(2,1)$, this series was surprisingly ignored by classical geometers, see [Russo00] for an historical discussion and precise references.

Example 2. (*Smooth n-dimensional varieties of minimal degree in $\mathbb{P}^{2n+1}$ whose secant variety fill the whole space have one apparent double point*). We construct two series of varieties $X_n^{n+2} \subset \mathbb{P}^{2n+1}$, each of which has one apparent double point. These are the smooth n-dimensional rational normal scrolls in $\mathbb{P}^{2n+1}$ of degree $n+2$ $S(\underbrace{1,\ldots,1}_{n-2},2,2)$, $n \geq 2$, or $S(\underbrace{1,\ldots,1}_{n-1},3)$, $n \geq 1$. These varieties are linearly normal, have $h^1(\mathcal{O}) = 0$ and their secant varieties fill the whole space. Moreover, these are the only smooth $X_n^{n+2} \subset \mathbb{P}^{2n+1}$, whose secant varieties fill the whole space by a classical theorem of Bertini and Del Pezzo. For $n \geq 2$ these are divisors of type $(2,1)$ on the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$ into $\mathbb{P}^{2n+1}$, while, for $n = 2$, $S(2,2)$ can be realized also as a divisor of type $(0,2)$ on the Segre 3-fold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. From this description one deduces that they have one apparent double point by applying an idea of Edge, see the following lemma 3.1.

Lemma 3.1. *(*[Edge32]*) Let $X_n^{a,b} \subset \mathbb{P}^{2n+1}$ be a smooth irreducible variety contained as a divisor of type (a,b) in the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$ into $\mathbb{P}^{2n+1}$, $n \geq 2$. Then $X_n^{a,b}$ has one apparent double point if and only if $(a,b) \in \{(2,1),(0,2),(1,2)\}$, i.e. if and only if $X_n^{a,b}$ is either a variety of minimal degree or one of the two Edge's varieties.*

Proof. For $p \notin \mathbb{P}^1 \times \mathbb{P}^n$ the entry locus of secant lines through p has the form $\Sigma_p(\mathbb{P}^1 \times \mathbb{P}^n) = \mathbb{P}^1 \times \mathbb{P}^1_p$ for some $\mathbb{P}^1_p \subset \mathbb{P}^n$ and spans a linear $\mathbb{P}^3_p$ (see for example [Edge32]). If $X_n^{a,b}$ is a divisor contained in $\mathbb{P}^1 \times \mathbb{P}^n$, the secant lines of $X_n^{a,b}$ passing through p are exactly the secant lines of $X_n^{a,b} \cap \mathbb{P}^3_p$ passing through p. For a general $p \in \mathbb{P}^{2n+1}$, $X_n^{a,b} \cap \mathbb{P}^3_p$ is a reduced, not necessarily irreducible curve and it is a divisor of type (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1_p$. Hence $X_n^{a,b}$ has one apparent double point if and only if $X_n^{a,b} \cap \mathbb{P}^3_p$ has one apparent double point for $p \notin \mathbb{P}^1 \times \mathbb{P}^n$ general; this happens if and only if $(a,b) \in \{(1,2),\ (2,1),\ (2,0),\ (0,2)\}$ as we remarked at the beginning of this section. If $(a,b) = (2,0)$, then $X_n^{2,0} = \mathbb{P}^n \amalg \mathbb{P}^n$ is reducible. □

In the next section we will construct more examples using a different tool and we will try to motivate the principle that essentially the unique way of arranging the secant lines of a variety so that through the general point there passes only one is to realize them as fibers of the rational map given by quadrics through the variety. Even if this seems to be only an empirical observation for the moment, clearly verified in every known example, we will return to it in the last section where we will discuss some open problems on the structure and classification of VOADP's.

4. Special subhomaloidal systems of quadrics and VOADP's

Let us introduce some definitions.

Definition 2. A linear system of hypersurfaces of $\mathbb{P}^r$ is said *homaloidal* if it defines a birational map onto the image and *subhomaloidal* if the (closure of a) general fiber of the associated rational map is a linear projective space. It is said *special* if the base locus scheme of the linear system is a smooth irreducible subvariety $X \subset \mathbb{P}^r$; a linear system is said *completely (sub)homaloidal* if (the closure of) every fiber is a linear projective space.

Let us take a variety $X_n \subset \mathbb{P}^{2n+1}$ defined by quadratic equations and having secant variety filling the whole space. Let us suppose that the linear system of quadrics containing X_n restricted to a general hyperplane section of X_n is a special homaloidal system of quadrics. Then X_n has one apparent double point. In fact the linear system $|H^0(\mathcal{I}_{X_n}(2))|$ contracts the secant lines to X to points and defines a rational map $\phi : \mathbb{P}^{2n+1} \dashrightarrow \mathbb{P}(H^0(\mathcal{I}_{X_n}(2)))$. If the restriction of ϕ to a general hyperplane is a birational map onto the image, then the general fibers of ϕ are necessarily lines and hence secant lines to X_n (a contracted line is a secant line to X_n); this proves that through a general point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to X_n and that $|H^0(\mathcal{I}_{X_n}(2))|$ is subhomaloidal.

In this way one can verify that Edge' s varieties and varieties of minimal degree are VOADP's. We will give below an idea of the proof that in this cases $|H^0(\mathcal{I}_{X_n}(2))|$ is subhomaloidal with general fiber a line.

Let us apply the above remark to construct some more examples.

The fact that the 3-fold described in example 3 has one apparent double point seems to be unknown before.

Example 3. (*Smooth 3-folds of degree 8 in $\mathbb{P}^7$ with one apparent double point*). Let $X = \mathbb{P}(\mathcal{E}) \subset \mathbb{P}^7$ be the scroll over $\mathbb{P}^2$ associated to the very ample vector bundle $\mathcal{E}$ of rank 2 given as an extension by the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \to \mathcal{I}_{p_1,\ldots,p_8}(4) \to 0,$$

where $p_1, \ldots, p_8$ are points in $\mathbb{P}^2$ not lying on a conic and such that no 4 are collinear. Such a vector bundle exists (see [Ionescu82]). A general hyperplane section of X corresponding to a general section of $\mathcal{E}$ is a smooth octic rational surface Y, the embedding of the blow-up of $\mathbb{P}^2$ at the p_i's given by the quartics through the p_i's. The surface Y is arithmetically Cohen-Macaulay and it is cut out by 7 quadrics for example by proposition 4.2. Moreover the linear system $|H^0(\mathcal{I}_Y(2))|$ defines a Cremona transformation of $\mathbb{P}^6$ (see [HKS92], [ST70] and also [MR01]). The variety X is an arithmetically Cohen-Macaulay 3-fold whose ideal is generated by 7 quadratic forms. We now show that $Sec(X) = \mathbb{P}^7$ and that X has one apparent double point. Let $\phi : \mathbb{P}^7 \dashrightarrow \mathbb{P}^6$ be the rational map given by the quadrics defining X; ϕ is defined outside X and contracts all the secant lines to X. Since restricting ϕ to a general hyperplane H we obtain a Cremona transformation, the general fiber of ϕ is a line and necessarily a secant line to X because ϕ is defined by the quadrics through X.

We would like to remark that by looking to the resolution of the ideal of Y, see [HKS92] section 3, one sees that X does not satisfy condition K_2 (see below) and that, by adapting the arguments of lemmas 3.4 and 3.5 of [HKS92], the quadrics through X define a subhomaloidal system, which is not completely subhomaloidal. Moreover, there exists $p \in \mathbb{P}^7 \setminus X$ for which the entry locus is not quadratic.

Example 4. (*A smooth 4-fold of degree 12 in $\mathbb{P}^9$ with one apparent double point*). Let $S_{10} \subset \mathbb{P}^{15}$ be the 10-dimensional spinorial variety. The ideal of S_{10} is generated by 10 quadrics and the linear system $|H^0(\mathcal{I}_{S_{10}}(2))|$ defines a rational map $\phi : \mathbb{P}^{15} \dashrightarrow \mathbb{P}^9$, whose image is the 8-fold quadric $Q_8 \subset \mathbb{P}^9$ and such that the closure of every fiber is a $\mathbb{P}^7$ (see [ESB89]). Taking a section of S_{10} with a general $\mathbb{P}^9 \subset \mathbb{P}^{15}$ we have a smooth 4-fold $X_4 \subset \mathbb{P}^9$ with $Sec(X_4) = \mathbb{P}^9$ and such that $|H^0(\mathcal{I}_{X_4}(2))|$ is a completely subhomaloidal linear system for which (the closure of) a general fiber is a secant line to X. Then X has one apparent double point, it has quadratic entry locus for every $q \in \mathbb{P}^9 \setminus X_4$ and the projection from a general tangent space is birational (for this last fact see section 7).

To study special (sub)homaloidal systems of hypersurfaces the following condition, introduced in [Vermeire01], showed to be very useful.

Definition 3. A subscheme $X \subset \mathbb{P}^r$ *satisfies condition K_d* if X is scheme theoretically cut out by forms $F_0, \ldots, F_k$ of degree d such that the trivial (or Koszul) relations among the F_i are generated by linear syzygies.

More generally, let $V \subseteq H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ be a linear system of forms of degree d with possibly (empty) base scheme X. Then the pair (X, V) *satisfies* *condition* K_d if the trivial relations among the elements of V are generated by linear syzygies. If V satisfies condition K_d, and if $M \simeq \mathbb{P}^k \subset \mathbb{P}^r$ is a linear space, then the restriction of V to M satisfies property K_d. If a smooth projective variety satisfies condition N_2 of Green, then $H^0(\mathcal{I}_X(2))$ clearly satisfies condition K_2. The following proposition is a consequence of the definitions in the case $d = 2$ (on $\mathbb{P}^1$ the only linear system of quadrics satisfying condition K_2 and without base points is the complete one) while in general it is proved using an idea contained in [HKS92]. We would like to remark that for geometrical applications the importance of this result is that under condition K_d we have completely subhomaloidal linear systems, i.e. (the closure of) every fiber is a projective linear space, not necessarily of the same dimension, eventually of dimension 0. For the nice proof we refer to [Vermeire01].

Proposition 4.1. *(*[Vermeire01]*) Let $X \subset \mathbb{P}^r$ be a smooth variety scheme theoretically defined by forms $F_0, \ldots, F_k$ satisfying condition K_d. Then the rational map $\phi : \mathbb{P}^r \dashrightarrow \mathbb{P}^k$ defined by the F_i's is an embedding off $Sec_d(X)$, the variety of d-secant lines. Moreover the closure of a positive dimensional fiber of ϕ is a linear space in $\mathbb{P}^r$ intersecting X in a hypersurface of degree d. In particular if $V =< F_0, \ldots, F_k >$, then the linear system $|V|$ is special and completely subhomaloidal.*

Special (sub)homaloidal systems of quadrics deserved great interest. Well known examples are given by quadrics through a rational normal curve of degree 4 (general representation of $\mathbb{G}(1,3)$, see [Semple31], [SR49]), by quadrics through a normal elliptic quintic curve (quadro-cubo transformation of $\mathbb{P}^4$, see [SR49]) or by quadrics through a Veronese surface in $\mathbb{P}^5$ (quadro-quadric involutory transformation of $\mathbb{P}^5$, see [SR49], [ESB89]), by quadrics through a rational octic surface in $\mathbb{P}^6$ (see [ST70],[HKS92]). These special homaloidal systems have also the property that the map they define is an isomorphism on $\mathbb{P}^r \setminus Sec(X)$ and more precisely that they are completely subhomaloidal. Several interesting examples of special completely subhomaloidal systems are discussed in [SR49], [ESB89] and [HKS92] and most of the examples of homaloidal systems are obtained from the subhomaloidal ones by restricting to a general linear space of dimension equal to the codimension of the generic fiber. Viceversa once we have a special homaloidal system and we know that the base locus can be extended non-trivially we could hope to find a special subhomaloidal system. We list the following result which includes as a special case the verification that most of the special linear system of quadrics listed above are completely (sub)homaloidal and whose proof can be found in [AR02].

Proposition 4.2. *(*[AR02]*) Let $X \subset \mathbb{P}^r$ be a smooth linearly normal variety. Suppose $h^1(\mathcal{O}_X) = 0$, if $dim(X) \geq 2$. Let s=$codim(X)$ and let d=$deg(X)$. If $d \leq 2s+1$, then X is arithmetically Cohen-Macaulay. If $d \leq 2s$, then the homogenous ideal of X is generated by quadratic forms. If $d \leq 2s-1$, then* *the syzygies of the generators of the homogenous ideal of X are generated by the linear ones and the quadrics through X give a special and completely subhomaloidal system. If $d \leq 2s-1$ and $Sec(X) \subsetneq \mathbb{P}^r$, then the associated rational map is an embedding on $\mathbb{P}^r \setminus Sec(X)$.*

Varieties with quadratic entry locus have a nice geometry and a lot of interesting properties; proposition 4.1 gives a tool to verify that some examples share this property. Proposition 4.1 can be rephrased saying that if $X \subset \mathbb{P}^r$ satisfies condition K_2, then its entry locus is quadratic for every $p \in Sec(X) \setminus X$.

Varieties with one apparent double point are a notable class of varieties for which the quadratic entry locus for a general point consists of a 0-dimensional quadric, i.e. 2 points on a secant line; all the known examples with the exception of example 3 have also the property that the entry locus is quadratic for every point $p \in Sec(X) \setminus X$.

Remark 5. In [AR3] we proved geometrically some results on varieties whose degree is small with respect to codimension by using Castelnuovo's argument that, in $\mathbb{P}^s$, a set of $k \leq 2s+1$ points in general linear position impose independent conditions to quadric hypersurfaces. We proved directly that regular varieties $X \subset \mathbb{P}^r$ whose degree d is less than or equal to $2s-1$ are the scheme-theoretic intersection of the quadrics through them and that $|H^0(\mathcal{I}_X(2))|$ is a special subhomalidal system, showing that this last result is elementary and completely geometric. The stronger fact that such varieties have homogenous ideal generated by quadratic forms whose first syzygies are generated by the linear ones gives that every fiber of the associated rational map is linear, i.e. it gives information about the entry locus of the variety.

Let us apply the above results as in [AR02].

Theorem 4.3. *([AR02]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth n-dimensional variety scheme theoretically defined by quadratic forms $F_0, \ldots, F_k$ satisfying condition K_2 and such that $Sec(X_n) = \mathbb{P}^{2n+1}$. Then X_n has one apparent double point. In particular a linearly normal regular variety $X_n \subset \mathbb{P}^{2n+1}$ with $Sec(X_n) = \mathbb{P}^{2n+1}$ and of degree $d \leq 2n+1$ has one apparent double point.*

1^{st} **Proof**. Let $p \in \mathbb{P}^{2n+1}$ be a general point and suppose that through p there pass at least two secant lines l_1, respectively l_1'. By the generality of p we can assume that they cut X_n at the distinct points q_1, q_2, respectively q_1', q_2'. Remember that since $Sec(X_n) = \mathbb{P}^{2n+1}$ through a general point of $\mathbb{P}^{2n+1}$ there passes a finite number of proper secant lines to X_n. Then the plane $\Pi_p = < l, l' >$ would intersect X_n, scheme theoretically, in a finite set of points $\Gamma \supseteq \{q_1, q_2, q_1', q_2'\}$ (otherwise through p there would pass infinitely many secant lines to X_n), which would be scheme theoretically defined by the restrictions of the F_i's to Π_p. This would imply that $\Gamma = \{q_1, q_2, q_1', q_2'\}$

is the complete intersection of two conics, hence the restrictions of the F_i's to Π_p would not satisfy condition K_2.

2^{nd} **Proof**. Let $Y = X_n \cap H$ be a generic hyperplane section. Then Y satisfy property K_2 and $Sec(Y) \subset H = \mathbb{P}^{2n}$. By proposition 4.1 the generic fiber of the rational map defined by the $F_i^{\prime}s$ is then a line (secant to X_n), since it cuts H only in one point; hence X_n has one apparent double point since through a generic point of $\mathbb{P}^{2n+1}$ there passes a unique secant line to it. $\square$

Let us construct further examples of varieties with one apparent double point.

Example 5. (*Lagrangian Grassmannians $G(3,6)$ over the 4 composition algebras have one apparent double point*). Let $X_6 \subset \mathbb{P}^{13}$ be the 6-dimensional symplectic Grassmanian, let $X_9 \subset \mathbb{P}^{19}$ be the Plücker embedding of the 9-dimensional Grassmanian $G(3,6)$, let $X_{15} \subset \mathbb{P}^{31}$ be the 15-dimensional orthogonal Grassmanian and finally let $X_{27} \subset \mathbb{P}^{55}$ be the 27-dimensional E_7-variety. These varieties have a uniform description as the Lagrangian Grassmannian $G^{Lag}(3,6)$ over the 4 composition algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, respectively $\mathbb{O}$ or as *twisted cubics over the cubic Jordan algebras* $Sym^3\mathbb{C}$, $M_3\mathbb{C}$, $Alt_6\mathbb{C}$, respectively $H_3\mathbb{O}$ and are also examples of *Legendre submanifolds* (see [Mukai99]).

These are homogenous varieties whose ideal is generated by quadratic forms satisfying condition K_2 and whose secant varieties fill the whole space. They have one apparent double point and the projection from a general tangent space is birational (for the last fact see section 7). The fact that these varieties have one apparent double point was communicated to us by F. L. Zak, whose proof is completely different. The fact that the projection from the tangent space is birational for these varieties was proved originally by Mukai by an ad hoc method using representation theory and the fact that these variety could be described as twisted cubics over the cubic Jordan algebras.

Remark 6. The same idea used in the second proof of theorem 4.3 can be applied to prove some classical results about varieties with one apparent triple point, i.e. varieties such that through the generic point of the space there passes a unique trisecant line. Take for example the projected Veronese surface $S \subset \mathbb{P}^4$; its ideal is generated by 7 cubic forms and the relations between these cubic forms are generated by linear syzygies (see for example [AR02]); hence these cubics satisfy condition K_3, define a special completely subhomaloidal linear system and give a rational map $\phi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^6$ which contracts the trisecant lines of S. These trisecant lines fill the whole space and we can show that through the general point of $\mathbb{P}^4$ there passes only one trisecant line by observing that the restriction of ϕ to a general hyperplane gives a birational map onto the image by proposition 4.1: the hyperplane section is a projected quartic rational curve, scheme theoretically defined by the seven cubics restricted to the hyperplane and the trisecant lines to the curve do not fill the hyperplane. Then a general fiber of ϕ is a trisecant line to S. The same technique works for the Bordiga surface of $\mathbb{P}^4$: it has ideal generated by 4 cubic forms, whose first syzygies are all linear (see for example [Ionescu82]), and the trisecant lines fill the whole space. Then the Bordiga surface has one apparent triple point and the cubic forms defining it form a special and completely subhomaloidal linear system. In [Severi01] Severi proves that the projected Veronese surface and the Bordiga surface are the only smooth surfaces in $\mathbb{P}^4$ having one apparent triple point.

5. Classification of OADP surfaces and 3-folds

The varieties described in examples 1,2,3,4 and 5 seem to be the only smooth OADP varieties known. The strong geometric properties of VOADP's described in section 2 and more properties which will be discussed in the following sections furnish the hope of classifying varieties with one apparent double point. Surely this will be an hard job requiring new ideas. The classification results known till now will be described in this section and the following one.

As we have seen it is easy to see that the only smooth OADP curve is the twisted cubic. In the two dimensional case the surfaces described in examples 1 and 2 are the scrolls $S(2,2)$ and $S(1,3)$ and the Del Pezzo surface of degree 5 in $\mathbb{P}^5$, $X_2^5 \subset \mathbb{P}^5$. In [Severi01], Severi firstly classifies surfaces in $\mathbb{P}^5$ without apparent double points showing in his famous theorem that there is only one, the Veronese surface; then Severi claims that the above three surfaces are the only smooth OADP surfaces. This classification was completed recently in [Russo00] by showing that this was the case by using the ideas contained in the original paper of Severi. Let us give the classification in the 2 dimensional case.

Theorem 5.1. *(*[Severi01], [Russo00]*) Let $X_2 \subset \mathbb{P}^5$ be an OADP surface. Then X_2 is projectively equivalent either to $S(2,2)$, or to $S(1,3)$ or to $X_2^5 \subset \mathbb{P}^5$.*

We will give a rough idea of the proof given in [Russo00] and then sketch the proof of the previous theorem by using a recent result of [CMR01]. These last results contain a lot of geometrical information about the tangential behaviour of OADP varieties.

In [Russo00] it is shown that either the conclusion is as in the theorem, getting it by applying Severi' s argument, or the surface $X_2 \subset \mathbb{P}^5$ contains a one dimensional family of (irreducible) plane curves of degree $\alpha \geq 3$ so that it has a lot of trisecant lines and Severi' s argument cannot be applied. To conclude the classification it is sufficient to exclude the existence of this last cases, see [Russo00], Lemma 3 and Proposition 8.

We present the following useful generalization (see also [AR02], proposition 9).

Proposition 5.2. *([CMR01], prop. 2.6). Let $X_n \subset \mathbb{P}^{2n+1}$ be an OADP-variety of dimension $n \geq 2$. Suppose X is covered by a 1-dimensional family $\mathcal{D}$ of divisors D of degree α, whose general element spans a $\mathbb{P}^n$. Then $\alpha = 2$, X is an Edge variety, hence X is either a divisor of type $(0,2)$ or $(1,2)$ on $\mathbb{P}^1 \times \mathbb{P}^n$, and $\mathcal{D}$ is cut out on X by divisors of type $(1,0)$.*

Proof. Let us take two general divisors $D_1, D_2 \in \mathcal{D}$. Since X is regular by 2.2, the two distinct algebraically equivalent divisors D_1 and D_2 are in fact linearly equivalent. Let $\mathcal{D}'$ be the linear pencil generated by D_1 and D_2 and let $Y \supset X$ be the variety swept out by $< D' >$ as D' varies in $\mathcal{D}'$. Since X is linearly normal it is easy to see that for every divisor $D' \in \mathcal{D}'$ we have $< D' >= \mathbb{P}^n$. Then Y is a rational normal scroll of dimension $n+1$ (see [EH87, Theorem 2]). Next we prove that Y is not a cone. Indeed, in this case, let V be the vertex of Y and let $s \in V$ be any point. Then either the general D' does not pass through s or s is a base point of the pencil $\mathcal{D}'$. In the last case the general D' is smooth at s because otherwise we would have $< D' >= T_s(D') = T_s(X)$. Take $D'_i \in \mathcal{D}'$, $i = 1, 2$, two general divisors and a general point $p_i \in D'_i$. Of course the points p_1, p_2 are general points of X. We claim that s is a base point of $\mathcal{D}'$ and that $\alpha = 2$. Indeed, if either $s \notin D'_i$, $i = 1, 2$, or if $s \in D'_1 \cap D'_2$ and $\alpha \geq 3$, the line $< s, p_i >$ would intersect the divisor D'_i in a point q_i different from s and p_i and the line $< q_1, q_2 >$ would cut the general secant line $< p_1, p_2 >$ in a point different from p_1 and p_2 and X would not be an *OADP*-variety. Now Zak' s Theorem on Tangencies, [Zak93, Corollary 1.8], implies that every quadric $D' \in \mathcal{D}'$ has at most a singular point. In particular $h^0(D', \mathcal{O}_{D'}(D')) \geq 2$ since $\mathcal{O}_{D'}(D')$ has a non zero section which vanishes. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D') \to \mathcal{O}_{D'}(D') \to 0,$$

together with $h^1(X, \mathcal{O}_X) = 0$, yields $h^0(X, \mathcal{O}_X(D')) \geq 3$. Then the general section of X with a linear space of dimension $n+3$ would be a smooth irreducible non-degenerate surface $S \subset \mathbb{P}^{n+3}$ containing a 2-dimensional family of conics. Therefore S would be either the Veronese surface in $\mathbb{P}^5$ or a projection of it (e.g. see the appendix). Thus $n = 2$ and X would be the Veronese surface, which is impossible. This contradiction proves that Y is smooth so that $Y = \mathbb{P}^1 \times \mathbb{P}^n$ and the conclusion follows from lemma 3.1. □

Let us briefly describe a special case of the geometrical idea of [CMR01] adapted to our setting and apply them to give a short proof of the classification of OADP-surfaces.

Before doing this let us remark the following. The problem of classifying VOADP's is clearly different from that of classifying n-dimensional subvarieties of $\mathbb{P}^{2n}$ having one improper node.

We construct an example of an algebraic surface in $\mathbb{P}^4$ having one improper node, which is not the projection of a smooth surface in $\mathbb{P}^5$ (see [SR49], pg. 143).

Example 6. Take two planes in $\mathbb{P}^4$ intersecting in one point P and two general cubic 3-folds through these planes. The residual intersection of these two 3-folds is a surface of degree 7 in $\mathbb{P}^4$ having one improper node in P and whose tangent cone is represented by the two planes.

Let $X_n \subset \mathbb{P}^{2n+1}$ be a VOADP. Let us take a general $(n-1)$-dimensional linear space $\Pi = \mathbb{P}^{n-1}$. In particular we have $\Pi \cap X_n = \emptyset$ and we can project X_n from Π onto an hypersurface $X'_n \subset \mathbb{P}^{n+1}$. The projection morphism $\pi : X_n \to X'_n$ is birational and has a "double point" locus given on X_n by the couples of points of X_n collinear with the center of projection Π. The image by π on X'_n of this locus is the singular locus of the hypersurface X'_n. Since X_n is an OADP variety and since Π is general through the general point of Π there passes a unique secant line to X_n so that the singular locus of X'_n is an irreducible rational variety of dimension $n-1$.

Let now x be a general point on X_n and let $T_x(X_n)$ be the projective tangent space to X_n at x. Roughly speaking, we will study how the projection $X_n \to \mathbb{P}^{n+1}$ *degenerates* when its centre Π tends to a general $(n-1)$-dimensional subspace Π_0 containing x, i.e. such that $\Pi_0 \cap X_n = \Pi_0 \cap T_x(X_n) = \{x\}$. To be more precise we will describe the *limit* of the *double point scheme* of $X_n \to \mathbb{P}^{n+1}$ in such a degeneration and show that a general tangent hyperplane section appears as a component of the *limit* of the double point scheme (or naturally of the singular locus of the *limit projection*). Since in the low dimensional case the smooth components of the *limit* of families of irreducible rational varieties can be controlled, one deduces that a general tangent hyperplane section is a(n) (irreducible) rational variety if X_n is not a scroll in $\mathbb{P}^{n-1}$.

Let us recall the following well known result which for $n = 2$ is a classical theorem of Castelnuovo and Kronecker. We will state it only for $X_n \subset \mathbb{P}^{2n+1}$ but the result is more general.

Proposition 5.3. *Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth irreducible non-degenerated variety. For $n \geq 3$ a general tangent hyperplane section is reducible if and only if X_n is a scroll into $\mathbb{P}^{n-1}$ over a curve. For $n = 2$ a general tangent hyperplane section is reducible if and only if X_2 is the Veronese surface or a scroll in lines over a curve.*

Then the study of the degeneration of the double locus schemes in the *limit projection* gives the following result in the low dimensional cases, which opens the way to the classification of OADP 3-folds. In the 3-dimensional case a detailed study of a general tangent curve section is also needed. See [CMR01] for a more general statement in arbitrary dimension.

Proposition 5.4. *(*[CMR01]*) Let $X_n \subset \mathbb{P}^{2n+1}$, $n = 2, 3$, be a VOADP. Let D be its general tangent hyperplane section. Assume X_n is not a scroll in $\mathbb{P}^{n-1}$'s over a curve. Then D is irreducible, rational and has a double point as its singularity. Moreover, for $n = 3$ the general hyperplane section H of X_3 is a rational surface.*

Then we can easily sketch the proof of the classification theorem in the 2 dimensional case. Either X_2 is a scroll over a curve, necessarily rational since $q(X_2) = 0$ and hence it is $S(2,2)$ or $S(1,3)$ (recall that a VOADP is linearly normal), or its general tangent hyperplane section D is an irreducible rational curve having a double point as its singularity. This implies that $p_a(D) = 1$ and hence that the geometric genus of a general hyperplane section of X_2 is 1, i. e. a general hyperplane section is a smooth elliptic curve $C \subset \mathbb{P}^4$. The curve C is linearly normal since X_2 is linearly normal and $q(X_2) = 0$; it easily follows that $deg(X_2) = deg(C) = 5$ by Riemann-Roch theorem and that X_2^5 is a Del Pezzo surface of degree 5.

In the 3-dimensional case a deep study of a general tangent curve section and of the (unreduced) scheme $Z_x = T_x(X_3) \cap X_3$, $x \in X_3$ a general point, together with the previous analysis gives a deep information on the structure of X_3. One is able to prove that either X_3 is a scroll in $\mathbb{P}^2$ over $\mathbb{P}^1$ or the scheme $(Z_x)_{red}$ consists of m, $1 \leq m \leq 3$, lines through x and that $d = deg(X_3) = 9 - m$. Putting together all the strong geometrical restrictions on the geometry of X_3, it is then possible to get the classification of smooth OADP 3-folds. The final result is the following.

Theorem 5.5. *(*[CMR01]*) Let $X_3 \subset \mathbb{P}^7$ be an OADP 3-fold. Then X_3 is projectively equivalent either to a scroll into lines over $\mathbb{P}^2$ described in example 3 (d=8) or to a Edge variety of dimension 3 (d=7,6), or to $S(1,1,3)$ (d=5) or to $S(1,2,2)$ (d=5).*

6. A problem posed by F. L. Zak and the classification of n-dimensional OADP varieties of degree $d \leq 2n+4$

The property of having one apparent double point is a very strong condition.

Let us list the linearly normal varieties $X_n \subset \mathbb{P}^{2n+1}$ having $Sec(X_n) = \mathbb{P}^{2n+1}$ and degree $d \leq 2n+1$ for $n = 1, 2, 3, 4$. It turns out that they all have one apparent double point.

Proposition 6.1. *(*[AR02]*) Let $X_n \subset \mathbb{P}^{2n+1}$, $n \leq 4$, be a linearly normal variety of degree $d \leq 2n+1$ and with $Sec(X_n) = \mathbb{P}^{2n+1}$. Then X_n is one of the following: $S(\underbrace{1,\ldots,1}_{n-1},3)$, $n \geq 1$, or $S(\underbrace{1,\ldots,1}_{n-2},2,2)$, $n \geq 2$, or an Edge variety of dimension $n \geq 2$. In particular such an X_n has one apparent double point.*

So one could ask why there are few linearly normal varieties $X_n \subset \mathbb{P}^{2n+1}$ with degree $d \leq 2n+1$. As we will see the answer is that there are very few because, if $Sec(X_n) = \mathbb{P}^{2n+1}$ (the "general case"), they have one apparent double point!

Let us make another remark. Let $X_n \subset \mathbb{P}^{2n+1}$ with $Sec(X_n) = \mathbb{P}^{2n+1}$ as always and suppose that through the general point of $\mathbb{P}^{2n+1}$ there passes at least two distinct secant lines, i.e. that X_n has more than one apparent double point. Let p a general point and let L_1 and L_2 be two secant lines through p. Then the plane $\Pi_p =< L_1, L_2 >$ cuts X_n at least in 4 points. Let us take a general $q \in \Pi_p$ and a secant line to X_n passing through q. Then the 3-plane $\Sigma_p =< \Pi_p, L_3 >$ cuts X_n at least in 6 points. Proceeding in this way we can construct a linear space Γ of dimension $n+1 = codim(X_n)$ which cuts X_n in at least $2n+2$ points. So either the degree of X_n is greater or equal to $2n+2$ or, varying the point p, we would obtain a huge family of degenerated varieties in X_n.

The above result and the empirical discussion above raise the following natural questions: *is it true that for every n the scrolls $S(\underbrace{1,\ldots,1}_{n-2},2,2)$, $S(\underbrace{1,\ldots,1}_{n-1},3)$ and the Edge varieties of dimension n are the only varieties $X_n \subset \mathbb{P}^{2n+1}$ with one apparent double point having degree $d \leq 2n+1$? Or equivalently, are these the only smooth regular linearly normal n-dimensional varieties of degree less than or equal to $2n+1$ with secant variety filling the whole space?*

More generally F. L. Zak asked: *is a linearly normal variety $X_n \subset \mathbb{P}^{2n+1}$ of degree less than or equal to $2n+1$ and with $Sec(X_n) = \mathbb{P}^{2n+1}$ a variety with one apparent double point ? Is a linearly normal variety $X_n \subset \mathbb{P}^{2n+1}$ of degree less than or equal to $2n+1$ and with $Sec(X_n) = \mathbb{P}^{2n+1}$ a regular variety, i.e. $h^1(\mathcal{O}_{X_n}) = 0$?*

See also the end of this section for a relation between the above problem and the topological results of Barth and Larsen on varieties of small codimension.

If $n \geq 5$ and if $X_n \subset \mathbb{P}^{2n+1}$ is a variety with one apparent double point of degree $d \leq 2n+1$, then Theorem 1 of [Ionescu85] combined with Iskovskih-Fujita classification of Del Pezzo manifolds (see [Fujita90]) implies that X_n is either a scroll over a curve or a scroll over a birationally ruled surface or a hyperquadric fibration over a curve; this opens the chance of answering the above questions. We summarize this consequence of Ionescu's theorem in the following proposition which was of fundamental importance in the classification of VOADP's of "small degree" .

Proposition 6.2. *(*[Ionescu85]*) Let $X_n \subset \mathbb{P}^{2n+1}$ be a non-degenerate linearly normal smooth variety of degree $d \leq 2n+3$, $n \geq 4$, and let H denote its hyperplane section. Then X_n is one of the following:*

i) a scroll over a curve;

ii) a hyperquadric fibration;

iii) a scroll over a birationally ruled surface;

If $d = 2n+4$ and X_n is not as above, then

iv) $-K_{X_n} = (n-2)H$, i.e. X_n in an n-dimensional Mukai variety of genus $g(X_n) = n+3$.

We gave an affirmative answer to the above questions by proving also some general results on varieties with one apparent double point. The results are summarized in the following theorems.

Theorem 6.3. *([AR02]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a variety with one apparent double point of degree $d \leq 2n+1$. Then X_n is either the twisted cubic in $\mathbb{P}^3$, $n = 1$, or, for $n \geq 2$, it is a smooth irreducible divisor of type $(2,1)$, $(0,2)$ or $(1,2)$ on the Segre variety $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$.*

Corollary 6.4. *([AR02]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth n-dimensional linearly normal variety of degree $d \leq 2n+1$ and with $Sec(X_n) = \mathbb{P}^{2n+1}$. Then X_n is a variety with one apparent double point and it is either the twisted cubic in $\mathbb{P}^3$, $n = 1$, or, for $n \geq 2$, a smooth irreducible divisor of type $(2,1)$, $(1,2)$ or $(0,2)$ on the Segre variety $\mathbb{P}^1 \times \mathbb{P}^n \subset \mathbb{P}^{2n+1}$. In particular X_n is a regular and simply connected variety.*

To prove the theorem, by Ionescu' s result quoted above, we needed to classify VOADP's having the structure described in proposition 6.2. The first step is an easy exercise.

Proposition 6.5. *Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth n-dimensional variety with one apparent double point which is a scroll in $\mathbb{P}^{n-1}$ over a smooth curve. Then X_n is $S(\underbrace{1,\dots,1}_{n-2},2,2)$, $n \geq 2$, or $S(\underbrace{1,\dots,1}_{n-1},3)$, $n \geq 1$. For $n \geq 2$ X_n is a divisor of type $(2,1)$ on $\mathbb{P}^1 \times \mathbb{P}^n$, the general one being $S(\underbrace{1,\dots,1}_{n-2},2,2)$. The scroll $S(2,2)$ can be realized also as a divisor of type $(0,2)$ on $\mathbb{P}^1 \times \mathbb{P}^2$.*

The following result was a fundamental step in the proof of theorem 6.3 and it shows once more the rigidity of varieties with one apparent double point. Now it is a special case of proposition 2.6 of [CMR01] (see 5.2).

Proposition 6.6. *Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth n-dimensional variety with one apparent double point which is a hyperquadric fibration over a smooth curve. Then X_n is a divisor of type $(0,2)$ or $(1,2)$ on the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^n$ into $\mathbb{P}^{2n+1}$.*

Some words about the proof of the corollary, for the details see [AR02]. Fist of all, applying once more Ionescu' s theorem and the classification in low degree one easily reduces to prove that a scroll over a curve or an hyperquadric fibration over a curve as in the statement of the corollary have $q(X_n) = h^1(\mathcal{O}_{X_n}) = 0$, i.e. that the base curve is rational. In fact in section 4 we saw that such a regular variety is a VOADP, see theorem 4.3.

We used the "double point formula" because in the cases we have to consider it has a particular nice form. Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth variety: the number of apparent double points of $X_n \subset \mathbb{P}^{2n+1}$, $\delta(X_n) \geq 0$, is given by the following formula:

$$d^2 - \sum_{k=0}^{n} \binom{2n+1}{k} s_{n-k}(X_n) H^k = 2\delta(X_n),$$

where $s_i(X_n)$ is the i^{th} Segre class of X_n and H is the class of a hyperplane section of X_n (the formula was known to Severi, see [Severi02]; see [PS76], [Laksov76] and [HR79] for proofs, discussions about the origins of this formula and generalizations). Since in our case we know the abstract structure of (X_n, H) we can make the calculation of the Segre classes $s_i(X_n)$ and compute their total degree $s_i(X_n) \cdot H^{n-i}$. In both cases this was done in [BB00].

If (X_n, H) is a hyperquadric fibration, X_n can be realized as a divisor in a $\mathbb{P}^n$-bundle $\mathbb{P}_C(\mathcal{F})$ over C of numerical type $2T + bF$, where T is the class of the tautological divisor and F the class of a fiber; by computing the Segre classes of X_n in $\mathbb{P}_C(\mathcal{F})$ the double point formula yields

$$\frac{d(d-4n+1)}{2} - g + 1 - 2(n^2-1)(q-1) = \delta(X_n) \geq 1,$$

where g is the sectional genus of X_n and $q = q(X_n)$. If $d \leq 2n+1$, then necessarily $q(X_n) = 0$ as desired.

In the second case $X_n \simeq \mathbb{P}_C(\mathcal{E})$ with $\mathcal{E}$ a locally free sheaf of rank n over the smooth curve C, $q(X_n) = g(C) = q$ and the embedding of $\mathbb{P}_C(\mathcal{E})$ is given by the complete linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. By computing the Segre classes of (X_n, H) in this case one obtains

$$d^2 - (2n+1)d - n(n+1)(q-1) = 2\delta(X_n) \geq 2,$$

which for $d \leq 2n+1$ gives $q = q(X_n) = 0$ as desired.

Let now briefly describe the classification of OADP varieties $X_n \subset \mathbb{P}^{2n+1}$ having degree d satisfying $2n+2 \leq d \leq 2n+4$. Once again this is possible because due to proposition 6.2 we know the abstract structure of X_n.

One easily sees that for $n \geq 4$ there are no VOADP's satisfying $2n+2 \leq d \leq 2n+4$ and having the structure of scroll over a rational surface, see [AR02]. Since we have completely classified VOADP's which are scroll or hyperquadric fibration over a curve, it remains to consider only the case of Mukai varieties.

Let us recall the following result proved in [CMR01]. It is obtained by applying the main theorem of [CMR01] to exclude the existence of Mukai 3-folds with one apparent double point and then by analyzing Mukai's classification to detect which Mukai manifolds of dimension at least 4 are VOADP's.

Proposition 6.7. *([CMR01]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a Mukai variety of degree $2n+4$, sectional genus $n+3$ and having one apparent double point. Then either $n = 4$ and $X_4 \subset \mathbb{P}^9$ is a linear section of the spinorial variety*

$S_{10} \subset \mathbb{P}^{15}$ described in example 4, or $n = 6$ and $X_6 \subset \mathbb{P}^{13}$ is the symplectic Grassmanian of degree 16 described in example 5.

Remark 7. The two Mukai manifold appearing in the proposition can be characterized as the unique varieties with one apparent double point having as general curve section a canonical curve, see [CMR01]. The last one is an equivalent definition of Mukai variety.

Let us summarize the results in the following theorem.

Theorem 6.8. *([AR02]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a VOADP having degree d satisfying $2n+2 \leq d \leq 2n+4$. Then either $n = 3$, $d = 8$ and X_3 is the 3-fold described in example 3 or $n = 4$, $d = 12$ and X_4 is a linear section of the spinorial variety $S_{10} \subset \mathbb{P}^{15}$ described in example 4 or $n = 6$, $d = 16$ and X_6 is the symplectic Grassmannian described in example 5.*

The positive answer to the question posed by Zak and the complete classification of linearly normal varieties $X_n \subset \mathbb{P}^{2n+1}$ of degree $d \leq 2n+1$ inspired the following generalization given by Ionescu combining his theorem cited above and some ad hoc methods coming from the classification theory of varieties of small invariants (adjunction map). We cite Ionescu's result for completeness. We refer to [Ionescu00] for the proof and the complete description of varieties of degree $d \leq r$ in $\mathbb{P}^r$.

Theorem 6.9. *([Ionescu00]) Let $X_n \subset \mathbb{P}^r$ be a smooth irreducible non-degenerated variety of degree $d \leq r$. Then X_n is a simply connected variety and all such varieties can be completely and explicitly described.*

By the theorem of Barth and Larsen, see [BL72], we know that if a smooth non-degenerated variety $X_n \subset \mathbb{P}^r$ is such that $r \leq 2n-1$, then X_n is simply connected without restrictions on the degree. If $r = 2n$ and if $d \leq 2n$, then the above result can be proved exactly as in [AR02] where the case $r = 2n+1$, $d \leq 2n+1$ was treated. Hence Ionescu' s theorem is a non-straightforward generalization of corollary 6.4 but we would like to remark that the generalization is completely due to him and that a non trivial and interesting part consists of the complete description of varieties $X_n \subset \mathbb{P}^r$ of degree $d \leq r$.

7. Geometric characterization of VOADP's: Bronowski theorem

Let $X_n \subset \mathbb{P}^N$, $N \geq (k+1)n + k$, $k \geq 1$ be a non-degenerate smooth variety of dimension n. Clearly $dim(S^k X_n) \leq (k+1)n + k$. We show that the property $dim(S^k X_n) = (k+1)n + k$ can be characterized by means of the projection of X_n from a general $k(n+1)-1$-plane L tangent to X_n in k general points $x_1, \dots, x_k$ onto a disjoint $\mathbb{P}^{N-k(n+1)}$ by using Terracini's lemma. It is easy to see that $\dim(S^{k-1} X_n) = k(n+1) - 1$ and that $L = < T_{x_1}(X_n), \dots, T_{x_k}(X_n) >$.

Proposition 7.1. *Let $X_n \subset \mathbb{P}^N$, $N \geq (k+1)n+k$, be a smooth n-dimensional non-degenerate variety. Let $\pi_L : X_n \dashrightarrow \mathbb{P}^{N-1-s_{k-1}(X)}$ be the projection from the linear space $L =< T_{x_1}(X_n), \dots, T_{x_k}(X_n) >$ of dimension $s_{k-1}(X) = \dim(S^{k-1}X)$ tangent to X_n at the general points x_i, $i = 1, \dots, k$, onto a disjoint $\mathbb{P}^{N-1-s_{k-1}(X)}$. Then π_L is a generically finite rational map if and only if $\dim(S^k X_n) = (k+1)n+k$ (which implies $s_{k-1}(X) = kn+k-1$).*

Proof. It follows immediately from Terracini's lemma that, for every $i \geq 1$, $\dim(S^i X) - \dim(S^{i-1}X) \leq n+1$ and that, for every $i \geq 2$, $\dim(S^i X) - \dim(S^{i-1}X) \leq \dim(S^{i-1}X) - \dim(S^{i-2}X)$. Then $\dim(S^k X) = (k+1)n+k$ if and only if $\dim(S^k X) - \dim(S^{k-1}X) = n+1$.

By definition of projection, letting $Y = \pi_L(X)$, we have $S(X, L) = S(Y, L)$ and $< Y > \cap L = \emptyset$. Let us calculate the dimensions in two ways by means of Terracini's lemma: $\dim(S^k X) = \dim(< T_x X, L >) = \dim(S(X, L)) = \dim(S(Y, L)) = \dim(Y) + \dim(L) + 1$, i.e. $\dim(Y) = \dim(S^k X) - \dim(S^{k-1}X) - 1$. Thus π_L is generically finite if and only if $\dim(Y) = n$, if and only if $\dim(S^k X) - \dim(S^{k-1}X) = n+1$, if and only if $\dim(S^k X) = (k+1)n+k$. □

In [Bronowski33] Bronowski claimed that varieties $X_n \subset \mathbb{P}^{(k+1)n+k}$, $k \geq 1$, with $S^k X_n = \mathbb{P}^{(k+1)n+k}$ and such that through the general point of $\mathbb{P}^{(k+1)n+k}$ there passes a unique $(k+1)$-secant $\mathbb{P}^k$ can be characterized by the birationality of π_L. Since we were unable to understand his proof and since many of the results contained in that paper are not totally correct (see also [Zak99]), we thought it were safer for the moment to name the above problem *Bronowski' s conjecture.*

The interesting implication for $k = 1$ of Bronowski' s conjecture was recently obtained in [CMR01] as an application of the method of degeneration of projection and more precisely it was obtained a bound for the degree of $\pi_L = \pi_{T_p(X_n)}$. The result is the following.

Theorem 7.2. *([CMR01]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth, irreducible, projective variety of dimension n whose number of apparent double points is $\delta(X_n) > 0$. Let $\alpha(X)$ be the degree of the projection of X to $\mathbb{P}^n$ from the tangent $\mathbb{P}^n$ to X_n at its general point. Then $0 < \alpha(X) \leq \delta(X)$.*

As a consequence we have the following very interesting and important result.

Corollary 7.3. *([CMR01]) Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth, irreducible, projective OADP-variety of dimension n. Then the projection of X_n to $\mathbb{P}^n$ from the tangent $\mathbb{P}^n$ to X_n at its general point is birational. In particular X_n is a rational variety.*

A special case of this corollary was obtained before in [AR02]. We quote it only to explain how the stronger condition of having quadratic entry locus for every $p \in SX_n \setminus X_n$ sometimes simplifies the arguments.

Proposition 7.4. ([AR02]) *Let $X_n \subset \mathbb{P}^{2n+1}$ be a VOADP having quadratic entry locus for every $q \in \mathbb{P}^{2n+1} \setminus X_n$. Then the projection from a general tangent space to X_n onto a disjoint $\mathbb{P}^n$ is birational. Moreover, if X_n is also scheme theoretically defined by quadric hypersurfaces, then it is an arithmetically Cohen Macaulay variety.*

It is very interesting to note how a geometrical condition as the above one imposes very strong restrictions to X_n.

It would be interesting to know which OADP varieties satisfy the condition of having quadratic entry locus for every $q \in \mathbb{P}^{2n+1} \setminus X_n$. It could be the case that OADP varieties are the scheme theoretic intersection of the quadrics through them because "essentially" the unique way of realizing geometrically a *congruence of ((multi)-secant) lines* to a smooth variety is to put them together as fibers of the rational map given by quadrics through it; so varieties with one apparent double point should also naturally carry a special subhomaloidal system of quadrics: the quadrics through them.

The following result should be compared with corollary 2.11 of [Vermeire01], which is an improvement of corollary 2 of [BEL91] under condition K_d only if $Sec_d(X_n) \subsetneqq \mathbb{P}^{2n+1}$. It should be noticed that example 3 shows that for a variety which is the scheme theoretic intersection of the quadrics through it, condition K_2 is only sufficient to have quadratic entry locus for every $q \in \mathbb{P}^{2n+1} \setminus X_n$.

Corollary 7.5. ([AR02]) *Let $X_n \subset \mathbb{P}^{2n+1}$ be a smooth n-dimensional variety scheme theoretically defined by quadratic forms satisfying condition K_2 and such that $Sec(X_n) = \mathbb{P}^{2n+1}$. Then the projection of X_n from a general tangent n-plane is birational. In particular X_n is a rational arithmetically Cohen-Macaulay variety with one apparent double point.*

To our knowledge projection from the tangent space, or more precisely by the linear space generated by tangent spaces at various points, was studied classically only in the paper of Bronowski cited above. So to our knowledge for the moment only for $k = 1$ one implication of Bronowski' s conjecture is known. One can easily see that for $n = 1$ the only curves $C \subset \mathbb{P}^{2k+1}$ such that through the general point there passes a unique $(k+1)$-secant $\mathbb{P}^k$ are rational normal curves of degree $2k+1$. In fact such a curve (or variety) has to be linearly normal and with its $(k+1)^{th}$ symmetric product rational. To our knowledge the classification of such $X_n \subset \mathbb{P}^{(k+1)n+k}$ with $n \geq 2$, $k \geq 2$ is not known.

8. Some applications of the geometry of VOADP's

In this section we show how the geometric properties of VOADP's can be used to prove some interesting results.

Let us begin with a general discussion on the rationality of smooth cubic hypersurfaces. Let $X^3_m \subset \mathbb{P}^{m+1}$ be a smooth cubic hypersurface. If $m = 1$, then the smooth cubic curve $X_1^3 \subset \mathbb{P}^2$ is not rational and neither unirational. For $m \geq 2$, Max Nöether proved that a smooth cubic hypersurface $X_m^3 \subset \mathbb{P}^{m+1}$ is unirational over the field of definition of a line contained in it. Recently this was generalized by Kollár, see [Kollár00], who showed that a cubic hypersurface X is unirational over a field k if and only if $X(k) \neq \emptyset$.

If k is algebraically closed, then it is well known that a smooth cubic surface over k is rational. An easy proof consists in proving that $X_2^3 \subset \mathbb{P}^3$ contains two skew lines L_1 and L_2. Then we can define a rational map $\psi : L_1 \times L_2 \dashrightarrow X_2^3$ by associating to the point (p, q) the third point of intersection between the line $< p, q >$ and X_2^3. Since through every point of $\mathbb{P}^3 \setminus (L_1 \cup L_2)$ there passes only one line intersecting L_1 and L_2, the map ψ is birational and hence X_2^3 is rational. Over an arbitrary field k, if X_2^3 contains two lines L_1 and L_2 defined over k, then ψ would be defined over k and X_2^3 would be rational over k.

Shepherd-Barron in [Shepherd-Barron92] gave a geometrical proof that a smooth quintic Del Pezzo surface X_2^5 defined over an arbitrary field k is k-rational (a result known to Enriques, see [Enriques1897], and also proved by Swinnerton-Dyer in [Swinnerton-Dyer72]), by using the fact that such a surface projects birationally onto a cubic surface with two skew lines defined over k. His method is based on the study of the subhomaloidal systems of quadrics through X_2^5 and closely related to some results reported here. Let us briefly recall his proof. By proposition 4.2 we know that X_2^5 is the scheme-theoretic intersection of 5 quadrics giving, after blowing-up X_2^5, a morphism $\phi : Bl_{X_2^5}(\mathbb{P}^5) \to \mathbb{P}^4$ whose general fiber is a secant line to X_2^5. Since ϕ is defined over a k, a general secant line $l \subset \mathbb{P}^5$ cuts X_2^5 in two points p_1 and p_2, "defined over k". The projection of X_2^5 from l gives a birational morphism of $X_2^5 \setminus \{p_1, p_2\}$ onto the image which extends to an isomorphism between the blow-up of X_2^5 in p_1 and p_2 and a cubic surface $X_2^3 \subset \mathbb{P}^3$. Since p_1 and p_2 are "defined over k", the resulting skew lines on X_2^3 are "defined over k". Since X_2^3 is then k-rational, X_2^5 will be k-rational.

A fundamental theorem of Clemens and Griffiths, [CG72], shows that over the complex field a smooth cubic hypersurface in $\mathbb{P}^4$ is not rational. On the other hand, in every space of odd dimension, let us say $\mathbb{P}^{2n+1}$, it is possible to construct examples of smooth cubic hypersurfaces by taking a smooth cubic hypersurface containing a pair of skew $\mathbb{P}^n$, generalizing the construction given above for $n = 1$. One immediately realizes that such hypersurfaces are special and special when n increases. A pair of skew $\mathbb{P}^n$ is an example of reducible smooth VOADP and the same construction can be generalized to smooth cubics hypersurfaces of $\mathbb{P}^{2n+1}$ containing a smooth irreducible VOADP, see [Russo00]. Let us recall the details and make some historical remarks.

Let X_{2n} be a (smooth) cubic hypersurface in $\mathbb{P}^{2n+1}$ containing a (smooth) variety W_n. We can define a rational map

$$\psi : W_n^{(2)} \dashrightarrow X_n,$$

which to the distinct points $p, q \in W_n$ associates the third point of intersection of the line $< p, q >$ with X.

Taking a general point $x \in X_{2n} \setminus W_n$, we see that x is in the image of ψ when it lies on a secant line to W_n. The number of preimages of x by ψ is equal to the number of secant lines going through x. Thus ψ is finite and dominant if the generic projection of W_n from $x \in X_{2n}$ has a finite (non zero) number of apparent double points.

In particular if W_n has only one apparent double point, then through the general point of X_{2n} there passes only a secant line to W_n, since the fundamental locus of the birational map $\beta_1^{-1} : \mathbb{P}^{2n+1} \dashrightarrow SB(W_n)$ has codimension at least 2. By proposition 2.2 we also know that $W_n^{(2)}$ is rational. Hence if $\delta(W_n) = 1$, X_{2n} is a rational smooth cubic hypersurface.

When the variety W_n is defined by quadratic equations giving a special homaloidal system on $\mathbb{P}^{2n+1}$ whose general fiber is a secant line to W_n, then the above map is simply the restriction to X_n of the map $\phi_{|H^0(\mathcal{I}_{W_n}(2)|} : \mathbb{P}^{2n+1} \dashrightarrow Z \subseteq \mathbb{P}(H^0(\mathcal{I}_{W_n}(2))$, inducing a birational isomorphism between X_{2n} and Z. It is clear that in this case Z is a rational variety birational to $W_n^{(2)}$.

Then we have the following classical result.

Proposition 8.1. *Let $X_{2n} \subset \mathbb{P}^{2n+1}$ be a smooth cubic hypersurface containing a smooth variety W_n having one apparent double point. Then X_{2n} is a rational variety.*

Remark 8. The above idea for proving rationality of these cubic hypersurfaces in $\mathbb{P}^{2n+1}$ is classical, see [Roth55], pg. 101.

Morin erroneously claimed that a general smooth cubic hypersurface in $\mathbb{P}^5$ contains a rational normal scroll of degree 4 and hence it is rational (see [Morin40]). Fano then proved that if a smooth cubic hypersurface $X_4 \subset \mathbb{P}^5$ contains a rational normal scroll of degree 4, then it contains also a Del Pezzo surface of degree 5 (and viceversa) and that the family of such surfaces contained in X_4 has dimension 2, respectively 5, contrary to the "expected" dimension 1, respectively 4.

So the locus of smooth cubic hypersurfaces in $\mathbb{P}^5$ containing smooth OADP surfaces has codimension one in the moduli space of smooth cubic hypersurfaces in $\mathbb{P}^5$ (see [Fano42], [Roth55], pg. 101, [Tregub93] and [Beauville99], propositon 9.2 a) and b)). Let us recall that the locus of smooth cubic hypersurfaces in $\mathbb{P}^5$ containing a pair of skew planes has codimension two in the corresponding moduli space.

Fano also constructs explicit birational maps from a smooth cubic hypersurface $X \subset \mathbb{P}^5$ as above to $\mathbb{P}^4$, considering (the restriction to X of) the linear systems of quadrics through the surfaces S (see [Fano42] and [Tregub84]).

Tregub describes a different codimension two locus in the moduli space of smooth cubic hypersurfaces in $\mathbb{P}^5$, which parametrizes cubic hypersurfaces containing a Veronese surface and a plane intersecting in three points and shows that each of these cubic hypersurfaces is rational (see [Tregub93]).

Let Π be the divisor of smooth cubic hypersurfaces of $\mathbb{P}^5$ parametrizing smooth cubic hypersurfaces containing a plane. Recently Hasset constructed a countably infinite collection of divisors in Π, each of which parametrizes rational cubic hypersurfaces (see [Hasset99]).

9. Some questions

In this section we would like to discuss some open problems and try to relate the results presented in the previous sections. We will present the problems, somehow, in a decreasing order of difficulty according to our taste.

Question 1. Classify OADP varieties of dimension $n \geq 4$.

Question 2. Let X_n be an arbitrary smooth variety such that $X_n^{(2)}$ is a rational variety. Is then X rational? (for $n = 1$ it is easy to see that it is true; for $n = 2$ it is a theorem of Mattuck).

Question 3. Construct new examples of VOADP's.

Question 4. For $n \geq 4$ classify all the VOADP's with quadratic entry locus for every $q \in \mathbb{P}^{2n+1} \setminus X_n$.

Question 5. Classify all varieties $X_n \subset \mathbb{P}^N$, $N \leq 2n$ (or $\dim Sec(X_n) \leq 2n$) with quadratic entry locus for every $q \in Sec(X_n) \setminus X_n$.

Question 6. Prove (or disprove for $k \geq 2$, $n \geq 2$!) Bronowski' s conjecture. Prove for $k = 1$ the open implication of the conjecture.

Question 7. Does there exist a bound for the degree d of a VOADP?

Bronowski' s theorem seems to suggest the bound $d \leq 2^n + 1$.

Question 8. Is an OADP variety arithmetically Cohen-Macaulay?

Question 9. Is it true that an OADP variety does not admit a proper trisecant line? If it were true what does it imply for the focal locus of the family of secant lines?

Question 10. Is an OADP variety the scheme theoretic intersection of the quadrics through it?

10. Appendix-Characterizations of the Veronese surface

Indicheremo con $X_n \subset \mathbb{P}^N$ una varietá irriducibile non degenerata di dimensione n definita su un corpo algebricamente chiuso k di caratteristica zero. La varietá X puo' avere singolaritá arbitrarie.

E' evidente dalla definizione che la superficie di Veronese $\nu_2(\mathbb{P}^2)$ ha varietá delle secanti una ipersuperficie (cubica) di $\mathbb{P}^5$. Anche le superficie di $\mathbb{P}^5$ che sono coni di vertice un punto hanno varietá delle secanti di dimensione 4 (facile applicazione del lemma di Terracini). In quanto segue vogliamo caratterizzare la superficie di Veronese come la unica superficie irriducibile, non cono, che ha varietá delle secanti di dimensione 4, un famoso teorema di Francesco Severi, [Severi01].

Sia $X \subset \mathbb{P}^5$ una superficie irriducibile tale che SX, la varietá delle secanti a X, non esaurisca tutto lo spazio. Ricordiamo che SX é irriducibile e che $\dim(SX) \leq \dim(X)+1 = 3$ implica che lo span lineare di X, $< X >$, abbia dimensione minore o uguale a 3. Allora, essendo $X \subset \mathbb{P}^5$ non degenerata nelle nostre ipotesi SX é una ipersuperficie irriducibile.

Consideriamo $(SX)^* \subset (\mathbb{P}^5)^*$ e sia $H \in (SX)_{reg}$, l' insieme dei punti non singolari. Allora per la riflessivitá, abbiamo

$$C_H(SX) = \overline{\{q \in (SX)_{reg} \quad : T_qSX = H\}} = (T_H(SX)^*)^* \simeq \mathbb{P}^{4-\dim((SX)^*)}.$$

Proposition 10.1. *Sia $X \subset \mathbb{P}^5$ una superficie non cono con $\dim(SX) = 4$. Sia $H = T_pSX =< T_xX, T_yX >$ con $p \in SX$ punto generale un iperpiano generale bitangente a X in x e y. Allora $C_H(SX) = \mathbb{P}^2$, $C_H(SX) \cap X = D$ é di dimensione 1 e la parte 1 dimensionale di D é una conica irriducibile passante per x e y.*

Proof. Sia $< x, y >$ come sopra una secante generale passante per p. Allora per il lemma di Terracini abbiamo $T_pSX =< T_xX, T_yX >$. Pertanto per $p' \in < x, y >$ punto generale $T_{p'}SX =< T_xX, T_yX >= T_pSX$ e quindi $< x, y > \subset C_H(SX)$. Sia $D = C_H(X) \cap X$ come sopra. Considerando un punto $q \in C_H(SX)$ generale, esiste una secante a D passante per q, i.e. $SD = C_H(SX)$ e per q passano infinite secanti essendo un punto generale di SX. Segue che $C_H(SX) = \mathbb{P}^2$ e che, essendo la generica secante di D non trisecante, la parte 1-dimensionale di D é una conica. Per mostrare che questa conica é irriducibile é sufficiente mostrare la seguente proposizione che finirá la dimostrazione. □

Proposition 10.2. *Sia $X_n \subset \mathbb{P}^N$, $N \geq n+3$, una varietá non degenerata con $\dim(SX) = n+2$, $n = \dim(X)$, e coperta da rette. Allora X é un cono.*

Proof. Sia $L \subset X$ una retta generale. La varietá irriducibile $S(L, X)$ é un cono il cui vertice contiene L e pertanto contiene propriamente X (altrimenti X sarebbe uno spazio lineare). Quindi $n+2 \geq \dim(S(L, X)) \geq n+1$. Se per la retta generale della famiglia avessimo $\dim(S(L, X)) = n+2$, allora $S(L, X) = SX$; indicato con $Vert(SX)$, il vertice di SX, da $Vert(SX) \supseteq L$ e dalla generalitá di L deduciamo $Vert(SX) \supset X$ e X sarebbe degenerata.

Concludiamo quindi che per L retta generale abbiamo $\dim(S(L,X)) = n+1$ e per il lemma di Terracini dato un punto generale $x \in X$ vale $T_xX \cap L = p_x$. Vogliamo mostrare che al variare di x il punto p_x non varia e che pertanto $p = p_x \subseteq \cap_{x\in X} T_xX$, i.e. X é un cono.

Consideriamo il cono di vertice L, con L retta generale, $S(L,SX) \supseteq SX$. Ragionando come prima abbiamo $\dim(S(L,SX)) = n+3$ (altrimenti $Vert(SX) \supseteq L$ e per la generalitá di L, lo spazio lineare $Vert(SX)$ conterrebbe X). Pertanto $<< T_xX, T_yX >, L >= \mathbb{P}^{n+3}$ per $x, y \in X$ generali, i.e. $p_{x,y} = L\cap < T_xX, T_yX > \supseteq L \cap T_xX = p_x$. Per la simmetria di x e y nella espressione otteniamo che $p_x = p_{x,y} = p_y = p$ e pertanto l' asserto. □

Proposition 10.3. *Sia $X \subset \mathbb{P}^N$, $N \geq 5$, una superficie non degenerata, che non sia un cono, e avente* $\dim(SX) = 4$. *Allora X contiene una famiglia irriducibile 2-dimensionale di coniche liscie che sono linearmente equivalenti tra loro e formano un sistema lineare completo di divisori di Caritier di dimensione 2. Inoltre se C é una tale conica, allora $C^2 = 1$. Il sistema lineare associato $|C|$ definisce un isomorfismo di X con $\mathbb{P}^2$ il cui inverso realizza X come una superficie proiettivamente equivalente alla superficie di Veronese $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$*

Proof. A meno di proiettare X in $\mathbb{P}^5$ possiamo supporre $N = 5$. Infatti se proiettiamo e otteniamo una superficie liscia, allora la superficie iniziale é liscia e isomorfa alla proiettata (un morfismo finito e birazionale con immagine liscia é un isomorfismo). Allora se la superficie ottenuta in $\mathbb{P}^5$ é lisica e non ottenuta per proiezione avremo necessariamente $N = 5$.

Dalla proposizione 10.1, segue dati due punti generali esiste una conica della famiglia costruita che li unisce e pertanto che le coniche costruite a partire da $C_H(SX)$ sono almeno una famiglia di dimensione 2. Mostriamo che due coniche generali che passano per un punto generale $x \in X$ sono divisori di Cartier linearmente equivalenti. Da questo dedurremo che, essendo due punti generali uniti da una conica della famiglia, due coniche generali della famiglia sono divisori di Cartier linearmente equivalenti. Consideriamo la proiezione dal piano tangente T_xX nel punto generale $x \in X$. Questo definisce una applicazione razionale $\pi_x : X \dashrightarrow \mathbb{P}^2$. La dimensione della immagine $\pi_x(X) = Y$ é uno: esiste una conica che passa per x e per un punto generale y di X che é contratta dalla mappa (la superficie X é non degenerata).

Affermiamo che Y é una conica liscia e che $(T_xX \cap X)_{red} = x$, i.e. $\pi_x : X \setminus x \to Y$ é un morfismo su una conica liscia. La prima affermazione é triviale, perche' due punti generali di X sono connessi da una conica e quindi l' immagine per proiezione sara' necessariamente la proiezione di una conica generale. Proiettando una tale conica generale potremmo ottenere solo una retta o una conica irriducibile contenute nell' immagine. Il primo caso é escluso perche' X é non degenerata.

Mostriamo che $(T_xX \cap X)_{red} = x$. Infatti se per un tale x generale avessimo $q_x \in T_xX \cap X$, $q_x \neq x$, allora la retta $L_x =< q_x, x >$ sarebbe contratta a un punto da $\pi_{x'}$ con $x' \in X$ punto generale perche' avremmo $\pi_{x'}(L_x) \subseteq \pi_{x'}(T_x X \cap X) \subseteq T_{\pi_{x'}(x)} Y' \cap Y' = \pi_{x'}(x)$, i.e. $L_x \cap T_{x'} X = p_{x'}$ (osserviamo che possiamo supporre che $\pi_{x'}$ sia definita anche in q_x perche' $\cap_{x \in X} T_x X = \emptyset$ e inoltre la retta $< x, x' >$ non é tangente) e non essendo X un cono il punto deve variare con $x' \in X$ punto generale. Allora si avrebbe che $S(L_x, SX) = SX$ e che quindi il vertice di SX, $Vert(SX)$, conterrebbe un punto generale di X e quindi tutto X.

Ritorniamo allo studio della mappa razionale π_x che sappiamo ora essere definita su $X \setminus x$. Se $y \in X \setminus x$ é un punto generale, considerando la retta tangente l' alla conica $Y = \pi_x(X)$ nel punto $\pi_x(y) = y'$, otteniamo che l' iperpiano di $\mathbb{P}^5$, $< T_x X, l' >= H_{x,y}$ contiene anche $T_y X$ e quindi per il lemma di Terracini $H_{x,y} =< T_x X, T_y X >= T_p SX$ con $p \in < x, y >$ punto generale. Allora sappiamo che essendo $x \in X$ liscio il luogo di contatto $C_{H_{x,y}}(SX) \cap X$ in un intorno di x é proprio la conica $C_{x,y}$ che unisce x e y. Da questo deduciamo che in un intorno di x, l' iperpiano $H_{x,y}$ é tangente alla superficie X lungo la conica $C_{x,y}$ esattamente, i.e. in un intorno di x, $H_{x,y} \cap X = 2C_{x,y}$ perche' la fibra della mappa π_x sul punto y' é liscia in un intorno di x e quindi é uguale a $C_{x,y}$ in un intorno di X. Da questo segue che $\pi_x^{-1}(y') = C_{x,y}$ fuori da x perche' le altre eventuali componenti dovrebbero essere contenute in $H_{x,y}$ e passare per x per essere contratte; quindi due coniche generali per x sono linearmente equivalenti e sono divisori di Cartier su X. Quindi la chiusura di una fibra qualsiasi di π_x é una conica che deve essere liscia perche' da $(T_x X \cap X)_{red} = x$ deduciamo che per x non passano rette. Da questo segue facilmente $H_{x,y} = 2C_{x,y}$ e quindi $C_{x,y}^2 = 1$.

Una superficie irriducibile di $\mathbb{P}^N$, $N \geq 4$ non puó contenere una famiglia di dimensione 3 di coniche, perché altrimenti per un punto generale passerebbero infinito a 2 coniche e proiettando otterremmo una superficie irriducibile con infinito a due rette che sarebbe necessariamente un piano (altrimenti avremmo $N = 3$).

Pertanto le coniche sono divisori di Cartier che fanno parte di un sistema lineare completo. Sia C una tale conica. Da $H = 2C$ deduciamo che C é un divisore ampio. Se $|C|$ avesse un punto base, non essendo X un cono, proiettando dal punto base avremmo una superficie che contiene infinito a 2 rette e X sarebbe degenerata in $\mathbb{P}^5$.

Considerata ora la mappa $\phi_{|C|} : X \to \mathbb{P}^2$, abbiamo che ϕ é birazionale e finita e quindi un isomorfismo e pertanto X é la superficie di Veronese di $\mathbb{P}^5$ perche' contiene coniche. $\square$

Incidentalmente osserviamo che abbiamo dimostrato il seguente fatto apparentemente piu' generale.

Proposition 10.4. *Sia $X \subset \mathbb{P}^N$, $N \geq 5$, una superficie non degenerata contenente una famiglia 2 dimensionale di coniche liscie. Allora X é proiettivamente equivalente alla superficie di Veronese di $\mathbb{P}^5$.*

Proof. Per due punti generali passa almeno una conica della famiglia e quindi $\dim(SX) = 4$. La conclusione segue applicando la proposizione precedente e osservando che una X come nella proposizione necessariamente non é un cono. Osserviamo che possiamo supporre che la famiglia sia solo di curve e a posteriori risulta essere di divisori di Cartier. □

Siamo finalmente pronti per enunciare le varie caratterizzazioni della superficie di Veronese di $\mathbb{P}^5$ come l' unica superficie irriducibile non cono avente alcune proprietá geometriche notevoli. Ricordiamo che, per il lemma di Terracini e per proprietá elementari della varietá delle secanti, per $X \subset \mathbb{P}^N$, $N \geq 5$ superficie non degenerata sono equivalenti $\dim(SX) = 4$ e la proprietá che dati due punti generali $x, y \in X$, $T_xX \cap T_yX \neq \emptyset$. Chiaramente un cono gode della ultima proprietá e quinda ha sempre varietá delle secanti di dimensione 4. Abbiamo quindi dimostrato il seguente classico teorema.

Theorem 10.5. *Sia $X \subset \mathbb{P}^N$, $N \geq 5$, una superficie non degenerata che non sia un cono. Allora X é proiettivamente equivalente alla superficie di Veronese di $\mathbb{P}^5$, $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ se e solo se una delle seguenti condizioni é soddisfatta:*

i) *dati due punti generali $x, y \in X$, $T_xX \cap T_yX \neq \emptyset$ (*[del Pezzo1887]*;*
ii) $\dim(SX) = 4$ *(*[Severi01]*;*
iii) *X contiene una famiglia di dimensione 2 di coniche (???).*

Ricordiamo che se $X \subset \mathbb{P}^N$ é una varietá non degenerata, allora $\dim(SX) \geq \dim(X) + 1 = n + 1$ e che se vale l' uguale abbiamo $N = n + 1$. Il caso successivo é quello delle varietá per cui $\dim(SX) = n+2$. Se $N = n+2$ non ci sono restrizioni e esistono infinite varietá di questo tipo. Se $N > n+2$ la risposta é contenuta nel seguente risultato.

Corollary 10.6. *(*[Edwards29]*) Sia $X_n \subset \mathbb{P}^N$, $N \geq n + 3$, una varietá irriducibile non degenerata con* $\dim(SX) = n + 2$, $n = \dim(X)$. *Allora o X_n é un cono su una curva oppure $N = n + 3$ e X_n é un cono su una superficie di Veronese di $\mathbb{P}^5$. Equivalentemente, $X_n \subset \mathbb{P}^N$, $N \geq n + 3$, varietá irriducibile non degenerata é un cono su una curva o un cono su una superficie di Veronese se e solo se contiene infinito a due quadriche di dimensione $n - 1$.*

Proof. Sia H un $\mathbb{P}^{N-n+2}$ generale in $\mathbb{P}^N$. Allora $X_n \cap H = X' \subset \mathbb{P}^{N-n+2}$ é una superficie irriducibile non degenerata in $\mathbb{P}^{N-n+2}$ con $N - n + 2 \geq n + 3 - n + 2 = 5$ e avente $\dim(SX') = 4$ e quindi X' é un cono o la superficie di Veronese di $\mathbb{P}^5$ e in quest' ultimo caso abbiamo $N = n + 3$. Se X' é un cono, allora X risulta essere un cono su una curva. Supponiamo $X' = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. Allora se $H' \subset \mathbb{P}^{n+3}$ é un $\mathbb{P}^6$ generale, $H' \cap X_n = X'' \subset \mathbb{P}^6$ é una varietá di dimensione 3 irriducibile il cui entry locus generico é una quadrica perche' l' entry locus della sua sezione iperpiana generica é una conica. Quindi X'' é coperto da rette e per la proposizione 10.2 otteniamo che X'' é un cono di vertice un punto sulla superficie di Veronese di $\mathbb{P}^5$, da cui segue che X é un cono sulla superficie di Veronese di $\mathbb{P}^5$. Osserviamo che questa ultima parte mostra che la superficie di Veronese ammette solo estensioni triviali, i.e. le uniche varietá di cui puo' essere sezione lineare generale sono i coni sulla superficie stessa. Una varietá che contiene infinito a due quadriche di dimensione $n-1$ é tale che per due punti generali di X_n passi una tale quadrica e quindi necessariamente $\dim(SX) \leq n+2$ da cui segue $\dim(SX) = n+2$. □

Questi risultati furono successivamente generalizzati in una altra direzione da Scorza nei lavori [Scorza08] e [Scorza09]. In [Zak93] si trovano le sistematizzazioni definitive dei risultati di Severi e Scorza, almeno nel caso liscio. Per completezza citiamo il seguente.

Theorem 10.7. *(*[Zak93]*, capitolo VI, teorema 2.1). Sia $X \subset \mathbb{P}^N$ una varietá liscia e non degenerata di dimensione n tale che* $\dim(SX) \leq 2n$. *Allora $N \leq \frac{n(n+3)}{2}$ e l' uguaglianza vale se e solo se $X = \nu_2(\mathbb{P}^n) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$ é la varietá di Veronese.*

Notevole é anche la caratterizzazione data da Ran della superficie di Veronese come la unica superficie liscia di $\mathbb{P}^5$ non contenuta in un 3-fold irriducibile che sia non singolare lungo la superficie ([Ran91]).

Remark 9. Varie dimostrazioni del teorema di Severi furono date successivamente e negli ultimi anni, estendendo il risultato anche a caratteristica positiva. Affermiamo che la dimostrazione fornita in queste note é la piu' "semplice" tra quelle conosciute all' autore nell' ipotesi di caratteristica zero e superficie irriducibile non necessariamente liscia. Possiamo affermare che la unica parte originale, se ne esiste una, sia la dimostrazione, ottenuta proiettando dal piano tangente che le coniche siano divisori di Cartier linearmente equivalenti sulla superficie a priori singolare X.

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Departamento de Matematica, Universidade Federal de Pernambuco, Cidade Universitaria, 50670-901 Recife–PE (Brasil)

E-mail address: `frusso@dmat.ufpe.br`