Singular t-ples of tensors and their Geometry.

Applied Algebraic Geometry
online
Bologna Ferrara Firenze Siena Trento
November 12, 2020
abridged version with no animations

Giorgio Ottaviani

University of Florence, Italy

Plan of the talk

1. The distance function in tensor spaces, how to define Tensor Eigenvectors.
2. Singular t-ples, generating function for the number of singular t-ples.
3. The critical space.
4. Tensors with eigenvectors in a given subspace.

How to define Eigenvectors ?

So, how to define Eigenvectors of a Tensor ?
Start with the symmetric matrix case, let $A \in \mathrm{S}ym^2 V$, V vector space.

Formula $Av = \lambda v$ requires linear map $A\colon V \to V^\vee$, we need a metric to identify $V \simeq V^\vee$. Which properties we want to generalize to define eigenvectors of $A \in \mathrm{S}ym^d V$?

There is an alternative geometric approach to Eigenvectors, which generalizes fairly generalizes to Tensors.

Symmetric Tensor space as a metric space

Let V be a real vector space of dimension $n+1$, let q be a nondegenerate symmetric bilinear positive definite form on V. The form q induces a nondegenerate symmetric bilinear positive definite form on $\mathrm{S}ym^d V$, by defining it on decomposable elements.

$$q(v^d) = q(v)^d$$

and then extending by linearity. Several terminologies: Frobenius, Bombieri-Weyl, Kostlan.

In quantum information this metric gives a continuous measure of entanglement, versus the discrete measure given by tensor rank.

Harmonic Decomposition and Orthogonally invariant metrics

Harmonic Decomposition

$$\mathrm{Sym}^d V = \oplus_{i=0}^{\lfloor d/2 \rfloor} q^i H_{d-2i}$$

where H_k is the space of homogeneous harmonic polynomials of degree k is the decomposition in $SO(V,q)$-irredicible summands.

Proposition

All $SO(V,q)$-invariant metrics are a linear combination of the Bombieri-Weyl metric induced on each summand.

Affine definition of Eigenvectors

First definition of eigenvectors, affine

Let $f \in \mathrm{S}ym^d V$. The critical points of the distance function $q(f - v^d)$ with $v \in V$ are the eigenvectors of f.

This is wonderful for real symmetric matrices, which have only real eigenvectors and real eigenvalues, but it opens a problem in tensor setting since there are complex critical values and it can exist isotropic eigenvectors for the Euclidean metric. Luckily the set of tensors with isotropic eigenvectors has measure zero.

Projective definition of Eigenvectors

Second definition of eigenvectors, projective

f defines a function on the unit sphere $S^n \subset V$. the critical points of this function are the eigenvectors.

Coordinate definition

$$f(x^{d-1}) = \lambda x$$

In invariant way

$$f(x, \dots, x, -) = \lambda q(x, -)$$

This is coherent with classical definition for matrices, indeed one minimizes the function

$$\frac{A(x^2)}{|x|^2}$$

$$\frac{A(x^2)}{\|x\|^2}$$

Here the eigenvalues are the values of the function on the critical points, in other words, since we evaluate on the sphere, are the values of $A(x^2)$ at the critical points on the sphere (note that antipodal points give the same value).
In the case d even, look at singular points of hypersurface $f - \lambda q^{d/2}$, hence one computes the "characteristic polynomial" $\mathrm{D}isc(f - \lambda q^{d/2})$, for any root λ_0 the singular point of $f - \lambda q^{d/2}$ are the eigenvectors.

The number of eigenvectors of a Symmetric Tensor

Theorem (Fornaess-Sibony, 1992)

The number of eigenvectors of general $f \in \mathrm{Sym}^d \mathbb{C}^{n+1}$ is equal to

$$\sum_{i=0}^{n} (d-1)^i = \frac{(d-1)^{n+1} - 1}{d-2}$$

This number is $\mathrm{EDdegree}$ *of d-Veronese embedding of $\mathbb{P}^n$.*

- Fornaess-Sibony proof uses Bezout Theorem
- Cartwright-Sturmfels give a toric proof (2013)
- Oeding-O. give another proof (2013) looking at eigenvectors as zero loci of section of the bundle $Q(d-1) := Q \otimes \mathcal{O}(d-1)$ where Q is the quotient bundle. Top Chern class $c_n(Q(d-1))$ gives the result.

This number is called the $\mathrm{EDdegree}$ of d-Veronese embedding of $\mathbb{P}^{m-1}$.

Reality questions for tensors, average EDdegree

Contrary to the matrix case, the eigenvectors of real symmetric tensors are not necessarily real. The space of tensors is divided in chambers, accordingly to the number of real eigenvalues.

The average EDdegree gives the expected value of the number of real critical points of the distance function, weighted according to the standard Gaussian distribution centered at the origin.
The rational normal curve C_n has

$$\mathrm{EDdegree}(C_n) = n \qquad \mathrm{avEDdegree}(C_n) = \sqrt{3n-2} \; [DHOST]$$

Bombieri-Weyl metric gives a probability space

The Bombieri-Weyl metric defines a measure, orthogonally invariant.

According to this measure, the expected number of real roots of a polynomial of degree d is $\sqrt{d}$ [Shub-Smale, Kostlan].

With different metrics the result is quite different !

The distance function in a general tensor space

Let V_i be real vector spaces equipped with a scalar product $q_i \colon V_i \times V_i \to \mathbb{R}$, equivalently a positive definite quadratic form $q_i \colon V_i \to \mathbb{R}$.
Example: $V_i \simeq \mathbb{R}^{n_i}$ with $q_i(x) = \sum x_i^2$.

They define a quadratic form on $V_1 \otimes \ldots \otimes V_d$ by $q(v_1 \otimes \ldots \otimes v_d) = q_1(v_1) \cdots q_d(v_d)$, then extended by linearity. This is called the *Bombieri norm*.

This gives a group embedding $SO(V_1) \times \ldots \times SO(V_d) \subset SO(V_1 \otimes \ldots \otimes V_d)$. There is an analogous Harmonic Decomposition and and analogous Proposition characterizing all Orthogonally invariant metrics.

Isotropic tensors

A tensor t is *isotropic* if $q(t) = 0$, they fill the isotropic quadric Q. The isotropic decomposable tensors fill a reducible variety, which is the union of cones over $\mathbb{P}(V_1) \times \ldots Q_i \ldots \times \mathbb{P}(V_d)$
where $Q_i \subset \mathbb{P}(V_i)$. **Example:** $(e_1 + \sqrt{-1}e_2) \otimes (f_1 + f_2)$ is isotropic.

In the symmetric setting, isotropic decomposable tensors make a nonreduced scheme, which is a multiple structure on $Q \subset \mathbb{P}(W)$.
Example: $(x + \sqrt{-1}y)^d$ is isotropic.

The critical points in the case of general matrices, the SVD

For general matrices A of size $n \times m$, with $n \leq m$, there are n critical points of the distance function to the Segre variety. They are pairs $\lambda v_i \otimes w_i$, with $q_V(v_i) = q_W(w_i) = 1$ such that

$$\begin{cases} Av_i = \sigma_i w_i \\ A^t w_i = \sigma_i v_i \end{cases}$$

which are called *singular pairs*

$A = \sum_i \sigma_i v_i \otimes w_i^t$ is the *Singular Value Decomposition* (SVD) of A.

The critical points in the case of general tensors

For general tensors A of size $m_1 \times \ldots \times m_d$ the critical points of the distance function to the Segre variety are t-ples $\lambda v_1 \otimes \ldots \otimes v_d$, such that

$$A(v_1 \otimes \ldots \widehat{v_i} \ldots \otimes v_d, -) = \lambda q_i(v_i, -)$$

which are called *singular t-ples*

The number of singular d-ples

Theorem (Friedland-O, $\mathrm{EDdegree}$ of Segre variety)

The number of singular d-ples of a general tensor t over $\mathbb{C}$ of format $m_1 \times \ldots \times m_d$ is the coefficient of $\prod_{i=1}^{d} t_i^{m_i-1}$ in the polynomial

$$\prod_{i=1}^{d} \frac{\hat{t}_i^{\,m_i} - t_i^{m_i}}{\hat{t}_i - t_i}$$

where $\hat{t}_i = \sum_{j \neq i} t_j$. This number is $\mathrm{EDdegree}(\mathbb{P}^{m_1-1} \times \ldots \times \mathbb{P}^{m_d-1})$

Theorem (Special case of binary tensors)

$$\mathrm{EDdegree}(\underbrace{\mathbb{P}^1 \times \ldots \times \mathbb{P}^1}_{d}) = d!$$

Theorem (Zeilberger)

Let $a_d(k_1, \ldots, k_d)$ be the number of critical points of format $\prod_{i=1}^{d}(k_i + 1)$ then

$$\sum_{k \in \mathbb{N}^d} a_d(k_1, \ldots, k_d)\mathbf{x}^k = \frac{1}{\left(1 - \sum_{i=2}^{d}(i-1)e_i(\mathbf{x})\right)} \prod_{i=1}^{d} \frac{x_i}{1 - x_i}$$

Theorem (Zeilberger, Pantone)

$$a_3(n, n, n) \sim \frac{2}{\sqrt{3}\pi} \frac{8^n}{n}$$

Emanuele Ventura will speak next time about generalizations to any number of modes, joint work with Sodomaco.

The Eckart-Young Theorem. Best rank k approximation for matrices

Theorem (Eckart-Young, 1936)

Let $A = \sum_{i=1}^{n} \sigma_i u_i \otimes v_i$ be the SVD of A, with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.
Then for any $k = 1, \ldots n-1$, $\sum_{i=1}^{k} \sigma_i u_i \otimes v_i$ is the best rank k approximation of A.

Let $X_r = \{M \in M_{n \times t} | \mathrm{rk}(M) \leq r\}$. It is closed, so for matrices, $\text{border rank} = \text{rank}$.

Theorem (All critical points)

Let $A = \sum_{i=1}^{n} \sigma_i u_i \otimes v_i$ be the SVD of A. All critical points of the distance function from A to X_k are $\sum_{i \in I_k} \sigma_i u_i \otimes v_i$, for any $I_k \subset \{1, \ldots, n\}$ of cardinality k. It follows $\mathrm{EDdegree}(X_k) = \binom{n}{k}$.

No analog of this Theorem is known for Tensors, in the sense that EDdegree of secant varieties in tensor spaces is known only in a few cases.

The critical points are not independent and lie in subspaces, I

Let $V = V_1 \otimes \ldots \otimes V_d$. Define the pairing (bilinear form with output a tensor, not a scalar) $[|]_I : V \times V \to \wedge^2 V_I$, which on decomposable elements is

$$[v_1 \otimes \ldots \otimes v_d, w_1 \otimes \ldots \otimes w_d] = \prod_{j \neq i} q_{V_j}(v_j, w_j) v_i \wedge w_i.$$

The *critical space* of $f \in V$ is

$$H_f := \{g \in V | \, [f|g]_I = 0 \forall I\}$$

For a matrix A, $H_A = \{B | A^t B, AB^t \text{ are symmetric}\}$

The critical points are not independent and lie in subspaces, II

Theorem (Draisma,O,Tocino, 2018)

All the critical points for f lie in H_f.

The critical points span the critical space if triangle inequality $\dim V_i \leq \sum_{j \neq i} \dim V_j$ is satisfied. It is the condition such that the dual variety of the Segre variety is a hypersurface, so that the hyperdeterminant is defined.
As a consequence, f is linear combination of its singular d-ples.

Banach Theorem

Theorem (Banach 1938)

Let t be a symmetric tensor. The closest rank one tensor to t may be chosen symmetric.

There is an alternative proof by Shmuel Friedland.

There are other critical points for the distance function, beyond the symmetric ones.

Symmetric Tensors with Eigenvectors in a given subspace

Theorem (O-Shahidi)

Let $L \subset \mathbb{C}^n$, $\dim L = d$. The variety

$$\kappa^s_{d,n,k}(L) = \{f \in \mathbb{P}(Sym^k(\mathbb{C}^n)) | f \textit{ has an eigenvector in } L\}$$

is irreducible, it has codimension $n-d$ and degree $\sum_{i=0}^{d-1} \binom{n-d+i}{i} (k-1)^i$.

Proof uses Chern classes.
The matrix case appears in a paper [O-Sturmfels, 2013].

Tensors with singular t-ples with first element in a given subspace

Theorem (O-Shahidi)

Let $L \subset \mathbb{C}^{n_1}$, $\dim L = d$. *The variety*
$\kappa_{d,n_1,\ldots,n_k}(L) =$
$= \{T \in \mathbb{P}(\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_k}) | T \textit{ has a singular } k\textit{-tuple } (v_1, \ldots, v_k) \textit{ with } v_1 \in L\}$
is irreducible, it has codimension $n_1 - d$ *and degree given by the coefficient of the monomial* $h^{n_1-d} v_1^{d-1} \prod_{i\geq 2} v_i^{(n_i-1)}$ *in the polynomial*

$$\prod_{i=1}^{k} \frac{[(\widetilde{v_i} + h)^{n_i} - v_i^{n_i}]}{(\widetilde{v_i} + h) - v_i}$$

where $\widetilde{v_i} = \Sigma_j v_j - v_i$..

Thanks

Thanks !!