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On the number of Waring decompositions for a generic polynomial vector. (English summary)

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A *power sum decomposition* of a complex homogeneous polynomial $F \in R = \mathbb{C}[x_0, \dots, x_n]$ of degree d is a collection of linear forms $\{\ell_1, \dots, \ell_k\}$ and scalars $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ such that $F = \lambda_1 \ell_1^d + \cdots + \lambda_k \ell_k^d$. More generally, suppose we are given several homogeneous forms $F_1, \dots, F_r \in R$, with each F_i homogeneous of degree d_i. A power sum decomposition of the vector $F = (F_1, \dots, F_r)$ is a collection of linear forms $\{\ell_1, \dots, \ell_k\}$ and scalars that express each F_i as a linear combination of d_ith powers of the $\ell_1, \dots, \ell_k$. Every vector of complex homogeneous polynomials admits a power sum decomposition. The primary interest is in decompositions with k as small as possible, called *Waring decompositions*.

It is natural to ask under what circumstances a vector F has a unique Waring decomposition, in which case one says F is *identifiable*. This has been especially studied in the case that F is general among vectors with a fixed dimension n, number of homogeneous forms r, and degrees $d_1, \dots, d_r$, i.e., the forms $F_1, \dots, F_r$ have general coefficients. Dimension counting gives a necessary condition (the dimension of the space of such vectors F must be equal to the dimension of the variety of power sum decompositions of the expected length k) which can be expressed as a simple arithmetic condition among the numbers n, r, and the d_i: namely, a necessary condition for identifiability of general vectors F is that $\sum_{j=1}^r \binom{d_j+n}{n}$ must be divisible by $r+n$. Cases that meet this condition are called *perfect*. The primary question is which perfect cases are *generically identifiable*. In addition, for perfect but generically non-identifiable cases, general vectors F have finitely many Waring decompositions (but more than one); a second question is the number of Waring decompositions in these cases.

For the case $r = 1$ (the classical case of Waring rank for a single homogeneous form), recent work of M. Mella [Trans. Amer. Math. Soc. **358** (2006), no. 12, 5523–5538; MR2238925; Proc. Amer. Math. Soc. **137** (2009), no. 1, 91–98; MR2439429] and F. Galuppi and Mella ["Identifiability of homogeneous polynomials and Cremona transformations", J. Reine Angew. Math., posted November 12, 2017; MR4036576] determined precisely which perfect cases are generically identifiable. The generically identifiable binary ($n = 1$) cases were determined by C. Ciliberto and F. Russo [Adv. Math. **200** (2006), no. 1, 1–50; MR2199628].

For $r, n > 1$, the generically identifiable cases are much more rare. Three generically identifiable cases were known classically. One of these classical cases, going back to Weierstrass, is equivalent to the fact that a general pair of quadratic forms has a unique simultaneous diagonalization, which is a close analogue to the well-known statement that a general ellipsoid has uniquely determined axes. (Ignoring distinctions between the complex and real cases, if we assume the quadratic forms are real and positive definite, the linear forms ℓ_i in the Waring decomposition of the pair of quadrics correspond to the axes of the ellipsoid given by the unit ball of one of the quadratic forms, after normalizing so that the unit ball of the other quadratic form is the unit sphere.)

In this paper, the authors find a new generically identifiable case and present evidence that there are no further generically identifiable cases. (It seems that they preferred not to explicitly make a conjecture in the paper. The closest they come is to say, in the abstract, that the new generically identifiable case is "likely the last one". In the reviewer's opinion, this deserves to be given serious consideration as a conjecture.)

The new case was *found* by using numerical algebraic geometry to conduct computational exploration. Once found, it was *proved* to be generically identifiable using the method of non-abelian apolarity introduced by J. M. Landsberg and G. M. Ottaviani [Ann. Mat. Pura Appl. (4) **192** (2013), no. 4, 569–606; MR3081636]. The authors also use non-abelian apolarity to give a uniform proof of generic identifiability for almost all of the previously known cases. Along the way they give a very good introduction and historical overview of the subject.

Some evidence that this may be the last case of generic identifiability is that the authors rule out generic identifiability for some cases. Using numerical algebraic geometry, they rule out generic identifiability for a lot of cases and also gather evidence that in non-identifiable cases the number of decompositions grows rapidly. Also, they prove that for the list of cases with consecutive degrees $(d_1, d_2) = (a, a+1)$ and $n = r = 2$, with a even, identifiability holds only for $a = 2$ (one of the classically known identifiable cases). In addition, for $a > 2$, the number of decompositions grows at least quadratically. This proof is not by numerical algebraic geometry, rather by reducing some questions about birational geometry to a consideration of intersections of plane curves. One more interesting result is that when generic identifiability holds, the variety parametrizing power sum decompositions with k terms (for k greater than the minimal value) is unirational.

This paper is a very nice introduction to questions around identifiability and gives a valuable historical overview. It poses an interesting question (or maybe conjecture) which seems worthy of attention. And it is a nice showcase of a variety of methods, from birational geometry and geometry of plane curves to computational methods such as numerical algebraic geometry. *Zach Teitler*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.