Introduction to derived algebraic geometry

Gabriele Vezzosi

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Plan of the talk

1. A quick introduction to Derived Algebraic Geometry
2. An example – the derived stack of vector bundles
3. Derived symplectic structures

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- Realize C^∞-intersection theory without transversality $\rightsquigarrow$ C^∞-derived cobordism (realized by D. Spivak (2009)).

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So local/affine objects of derived algebraic geometry are cdga's defined up to quasi-isomorphism.

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What is a 'homotopy theory' ? Roughly, a category M together with a distinguished class of maps w in M, such that we can define not only Hom-set (between objects in M) up to maps in w (i.e. $Hom_{w^{-1}M}(-,-)$) but a whole mapping space (top. space or simpl. set) of maps up to maps in w (i.e. $Map_{(M,w)}(-,-)$) & homotopy ve rsions of lim/colim

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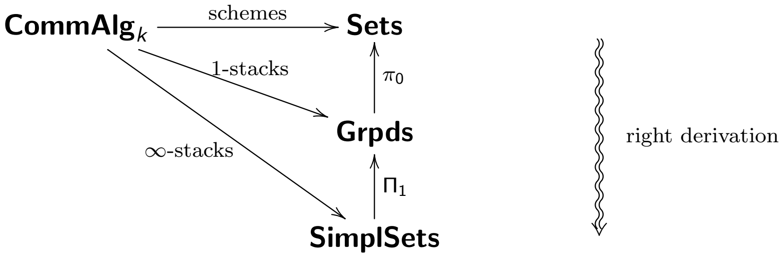

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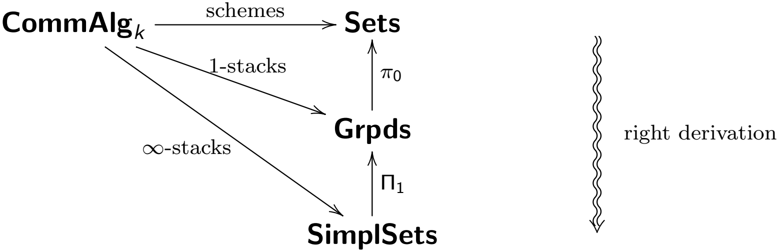

right derivation $\equiv$ adjoining homotopy colimits ($\Rightarrow$ can take quotients) $\rightsquigarrow$ promote the target categories to a homotopy theory (that of **SimplSets** or, eq.ly, topological spaces).

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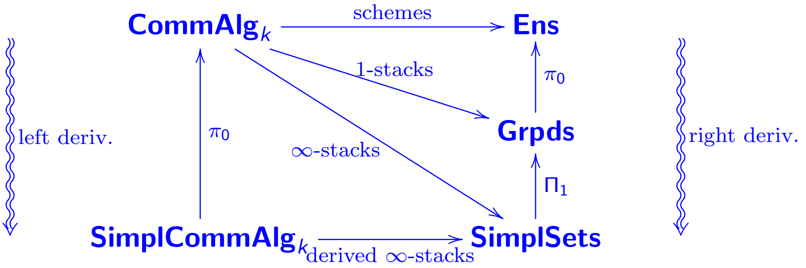

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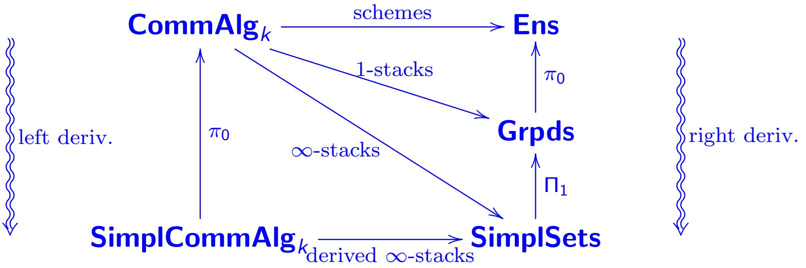

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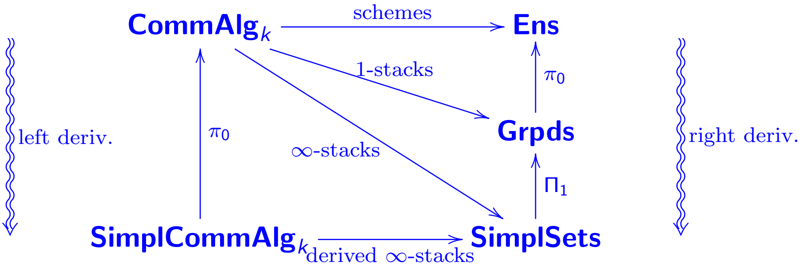

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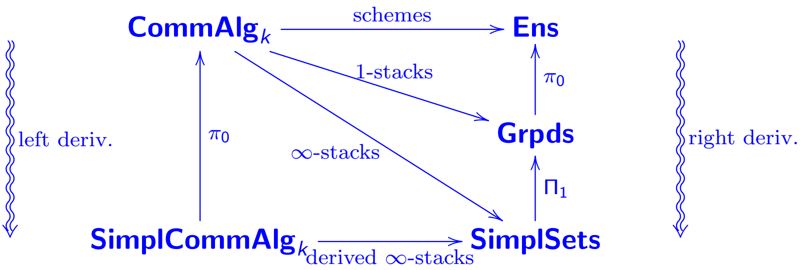

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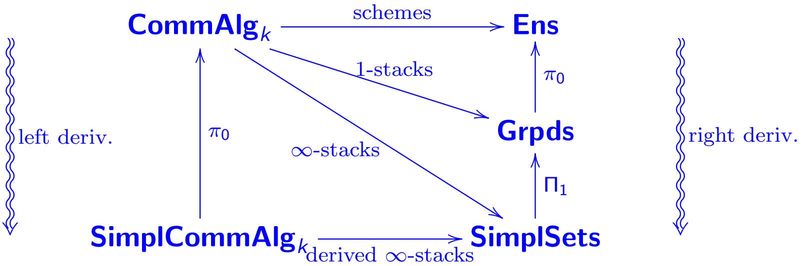

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Rmk. This is not an ad hoc definition: it is an elementary characterization of a more conceptual definition (via derived infinitesimal lifting property).

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- The category of derived stacks is $\mathbf{dSt}_k := \mathrm{Ho}(\mathbf{dSPr}_k)$

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Using atlases (and representability) $\rightsquigarrow$ extend notion of smooth, étale, flat, etc. to maps between geometric derived stacks.

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$\rightsquigarrow$ in derived algebraic geometry objects are very much transverse (no moving-lemmas needed).

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$\rightsquigarrow$ this is a global realization of Kontsevich hidden smoothness philosophy.

Derived symplectic structures

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I'll use the blackboard if I'll get to this...