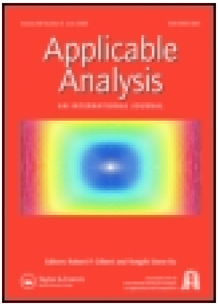

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Paolo Marcellini [a] & Carlo Sbordone [b]

[a] Istituto Maternatico , "U. Dini", Viale Morgagni 67a, Firenze, 50134, Italy

[b] Istituto Maternatico , "R. Caccioppoli", Via Mezzocannone 8, Napoli, 80134, Italy

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Homogenization of Non-Uniformly Elliptic Operators†

PAOLO MARCELLINI

Istituto Matematico "U. Dini", Viale Morgagni 67a, 50134, Firenze, Italy.

CARLO SBORDONE

Istituto Matematico "R. Caccioppoli", Via Mezzocannone 8, 80134, Napoli, Italy.

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An elliptic operator $\mathscr{A} = -\alpha_{ij}D_iD_j$ with constant coefficients is associated with any non-uniformly elliptic operator $A = -D_ia_{ij}(x)D_j$ with periodic coefficients ($\mathscr{A}$ is called the homogenization of A), such that the solutions of Dirichlet's problems for $A_\varepsilon = -D_ia_{ij}(x\varepsilon^{-1})D_j$ converge in L^2 (as $\varepsilon \to 0$) to the solution of the same problem for $\mathscr{A}$. The constants α_{ij} can be determined by solving a differential problem relative to A. These results (which are also proved for obstacle problems) extend those obtained by several authors when A is uniformly elliptic.

§1. INTRODUCTION

Let Y be an open interval in R^N and $[a_{ij}]$ a symmetric matrix of Y-periodic real functions in $L^1_{\mathrm{loc}}(R^N)$, such that $a_{ij}(x)\xi_i\xi_j \geqq 0\ \forall x,\ \xi \varepsilon R^N$.

Let us consider the operators

$$A = -D_ia_{ij}(x)D_j, \quad A_\varepsilon = -D_ia_{ij}(x\varepsilon^{-1})D_j,\ \varepsilon > 0 \tag{1.1}$$

and denote by u_ε the variational solution (assuming that it exists) of the Dirichlet problem

$$u_\varepsilon \in H^1_0(\Omega) \quad A_\varepsilon u_\varepsilon = \phi, \tag{1.2}$$

where Ω is a bounded open set in R^N and $\phi \varepsilon L^2(\Omega)$.

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To homogenize A means to find an operator

$$\mathscr{A} = -\alpha_{ij} D_i D_j \tag{1.3}$$

with constant coefficients $\alpha_{ij} \in R$, such that, for any $\phi \varepsilon L^2(\Omega)$, u_ε converges in some sense to u, the solution of the Dirichlet problem

$$u \varepsilon H_0^1(\Omega) \quad \mathscr{A} u = \phi. \tag{1.4}$$

The motivation for this problem lies in the study of physical models with periodic structure (Sanchez Palencia [19] and Babuška [2, 3]). We quote from [3]: "Homogenization is an approach which studies the macro-behaviour of a medium by its microproperties. The origin of this word is related to the question of a replacement of the *heterogeneous* material by an equivalent *homogeneous* one".

De Giorgi and Spagnolo [11], by using the technique of G-convergence, proved the following theorem:

THEOREM *If* $[a_{ij}]$ *is uniformly elliptic (i.e. there exist* $0 < \lambda \leqq \Lambda$ *such that*

$$\lambda |\xi|^2 \leqq a_{ij}(x) \xi_i \xi_j \leqq \Lambda |\xi|^2 \quad \forall x, \xi \varepsilon R^N), \tag{1.5}$$

then, $\forall \phi \varepsilon L^2(\Omega)$, *the solution* u_ε *of* (1.2) *tends in* $L^2(\Omega)$ *(as* $\varepsilon \to 0$*) to the solution* u *of* (1.4), *where* α_{ij} *is the elliptic matrix defined by*

$$\alpha_{ij} \xi_i \xi_j = |Y|^{-1} \ \mathrm{Inf}\{\textstyle\int_Y a_{ij}(u_{x_i} + \xi_i)(u_{x_j} + \xi_j) dx : u \in C^1, \ u Y\text{-periodic}\}.$$

This result was also obtained in [2], [19]; was extended to obstacle problems (Bensoussan–Lions–Papanicolaou [5], Boccardo–Capuzzo Dolcetta [8], Boccardo–Marcellini [9]), to non symmetric operators of order $2m$ $(m \geqq 1)$ (Tartar [22]) and to more general homogenization problems (Babuška [2, 3], Bensoussan–Lions–Papanicolaou [6], Lions [12, 13], Marcellini [16]) (see also the book [7]).

In this paper we study the homogenization of non-uniformly elliptic operators, (i.e. of operators which do not satisfy (1.5)). We replace this condition with suitable summability properties of the minimum and maximum eigenvalue of the matrix $[a_{ij}]$ such as (3.1), of the type considered in Murthy–Stampacchia [18] and Trudinger [23]. We prove the result corresponding to the above theorem for the Dirichlet problem (Theorem 5.2) and also for the obstacle problem (Theorem 6.1). These results have been announced in [24].

The proofs are derived by means of a compactness result (Theorem 3.1) with respect to Γ^--convergence (see the Definition 2.1) given in Marcellini–Sbordone [17]. The main difficulty is the extension of the Γ^--convergence from C^1 to the Sobolev space containing all the solutions. This is done in §4. The extension problem for more general classes of functionals, for example without periodicity of the coefficients, is still open.

§2. DEFINITION AND PROPERTIES OF Γ^--CONVERGENCE

In the following we consider topological spaces (V,τ) and sequences of functionals $F_h: V\to[0, \infty]$ satisfying one of the properties:

(2.1) (V, τ) *verifies the first countability axiom*

$$(2.2)\begin{cases} V \text{ is a reflexive and separable Banach space with dual } V^*,\ \tau=w-V \\ =\text{weak topology in } V;\ F_h \text{ convex, l.s. (lower semicontinuous)}, F_h(0)\leqq M \\ <\infty,\ F_h(v)\geqq\alpha(v), \text{ with } \alpha\colon V\to[0,\infty] \text{ such that } \lim\limits_{\|v\|\to\infty} \alpha(v)\|v\|^{-1}=\infty. \end{cases}$$

DEFINITION 2.1 Let (V,τ) and F_h satisfy (2.1) or (2.2); then $F: V\to[0,\infty]$ is the $\Gamma^-(\tau)$ limit of F_h on $V(F=\Gamma^-(\tau)-\lim F_h)$ *iff*

i) $\forall v\varepsilon V\ \exists v_h\varepsilon V: v_h\overset{\tau}{\to}v$ and $F(v)=\lim\limits_h F_h(v_h)$

ii) $\forall v_h,\ v\varepsilon V,\ v_h\overset{\tau}{\to}v\Rightarrow F(v)\leqq\lim_h \inf F_h(v_h)$.

Remark 2.2 The convergence (i), (ii) can also be defined for arbitrary (V,τ) and F_h, and in this case (see [1]) it is called "sequential Γ^--convergence". But, assuming (2.1) or (2.2), it coincides with Γ^--convergence as defined in [10] (cfr. [10] Proposition 3.1 and [1] Proposition 1.4). We observe that in the case (2.2) it is called G-convergence in [9, 14, 15].

THEOREM 2.3 ([9], *Theorem* 2.7), *If F_h, F satisfy* (2.2), *then* $F=\Gamma^-$-$(w-V)\lim F_h$ *on V iff* $\forall v^*\varepsilon V^*\ u_h(v^*)\to u(v^*)$ *in* $w-V$, *where* $u_h(v^*)$ *(resp.* $u(v^*)$*) is the minimum point in V of* $v\to F_h(v)-\langle v^*, v\rangle$ *(resp.* $F(v)-\langle v^*, v\rangle$*).*

PROPOSITION 2.4 ([14] *Proposition* 9). *Let F_h satisfy* (2.2); *then there exist a subsequence* (F_{h_r}) *of* (F_h) *and an F such that* $F=\Gamma^-$-$(w-V)\lim F_{h_r}$ *on V.*

THEOREM 2.5 ([9] *Theorem* 3.6). *Let F_h satisfy* (2.2), $F: V\to[0,\infty]$ *and let* $K_0\subseteq V$ *be such that:*

j) *K_0 is dense in* $\{v\varepsilon V: F(v)<\infty\}$, *with respect to a topology σ stronger than* $w-V$ *and F is σ-continuous.*

jj) $\forall v\varepsilon K_0\ \exists v_h\to v$ *in* $w-V$ *such that* $F(v)=\lim_h F_h(v_h)$.

jjj) $\forall v_h,\ v\varepsilon V,\ v_h\to v$ *in* $w-V\Rightarrow F(v)\leqq\lim_h$ *inf* $F_h(v_h)$.

Then: $F=\Gamma^-$-$(w-V)\lim F_h$ *on V.*

§3. Γ^--CONVERGENCE ON THE SPACE $C^1=C^1(R^N)$

In this section we consider non-uniformly elliptic matrices $[a_{ij}]$ satisfying $\forall i, j=1,\ldots, N$; $\forall x, \xi \varepsilon R^N$

$$\begin{cases} a_{ij}=a_{ji},\ 0\leqq m(x)|\xi|^2\leqq a_{ij}(x)\xi_i\xi_j\leqq M(x)|\xi|^2 \\ \|m^{-1}\|_{L^r(\Omega)}+\|M\|_{L^s(\Omega)}\leqq Q(\Omega);\ r,s\in]1,\infty],\ r^{-1}+s^{-1}<2N^{-1}, \end{cases} \tag{3.1}$$

where Ω is a bounded open set in R^N and $Q(\Omega)$ a fixed increasing real function of Ω.

THEOREM 3.1 ([17] *Corollary* 2.9). *Let* $[a_{ij,h}]$ *be a sequence of symmetric matrices such that*

$$\begin{cases} 0\leqq m_h(x)|\xi|^2\leqq a_{ij,h}(x)\xi_i\xi_j\leqq M_h(x)|\xi|^2 \\ \|m_h^{-1}\|_{L^r(\Omega)}+\|M_h\|_{L^s(\Omega)}\leqq Q(\Omega),\ (r, s \textit{ as in } (3.1),\ Q \textit{ fixed}). \end{cases} \tag{3.2}$$

There exist a subsequence, which we still denote $[a_{ij,h}]$, *and* α_{ij} *verifying* (3.1) *such that, if* $\forall u \varepsilon C^1$

$$F_h(\Omega, u)=\int_\Omega a_{ij,h}u_{x_i}u_{x_j},\ F(\Omega,u)=\int_\Omega \alpha_{ij}u_{x_i}u_{x_j}, \tag{3.3}$$

then for any $q\varepsilon[1,\infty]$ *and any bounded open set* Ω *in* R^N,

$$\begin{aligned} F(\Omega,u)&=\Gamma^-(L^q(\Omega))\lim F_h(\Omega,u) \\ &=\Gamma^-(L_0^q(\Omega))\lim F_h(\Omega,u) \textit{ on } C^1.\dagger \end{aligned} \tag{3.4}$$

We note that, under the assumptions of Theorem 3.1, one has

$$Q(\Omega)^{-1}\|Du\|^2_{L^p(\Omega)}\leqq F_h(\Omega,u)\leqq Q(\Omega)\|Du\|^2_{L^{2s/(s-1)}(\Omega)}.\ddagger \tag{3.5}$$

In the following we propose to extend the Γ^--convergence result of Theorem 3.1 from C^1 to $H_0^{1,p}(\Omega)$. For any bounded open set Ω in R^N, we denote by $\tilde F_h(\Omega, u)$ the convex function obtained as the lower semicontinuous envelope on the space $H^{1,p}(\Omega)$ of the convex functional

$$u\varepsilon H^{1,p}(\Omega)\to\begin{cases} F_h(\Omega,u) & \text{if } u\varepsilon C^1 \\ +\infty & \text{if } u\varepsilon H^{1,p}(\Omega)\backslash C^1; \end{cases} \tag{3.6}$$

that is

$$\tilde F_h(\Omega, u)=\inf\{\lim_k \inf F_h(\Omega, u_k): u_k\varepsilon C^1,\ u_k\to u \text{ in} H^{1,p}(\Omega)\}. \tag{3.7}$$

†We denote by $L_0^q(\Omega)$ the topology on C^1 induced by the extended metric $d(u,v)=\|u-v\|_{L^q(\Omega)}$ if $\operatorname{spt}(u-v)\subset\Omega$, $d(u,v)=+\infty$ otherwise.

‡We set $p=2r/(r+1)$ if $r<\infty$, $p=2$ if $r=\infty$, $2s/(s-1)=2$ if $s=\infty$.

We note that $\tilde{F}_h(\Omega, u) = F_h(\Omega, u) \quad \forall u\varepsilon C^1$.

The strictly convex l.s. functions

$$\phi_h(\Omega, u) = \tilde{F}_h(\Omega, u) + \|u\|^2_{L^p(\Omega)} \tag{3.8}$$

satisfy $\phi_h(\Omega, 0) = 0$ and $\phi_h(\Omega, u) \geqq \min\{Q(\Omega)^{-1}, 1\}\|u\|^2_{H^{1,p}(\Omega)} \quad \forall u\varepsilon H^{1,p}(\Omega)$. Therefore by Proposition 2.4, there exist a subsequence (ϕ_{h_r}) of (ϕ_h) and a convex l.s. function ϕ such that

$$\phi(\Omega, u) = \Gamma^- \text{-} (w - H^{1,p}(\Omega)) \lim \phi_{h_r}(\Omega, u) \text{ on } H^{1,p}(\Omega). \tag{3.9}$$

LEMMA 3.2 *If $\phi(\Omega, u)$ is as in (3.9), then*

$$\phi(\Omega, u) = F(\Omega, u) + \|u\|^2_{L^p(\Omega)} \quad \forall u\varepsilon C^1. \tag{3.10}$$

Proof For $u\varepsilon C^1$, let $u_r\varepsilon C^1$ be such that $u_r \to u$ in $L^p(\Omega)$ and $F(\Omega, u) = \lim_r F_{h_r}(\Omega, u_r)$. In particular (u_r) is $H^{1,p}(\Omega)$-bounded and so $u_r \to u$ in $w - H^{1,p}(\Omega)$. Then by (3.9)

$$\phi(\Omega, u) \leqq \lim_r \inf(F_{h_r}(\Omega, u_r) + \|u_r\|^2_{L^p(\Omega)}) = F(\Omega, u) + \|u\|^2_{L^p(\Omega)}. \tag{3.11}$$

Let now (u_r) converge weakly in $H^{1,p}(\Omega)$ to $u\varepsilon C^1$ and $\phi(\Omega, u) = \lim_r \phi_{h_r}(\Omega, u_r)$. It is easy to find $v_r\varepsilon C^1$ such that $|\phi_{h_r}(\Omega, v_r) - \phi_{h_r}(\Omega, u_r)| < 1/r$, $\|v_r - u_r\|_{H^{1,p}(\Omega)} < 1/r$, so that

$$F(\Omega, u) + \|u\|^2_{L^p(\Omega)} \leqq \lim_r \inf \phi_{h_r}(\Omega, v_r) = \phi(\Omega, u). \tag{3.12}$$

(3.10) follows from (3.11) and (3.12).

For any $k\varepsilon N$ let $\Omega_k = \{x\varepsilon R^N : |x| < k\}$ and let $\phi_h(\Omega_k, u)$ be as in (3.8). By Proposition 2.4, with a diagonal process it is possible to find a subsequence (ϕ_{h_r}) of $(\phi_h\}$ such that

$$\phi(\Omega_k, u) = \Gamma^- - (w - H^{1,p}(\Omega_k)) \lim_r \phi_{h_r}(\Omega_k, u) \text{ on } H^{1,p}(\Omega_k) \quad \forall k\varepsilon N. \tag{3.13}$$

LEMMA 3.3 *Let $\phi(\Omega_k, u)$ be as in (3.13). Then the function*

$$\phi(u) = \sup_k \phi(\Omega_k, u) \quad \forall u\varepsilon H^{1,p}(R^N) \tag{3.14}$$

is convex and l.s. on $H^{1,p}(R^N)$ and (α_{ij} being as in Theorem 3.1):

$$\phi(u) = \int_{R^N} \alpha_{ij} u_{x_i} u_{x_j} + \|u\|^2_{L^p(R^N)} \quad \forall u\varepsilon C^1 \cap H^{1,p}(R^N). \tag{3.15}$$

Proof Clearly ϕ is convex and l.s. and by $\phi(\Omega_k, u) \leqq \phi(\Omega_{k+1}, u) \quad \forall k\varepsilon N$, $\forall u\varepsilon H^{1,p}(R^N)$, we deduce that the supremum in (3.14) is a limit as $k \to \infty$. By Lemma 3.2 and the monotone convergence theorem we have (3.15).

Remark 3.4 In the next section we shall prove that $F(\Omega, u)=\Gamma^- -(w-H_0^{1,p})\lim F_{h_r}(\Omega, u)$ on $H_0^{1,p}(\Omega)$. This will be a consequence of the fact that (3.15) holds on $H^{1,p}$ (see 4.9). This follows from (3.15) if, e.g. the F_h are uniformly elliptic functionals ($r=s=\infty$), as in this case F and ϕ are $H^{1,2}$-continuous and C^1 is dense in $H^{1,2}(\Omega)$. In the general case, for the same reason, one has that $\phi(\Omega,u)=F(\Omega,u)+\|u\|^2_{L^p(\Omega)}$ on $H^{1,2s/(s-1)}(\Omega)$; the main result of next section is (3.15) on $H_0^{1,p}(\Omega)\supseteq H_0^{1,2s/(s-1)}(\Omega)$.

§4. Γ^--CONVERGENCE ON THE SPACE $H_0^{1,p}$

We begin this section with an abstract result on convex functions which will be useful in the sequel. It is a generalization of Jensen's inequality.

PROPOSITION 4.1 *Let Ω be an open set in R^N, $\phi: V=H^{1,p}(\Omega)\to[0, \infty]$ convex and l.s. function; $\alpha: R^N\to[0, \infty[$ such that $\int_{R^N}\alpha(y)dy=1$. If $v: R^N\to V$ is measurable and $u=\int_{R^N}\alpha(y)v(y)dy\varepsilon V$ then*

$$\phi\left(\int_{R^N}\alpha(y)v(y)dy\right)\leqq\int_{R^N}\alpha(y)\phi(v(y))dy.$$

Proof Using known properties of ϕ, if $\phi(u)<\infty$ then $\forall\varepsilon>0\exists v^*\varepsilon V^*$, $a\varepsilon R$ such that

$$\phi(v)\geqq\langle v^*, v\rangle+a \quad \forall v\varepsilon V; \quad \phi(u)<\langle v^*, u\rangle+a+\varepsilon;$$

from which it follows that

$$\phi(u)-\phi(v)<\langle v^*, u-v\rangle+\varepsilon \quad \forall v\varepsilon V. \tag{4.1}$$

In particular, setting in (4.1) $v=v(y)$, multiplying by $\alpha(y)$ and integrating over R^N, one has

$$\begin{aligned}\phi(u)-\int_{R^N}\alpha(y)\phi(v(y))dy&<\int_{R^N}\alpha(y)\langle v^*, u-v\rangle dy+\varepsilon\\ &=\langle v^*, u\rangle-\left\langle v^*, \int_{R^N}\alpha(y)v(y)dy\right\rangle+\varepsilon=\varepsilon.\end{aligned}$$

Since ε is arbitrary, we have the assertion.
If $\phi(u)=\infty$, then for any $k\exists v^*\varepsilon V^*$ and $a\varepsilon R$ such that

$$\phi(v)\geqq\langle v^*, v\rangle+a\forall v\varepsilon V; \ \langle v^*, u\rangle+a>k,$$

by which $\phi(v)>\langle v^*, v-u\rangle+k$. Since k is arbitrary, the assertion follows as in the previous case.

COROLLARY 4.2 *Let Ω be an open set in R^N, $V=H^{1,p}(\Omega)$, $\phi: V\to[0, \infty]$ convex and l.s.; $\alpha_k: R^N\to[0, \infty[$ with $\int\alpha_k(y)dy=1$. Let $v: R^N\to V$ be such that $v_k=\int_{R^N}\alpha_k(y)v(y)dy\varepsilon V$ and $v_k\to w$ in V. Then if $\phi(v(y))=\phi(w)$ $\forall y\varepsilon R^N$, we have $\phi(w)=\lim_k \phi(v_k)$.*

Proof By Proposition 4.1 we deduce $\phi(v_k)\leqq\phi(w)\quad\forall k$, so that $\lim_k \sup\phi(v_k)\leqq\phi(w)$. Since ϕ is l.s., this proves the assertion.

In the following we consider sequences of periodic matrices. Let $[a_{ij}]$ be as in (3.1) with the further assumption that there exist an open interval $Y\subseteq R^N$ such that a_{ij} is Y-periodic $\forall i, j$. Let (τ_h) be a divergent sequence of positive numbers and set $\forall h\varepsilon N$

$$a_{ij,h}(x)=a_{ij}(\tau_h x)\qquad i,j=1,\ldots, N,\ x\varepsilon R^N. \tag{4.2}$$

In order to prove that the $[a_{ij,h}]$ verify (3.2), let Y' an integer multiple of Y containing Ω and $k_h=[\tau_h]+1$; then we have for h large

$$\int_\Omega M_h^s dx\leqq\int_{k_h/\tau_h Y'} M_h^s dx=1/\tau_h^N\int_{k_h Y'} M^s dx\leqq 2^N Q(Y')^s.$$

LEMMA 4.3 *Let $[a_{ij,h}]$ be defined by (4.2) with a_{ij} Y-periodic and satisfying (3.1). If α_{ij} is defined by (3.3), (3.4); then α_{ij} are real constants and $\alpha_{ij}\xi_i\xi_j\geqq\lambda|\xi|^2$ with $\lambda>0$.*

Proof It is sufficient to prove that

$$F(R^N, u)=F(R^N, u(y))\qquad\forall u\varepsilon C_0^1,\quad\forall y\varepsilon R^N, \tag{4.3}$$

where $u(y)(x)=u(x-y)$. For $y\varepsilon R^N$ let $k_h\varepsilon N^N$ be such that $y_h=k_h/\tau_h\to y$ and $a_{ij}(x+k_h)=a_{ij}(x)\quad\forall h\varepsilon N,\ x\varepsilon R^N$. Then

$$a_{ij,h}(x+y_h)=a_{ij,h}(x). \tag{4.4}$$

If (u_h) is a sequence in C^1 converging to $u\varepsilon C_0^1$ in L_0^q such that $F(R^N, u)=\lim_h F(R^N, u_h)$; from (4.4) one has

$$\begin{aligned} F(R^N, u)&=\lim_h\int_{R^N} a_{ij,h}(x+y_h)D_i u_h(x)D_j u_h(x)dx\\ &=\lim_h\int_{R^N} a_{ij,h}(x)D_i u_h(x-y_h)D_j u_h(x-y_h)dx\geqq F(R^N, u(y)), \end{aligned}\tag{4.5}$$

as $u_h(y_h)$ converge to $u(y)$ in $L_0^q(\Omega')$, where Ω' is a bounded open set containing $spt(u_h(y_h))\quad\forall h$. The opposite inequality of (4.5) being obtained by symmetry, the lemma is proved.

LEMMA 4.4 *Under the assumptions of preceding lemma, let ϕ be defined on $H^{1,p}(R^N)$ by (3.14). Set $u(y)(x)=u(x-y)$; we have*

$$\phi(u(y))=\phi(u)\qquad\forall y\varepsilon R^N,\ u\varepsilon H^{1,p}(R^N). \tag{4.6}$$

Proof For any $y \varepsilon R^N$ let $y_r \to y$ with $|y_r| < k_0$ and $a_{ij,h_r}(x+y_r) = a_{ij,h_r}(x)$. For $u \varepsilon H^{1,p}(R^N)$ let $u_r \to u$ in $w - H^{1,p}(\Omega_{k+k_0})$ and $\lim_r \phi_{h_r}(\Omega_{k+k_0}, u_r) = \phi(\Omega_{k+k_0}, u)$. From the obvious relation $\phi_{h_r}(\Omega_{k+k_0}, u_r) \geqq \phi_{h_r}(\Omega_k, u_r(y_r))$ it follows that

$$\phi(\Omega_{k+k_0}, u) \geqq \lim_r \inf \phi_{h_r}(\Omega_k, u_r(y_r)) \geqq \phi(\Omega_k, u(y)), \qquad (4.7)$$

since $u_r(y_r) \to u(y)$ in $w - H^{1,p}(\Omega_k)$. Passing to the limit in (4.7) as $k \to \infty$, we have $\phi(u) \geqq \phi(u(y))$ and the result follows by symmetry.

LEMMA 4.5 *Let $a_{ij,h}$ be defined by* (4.2), F_h, F *as in* (3.3), (3.4), *and* $\tilde{F}_h$ *as in* (3.7). *Then there exists a subsequence* $(\tilde{F}_{h_r})$ *of* $(\tilde{F}_h)$ *such that* $u_r \to u$ $w - H_0^{1,p}(\Omega) \Rightarrow F(\Omega, u) \leqq \lim_r \inf \tilde{F}_{h_r}(\Omega, u_r)$.

Proof Let $u \varepsilon H^{1,p}(R^N)$ and (α_k) be a sequence such that $\alpha_k * u \varepsilon C^1$, $\alpha_k * u \to u$ in $H^{1,p}(R^N)$. Using Lemma 4.4 and Corollary 4.2 we have

$$\phi(u) = \lim_k \phi(\alpha_k * u) \qquad u \varepsilon H^{1,p}(R^N). \qquad (4.8)$$

Formula (4.8) holds also for $F(R^N, u) + \|u\|^2_{L^p(R^N)}$ by Lemma 4.3.

Replacing u in (3.15) by $\alpha_k * u$ and passing to the limit, we have

$$\phi(u) = F(R^N, u) + \|u\|^2_{L^p(R^N)} \qquad \forall\, u \varepsilon H^{1,p}(R^N). \qquad (4.9)$$

Now, for a fixed Ω let $\Omega_{k_0} \supseteq \Omega$ and $u_r, u \varepsilon H_0^{1,p}(\Omega)$ such that $u_r \to u$ in $w - H_0^{1,p}(\Omega)$. Then $\forall k \geqq k_0$ we have from (3.9)

$$\lim_r \inf \tilde{F}_{h_r}(\Omega, u_r) = \lim_r \inf \tilde{F}_{h_r}(\Omega_k, u_r) \geqq \phi(\Omega_k, u) - \|u\|^2_{L^p(\Omega)}.$$

By passing to the limit as $k \to \infty$ and using (4.9) we deduce

$$\lim_r \inf \tilde{F}_{h_r}(\Omega, u_r) \geqq \phi(u) - \|u\|^2_{L_p(\Omega)} = F(\Omega, u).$$

We can now state and prove the principal result of this section.

THEOREM 4.6 *Under the assumptions of the preceding lemma we have*

$$\alpha_{ij} \int_\Omega u_{x_i} u_{x_j} dx = F(\Omega, u) = \Gamma^- \text{-} (w - H_0^{1,p}(\Omega)) \lim \tilde{F}_{h_r}(\Omega, u) \quad \textit{on} \;\; H_0^{1,p}(\Omega).$$

Proof The proof comes from Theorem 2.5 with $K_0 = C_0^1(\Omega)$. In fact (j) is verified since, $[\alpha_{ij}]$ being positive definite, one has $\{u \varepsilon H_0^{1,p}(\Omega) : F(\Omega, u) < \infty\}$ $= H_0^{1,2}$; $C_0^1(\Omega)$ is dense in $H_0^{1,2}(\Omega)$ and $F(\Omega, u)$ is $H^{1,2}(\Omega)$-continuous.

Moreover by Theorem 3.1, $\forall u \varepsilon C_0^1(\Omega) \exists (u_r) \subset C_0^1(\Omega)$ such that

$$\lim_r \|u_r - u\|_{L^p} = 0, \quad \lim_r \tilde{F}_{h_r}(\Omega, u_r) = F(\Omega, u). \tag{4.10}$$

From (4.10) and (3.5) we deduce that (u_r) is $H_0^{1,p}(\Omega)$-bounded and so $u_r \to u$ in $w - H_0^{1,p}(\Omega)$. So (jj) of Theorem 2.5 is checked. Lemma 4.5 gives (jjj).

§ 5. THE HOMOGENIZATION

Let $[a_{ij}]$ be a Y-periodic matrix (Y an open interval in R^N) satisfying (3.1). Let us consider the family of second order non-uniformly elliptic operators $A_\varepsilon = -D_i a_{ij}(x\varepsilon^{-1})D_j$. The aim of this section is to prove that for any $\phi \varepsilon L^2(\Omega)$ the variational solution $u_\varepsilon = u_\varepsilon(\phi)$ of the problem

$$u_\varepsilon \varepsilon H_0^{1,p}(\Omega): A_\varepsilon u_\varepsilon = \phi \tag{5.1}$$

converges in $L^2(\Omega)$ to the solution $u = u(\phi)$ of the problem

$$u \varepsilon H_0^{1,p}(\Omega): \mathscr{A} u = \phi. \tag{5.2}$$

$\mathscr{A}$ is an uniformly elliptic operator of the form $\mathscr{A} = -\alpha_{ij} D_i D_j$ whose coefficients it is possible to compute by solving a differential problem on Y relative to the operator $A = -D_i a_{ij}(x) D_j$. This generalizes the similar theory ([2, 3, 5, 11, 13, 19, 21]) for the case $r = s = \infty$.

We begin with some notations and a lemma.

Let W_Y be the completion with respect to the norm $\|u\|_{W_Y} = \|Du\|_{L^p(Y)}$ of the space of C^1-functions u which are Y-periodic and have $\int_Y u\,dx = 0$.

LEMMA 5.1 *Let $[a_{ij}]$ be an Y-periodic matrix satisfying* (3.1). *Let $a_{ij,h}$ be as in* (4.2), *F_h and F as in* (3.3), (3.4), *$\tilde{F}_h$ as in* (3.7). *Set*

$$\psi_h(u) = \tilde{F}_h(Y, u + \langle \xi, x \rangle), \; \psi(u) = F(Y, u + \langle \xi, x \rangle) \qquad \xi \varepsilon R^N. \tag{5.3}$$

Then there exists (ψ_{h_r}) such that $\psi = \Gamma^- \text{-} (w - W_Y) \lim_r \psi_{h_r}$ on W_Y.

Proof By utilizing (3.5) it is easy to check that

$$\psi_h(u) \geqq Q(\Omega)^{-1} \|u + \langle \xi, x \rangle\|_{W_Y}^2 \quad \forall u \varepsilon W_Y, \; \psi_h(0) \leqq Q(\Omega) |\xi|^2 |Y|^{(s-1)/s}. \tag{5.4}$$

Then, by Proposition 2.3 there exists a subsequence (ψ_{h_r}) of (ψ_h) and a convex l.s. function χ verifying (5.4) such that $\chi = \Gamma^- - (w - W_Y) \lim_r \psi_{h_r}$ on W_Y. Let us prove first that $\chi(u) = \psi(u)\ \forall u \varepsilon C^1 \cap W_Y$.

If $u \varepsilon C^1 \cap W_Y$, then by (3.4) there exists a sequence $(v_r) \subset C^1$ such that $v_r \to u$

$+\langle\xi, x\rangle$ in $L^p(Y)$, $\operatorname{spt}[v_r-(u+\langle\xi, x\rangle)]\subset Y$ and $F(Y, u+\langle\xi, x\rangle)=\lim_r \tilde{F}_{h_r}(Y, v_r)$. Setting $u_r=v_r-\langle\xi, x\rangle$, we have $u_r\varepsilon W_Y\cap C^1$, u_r converges to u strongly in $L^p(Y)$ and in $w-W_Y$ (as $(\psi_{h_r}(u_r))$ is bounded), and $\psi(u)=\lim_r \psi_{h_r}(u_r)$. So $\chi(u)\leqq\lim_r \psi_{h_r}(u_r)=\psi(u)$.

Moreover if $(u_r)\subset W_Y$ is such that $u_r\to u$ in $w-W_Y$ and $\chi(u)=\lim_r \psi_{h_r}(u_r)$, then $u_r\to u$ in $L^p(Y)$ and therefore

$$\psi(u)=F(Y, u+\langle\xi, x\rangle)\leqq\liminf_r \tilde{F}_{h_r}(Y, u_r+\langle\xi, x\rangle)=\lim_r \psi_{h_r}(u_r)=\chi(u).$$

Setting $u(y)(x)=u(x-y)$ $\forall u\varepsilon W_Y$, we have, as in the proof of Lemma 4.4 $\chi(u(y))=\chi(u)$ for any u; and so, by Corollary 4.2 we deduce $\chi(u)=\psi(u)\,\forall\, u\varepsilon W_Y$.

Now we are able to prove the homogenization theorem for problems (5.1), (5.2). The proof of Theorem 5.2 is similar to that of [11] and so we will not enter in all the details.

THEOREM 5.2 *Let* $[a_{ij}]$ *be a Y-periodic matrix satisfying* (3.1) *and* Ω *a bounded open set in* R^N. *For any* $\phi\varepsilon L^2(\Omega)$ *let* $u_\varepsilon(\phi)$ *be the function in* $H_0^{1,p}(\Omega)$ *which minimizes the functional*

$$\int_\Omega a_{ij}(x\varepsilon^{-1})u_{x_i}u_{x_j}dx-2\int_\Omega \phi u\, dx(\dagger). \tag{5.5}$$

Then, as $\varepsilon\to 0$, $u_\varepsilon(\phi)$ *converges weakly in* $H_0^{1,p}(\Omega)$ *and strongly in* $L^2(\Omega)$ *to the function in* $H_0^{1,2}(\Omega)$ *which minimizes the functional*

$$\alpha_{ij}\int_\Omega u_{x_i}u_{x_j}dx-2\int_\Omega \phi u\, dx, \tag{5.6}$$

where $[\alpha_{ij}]$ *is the symmetric elliptic constant matrix defined by*

$$\alpha_{ij}\xi_i\xi_j=|Y|^{-1}\operatorname{Min}\{\textstyle\int_Y a_{ij}(u_{x_i}+\xi_i)(u_{x_j}+\xi_j)dx : u\varepsilon W_Y(\dagger)\}. \tag{5.7}$$

Proof By the compactness Theorem 3.1, Lemma 4.3 and Theorem 4.6, there exist an increasing sequence $\tau_h\to\infty$ and a symmetric elliptic constant matrix $[\alpha_{ij}]$ such that, with the notations (4.2), (3.3), (3.7): $F(\Omega, u)=\Gamma^-(w-H_0^{1,p}(\Omega))\lim_h \tilde{F}_h(\Omega, u)$ on $H_0^{1,p}(\Omega)$. For any $\phi\varepsilon L^2(\Omega)$, by Theorem 2.3, the function $u_h(\phi)$ which minimizes $\tilde{F}_h(\Omega, v)-2\int_\Omega \phi v\, dx$ on $H_0^{1,p}(\Omega)$ converges weakly in $H_0^{1,p}(\Omega)$, and strongly in $L^2(\Omega)$, to the function $u(\phi)$ which minimizes on $H_0^{1,p}(\Omega)$ (or on $H_0^{1,2}(\Omega)$) the functional $F(\Omega, v)-2\int_\Omega \phi v\, dx$.

If we prove (5.7), by the uniqueness of the limit matrix α_{ij} and using a compactness argument, we have that $u_\varepsilon(\phi)\to u(\phi)$ in $w-H_0^{1,p}$ and strongly in $L^2(\Omega)$.

†The functional is defined on C^1 and extended by semicontinuity as in (3.7).

Let us prove (5.7). Since under our assumptions $\Gamma^- -(w-V)$ convergence implies the convergence of minimum values ([14], Proposition 6 (i)), we deduce from Lemma 5.1 that

$$\text{Min}\,\{\psi(u): u\varepsilon W_Y\} = \lim_r \text{Min}\,\{\psi_{h_r}(u): u\varepsilon W_Y\}. \tag{5.8}$$

By the definition of ψ (see (5.3)) and the fact that α_{ij} are constants, the left side in (5.8) is equal to $|Y|\alpha_{ij}\xi_i\xi_j$. Moreover (cfr. [11, 16]) the right side of (5.8) is equal to

$$\text{Inf}\{\textstyle\int_Y a_{ij}(D_i u+\xi_i)(D_j u+\xi_j)\,dx : u\varepsilon W_Y \cap C^1\}. \tag{5.9}$$

So we have (5.7) and the theorem.

§6. CONVERGENCE OF SOLUTIONS OF OBSTACLE PROBLEMS

Let us consider the following closed convex sets in $H_0^{1,p}(\Omega)$:

$$K_1 = \{v\varepsilon H_0^{1,p}(\Omega): v \geqq \psi \quad \text{on} \quad E\} \tag{6.1}$$

$$K_2 = \{v\varepsilon H_0^{1,p}(\Omega): v \geqq \psi \quad \text{on} \quad \Omega\}, \tag{6.2}$$

where E is a compact of Ω, $\psi\varepsilon L^\infty(\Omega) \cap H^{1,p}(\Omega)$ and $\psi \leqq 0$ on Ω. The inequality $v \geqq \psi$ means that there exists $(v_h) \in C^1$ such that $v_h \geqq 0$ and v_h converges to $v-\psi$ in $H^{1,p}(\Omega)$.

THEOREM 6.1 *Under the assumptions of Theorem* 5.2, *if* $u_\varepsilon(\phi)$ $(\phi\varepsilon L^2(\Omega))$ *is the vector which minimizes the functional* (5.5) *over* $K_1\,(K_2)$, *then, as* $\varepsilon\to 0$, $u_\varepsilon(\phi)$ *converges to* $u(\phi)$ *in* $w-H_0^{1,p}(\Omega)$ *and strongly in* $L^2(\Omega)$, *where* $u(\phi)$ *is the vector which minimizes the functional* (5.6) *over* $K_1\,(K_2)$, *and* $[\alpha_{ij}]$ *is given by* (5.7).

Proof Let $\delta_{K_1}(v)=0$ if $v\varepsilon K_1$ and $\delta_{K_1}(v)=\infty$ if $v\notin K_1$. If $\tau_h\to\infty$ we define $a_{ij,h}$ as in (4.2) and F, $\tilde F_h$ as in (3.3), (3.7). We prove that

$$F+\delta_{K_1} = \Gamma^- -(w-H_0^{1,p}(\Omega))\lim\,(\tilde F_h+\delta_{K_1}); \tag{6.3}$$

and this, by Theorem 2.3, proves the result relative to the convex K_1.

We check (6.3) by using Theorem 2.5. If $v, v_h\varepsilon H_0^{1,p}(\Omega)$ and $v_h\to v$ in $w-H_0^{1,p}(\Omega)$, we deduce from Theorems 2.3, 5.2 and (ii) of Definition 2.1 that $F(v) \leqq \lim_h \inf \tilde F_h(v_h)$ and, since δ_{K_1} is l.s., $F(v)+\delta_{K_1}(v) \leqq \lim_h \inf(\tilde F_h(v_h) + \delta_{K_1}(v_h))$. This gives (jjj) of Theorem 2.5. We choose in (j)

$$K_0 = \{v\varepsilon C_0^1(\Omega): \exists\,\varepsilon=\varepsilon(v)>0,\ v>\psi+\varepsilon \quad \text{on} \quad E\}; \tag{6.4}$$

in fact, since $[\alpha_{ij}]$ is positive definite, $\{v\colon F(v)+\delta_{K_1}(v)<\infty\}=K_1\cap H_0^{1,2}(\Omega)$ and K_0 is dense in this set with respect to $H_0^{1,2}$-norm, while F is $H_0^{1,2}$-continuous.

For $v\varepsilon K_o$, let

$$(v_h)\subset C_0^1(\Omega)\colon\lim_h\|v_h-v\|_{L^\infty}=0,\ \lim_h\tilde{F}_h(v_h)=F(v); \tag{6.5}$$

this is possible by choosing $q=\infty$ in Theorem 3.1. As $v\varepsilon K_0$ we have $v_h\varepsilon K_1$ for h large, and so $\delta_{K_1}(v_h)=0$. Therefore (v_h) satisfy (jj) of Theorem 2.5, since v_h converge to v in $w-H_0^{1,p}(\Omega)$, (v_h) being bounded in $H^{1,p}(\Omega)$ by (6.5) and (3.5). This completes the proof for K_1.

In the case of K_2, as in the previous one, we prove that

$$v_h\to v \text{ in } w-H_0^{1,p}(\Omega)\Rightarrow F(v)+\delta_{K_2}(v)\leqq\lim_h\inf[\tilde{F}_h(v_h)+\delta_{K_2}(v_h)]. \tag{6.6}$$

Let $K_0=\{v\varepsilon C^1(\bar{\Omega})\cap H_0^{1,p}(\Omega)\colon\forall E\subset\subset\Omega\exists\varepsilon=\varepsilon(v,E)>0,\ v>\psi+\varepsilon \text{ on } E\}$.

If (v_h) verifies (6.5), let us set $w_h=\max\{v_h,\psi\}$. One can verify (e.g. as in the proof of Theorem 4.5 in [9]) that $w_h\to v$ in $w-H_0^{1,p}(\Omega)$ and $F(v)=\lim_h F_h(w_h)$. Since $w_h\varepsilon K_2$, we have

$$F(v)+\delta_{K_2}(v)=\lim_h[F_h(w_h)+\delta_{K_2}(w_h)]\quad\forall v\varepsilon K_0. \tag{6.7}$$

It is easy to check that K_0 is $H_0^{1,2}(\Omega)$-dense into the set $\{w\colon F(w)+\delta_{K_2}(w)<\infty\}=K_2\cap H_0^{1,2}(\Omega)$. By this and (6.6), (6.7) we deduce, by Theorem 2.5:

$$F+\delta_{K_2}=\Gamma^- -(w-H_0^{1,p}(\Omega))\lim_h[\tilde{F}_h+\delta_{K_2}]. \tag{6.8}$$

Using Theorem 2.3, we obtain the result.

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