On the second differentiability of convex surfaces

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Abstract

Properties of pointwise second differentiability of real–valued convex functions in $I\!R^n$ are studied. Some proofs of the Busemann–Feller–Aleksandrov theorem are reviewed and a new proof of this theorem is presented.

Running head: Second differentiability of convex surfaces.

1 Introduction

This paper deals with results due to H. Busemann, W. Feller and A. D. Aleksandrov, regarding the regularity of convex surfaces. Over a period of about thirty years, these results have been proved by other authors mainly with functional analysis techniques; we present a new approach of geometric type. The basic theorem is the following:

Let u be a convex real–valued function defined in an open convex subset Ω of $I\!R^n$; then u is twice differentiable almost everywhere and the gradient of u is differentiable almost everywhere in Ω.

We say that ∇u, the gradient of u, is differentiable in a point x, if ∇u exists at x and a $n \times n$ symmetric matrix H exists such that

$$y = \nabla u(x) + (z - x)H + o(\|z - x\|)\,,$$

for every z and every y in the subgradient of u at z, denoted by $\partial u(z)$.

We recall that $\partial u(z) = \{y \in I\!R^n \,:\, u(x) - u(z) \geq < y\,, x - z > \ \forall x \in I\!R^n\}$ (here $< \cdot, \cdot >$ stands for the scalar product in $I\!R^n$).

This result is important not only in differential geometry and real analysis, but also in optimal control theory, partial differential equations and other fields.

A number of proofs of this theorem have appeared since the original work of Busemann and Feller. We will describe some of them.

Busemann and Feller in 1935 [4], (see also [3], chapter 1) proved that a convex function u of two variables has the following property: $\frac{\partial^2 u}{\partial \nu^2}$ exists a.e. for any direction $\nu = (\nu_1, \nu_2)$, and

$$\frac{\partial^2 u}{\partial \nu^2} = \sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x_i \partial x_j} \nu_i \nu_j \,, \tag{1}$$

where x_1 and x_2 are orthogonal coordinates.

In their proof they introduce the *indicatrix* of u at a point, defined as follows. Let x be a point such that the graph of u has a tangent plane π at $(x, u(x))$; the plane parallel to π lying above π at a distance h, intersects the graph of u in a convex curve γ_h. The authors prove that for almost every x, the curve γ_h, rescaled by a certain factor, converges, as h tends to zero, to a convex curve γ, namely the indicatrix. Where such convergence holds, $\frac{\partial^2 u}{\partial \nu^2}$ exists for every ν. Then by an argument based on the Vitali covering theorem, they prove that the second order mixed derivatives of u exists a.e., this establishes (1).

Four years later Aleksandrov [1] presented the result in the more general form stated at the beginning of this paper. In his paper he first generalizes the validity a.e. of (1) to the n–dimensional case and calls the right term of (1) *generalized second differential*. Then he proves that the generalized second differential is in fact the second differential in the ordinary sense where it exists.

We remark that the statement that a convex function is twice differentiable a.e., known as Aleksandrov's theorem, is proved but not stated as a theorem in his paper. In fact the emphasis is put on the fact that the gradient of a convex function is a.e. differentiable.

A different approach, based on classical results in measure and distribution theory, was introduced by Rešetnjak in 1968 [10]. It is also followed by Krylov in [7] and it is presented in the book of Evans and Gariepy [6]. The starting point is to observe that the generalized second derivatives of a convex function are measures (this follows from the Riesz representation theorem and the positivity of the hessian matrix of smooth convex functions). By the Lebesgue decomposition theorem, the regular parts of these measures are integrable functions and form a generalized second differential. Convexity implies, via a non trivial argument, that such differential is a.e. a pointwise second differential in the ordinary sense.

The paper of Bangert ([2], 1979) gives a thorough discussion of the differentiability properties of convex functions and includes another proof of the Busemann-Feller-Aleksandrov theorem. Let P be the manifold of the points having distance 1 from the epigraph of u. A map F from $P \to I\!R^n$ is defined as follows: for each $p \in P$ $F(p)$ is obtained first choosing the point of the epigraph of u having minimal distance from p and then projecting this point on the domain of u. F is non expansive and differentiability properties of F, obtained by Rademacher's and Sard's theorems, are used to prove that the gradient of u is differentiable a.e. .

Another way of proving the theorem rests on the property of the subgradient ∂u of a convex function of being a maximal monotone operator. Rockafellar ([11], 1985) observes that using this property and the differentiability a.e. of maximal monotone operators, proved by Mignot [8] in 1976, one can obtain a new proof of the theorem. The method of Mignot is used also by Crandall, Ishii and Lions ([5], Theorem A.2 of the appendix) in a simpler ,though longer, form. The above mentioned property of ∂u implies that, if $J = (I + \partial u)$ (I identity operator in $I\!R^n$), then J^{-1} exists and is a non expansive mapping. As in the paper of Bangert, differentiability properties of J^{-1} are used to prove that the gradient of u is differentiable a.e. . We remark the last two mentioned proofs are related, since J^{-1} and the map F introduced by Bangert are basically the same map, although expressed in different terms.

The proof given in the present paper is somewhat similar to the one given by Busemann, Feller and Aleksandrov: it is based essentially on geometric arguments, and the Vitali's covering theorem.

The behavior of u at a point x is studied by means of a family of functions obtained by a certain rescaling of u centered in x. As the rescaling parameter tends to zero, for almost every x such functions converge to a convex function w whose level sets are all homothetic to the indicatrix introduced by Busemann and Feller. w is a quadratic function on each radius through x and it approximates uniformly u up to the second order. For these reasons w appears to be the second order term in the Taylor expansion of u, i.e. the ordinary second differential, and in fact this happens at almost every x. To prove it we observe that the measure of the subgradient of u is a.e. differentiable with respect to the Lebesgue measure. This implies a characterization of the level sets of w by a geometric property which enables us to conclude that all their planar sections are ellipses. Consequently w is a quadratic form.

We would like to point out that the geometric property satisfied by the level sets of w is involved in other extremal problems for convex bodies, like in the Blaschke–Santaló inequality (see Remark 3 in the next section).

2 Proof of the theorem

We will throughout assume, without loss of generality, that $u-|x|^2$ is convex and $\Omega = I\!R^n$. Furthermore, for a direction ν, we define

$$\frac{\partial u}{\partial \nu}(x) := \lim_{h\to 0^+} \frac{u(x+h\nu)-u(x)}{h};$$

by the convexity of u, such a limit exists for all x and ν and is a.e. equal to the directional derivative along ν. We denote by D the set where u is differentiable and notice that the complement of D is negligible (has zero measure).

A central role in our proof is played by certain functions $w_{x,h}$, which are obtained by rescaling u in a point: for a fixed x in D and a positive h, define

$$w_{x,h}(y) := \frac{u(x+h(y-x))-u(x)-h\nabla u(x)(y-x)}{h^2}\,. \tag{2}$$

Clearly $w_{x,h}$ is strictly convex for any h.

Lemma 1 *As h tends to zero $w_{x,h}$ converges uniformly to a function w_x in the unit ball centered in x for almost every x, and*

$$u(x+h\nu) = u(x) + h\nabla u(x)\nu + h^2 w_x(x+\nu) + o(h^2) \quad , \tag{3}$$

for every unit vector ν.

Proof Fix a direction ν. The functions

$$\begin{aligned}\frac{\partial^-}{\partial\nu}\Big(\frac{\partial u}{\partial\nu}\Big)(x) &:= \liminf_{h\to 0}\frac{\frac{\partial u}{\partial\nu}(x+h\nu)-\frac{\partial u}{\partial\nu}(x)}{h} \quad ,\\ \frac{\partial^+}{\partial\nu}\Big(\frac{\partial u}{\partial\nu}\Big)(x) &:= \limsup_{h\to 0}\frac{\frac{\partial u}{\partial\nu}(x+h\nu)-\frac{\partial u}{\partial\nu}(x)}{h} \quad ,\end{aligned}$$

are measurable; in fact the ratios involved are measurable, and the limit can be taken on rational values of h since the numerators are monotone in h. Then the set $N_\nu := \{x : \frac{\partial^+}{\partial\nu}(\frac{\partial u}{\partial\nu})(x) > \frac{\partial^-}{\partial\nu}(\frac{\partial u}{\partial\nu})(x)\}$ is also measurable. Since a convex function of one variable has second derivative a.e., Fubini's theorem tells us that N_ν is negligible.

Let $\mathcal{D}$ be a countable dense subset of the set of all directions in $I\!R^n$. Repeating the above argument we see that $\frac{\partial^2 u}{\partial \nu^2}(x)$ exists for any $\nu \in \mathcal{D}$ and for almost every x in $I\!R^n$. If x is such a point the definition of $w_{x,h}$ shows that, for any $\nu \in \mathcal{D}$, we have

$$w_{x,h}(x+t\nu) = \frac{t^2}{2}\frac{\partial^2 u}{\partial \nu^2}(x) + \frac{o(t^2h^2)}{h^2} \qquad \forall \nu \in \mathcal{D},$$

for every positive t. This means that for every ν in $\mathcal{D}$, $w_{x,h}$ converges on the line through x parallel to ν. By convexity $w_{x,h}$ converges to a function w_x uniformly on compact sets. Furthermore, definition (2) implies that for any ν and $t > 0$

$$w_x(x+t\nu) = t^2 w_x(x+\nu)\,. \tag{4}$$

Formula (3) follows from the uniform convergence and the definition of $w_{x,h}$ and so Lemma 1 is proved.

Now we turn to the two dimensional case.

LEMMA 2 *Let $n = 2$; for almost every x, w_x is a symmetric quadratic form.*

PROOF For any subset A of $I\!R^2$ we define $\mu(A) := m_e(\partial u(A))$, where m_e is the Lebesgue outer measure in $I\!R^2$; μ is an outer measure, i.e. is nonnegative and countably subadditive. As u is strictly convex,

$$\begin{aligned} A \cap B = \emptyset \quad &\Rightarrow \quad \partial u(A) \cap \partial u(B) = \emptyset; \\ A \,\text{open} \quad &\Rightarrow \quad \partial u(A) \text{ open.} \end{aligned}$$

These relations imply that Borel sets of $I\!R^2$ are measurable with respect to μ. Then, as a consequence of Vitali's covering theorem (see for instance Saks [12], chapter 4) we have the following differentiability property:

For almost every x in $I\!R^2$ the limit

$$\lim_{diam(E)\to 0} \frac{\mu(E)}{m(E)}$$

exists and is finite. Here m is the Lebesgue measure and E is a closed set containing x, such that the ratio $\frac{m(E)}{m(J)}$ is bounded away from zero whenever J is a cube containing E.

Let x be a point in D where this property holds, and such that the functions $w_{x,h}$ defined above converge to w_x. We can assume that $\nabla u(x) = 0$,

$u(x) = 0$, $x = 0$. Consider the form of w_x in polar coordinates ρ, θ; from (4) it follows that

$$w_x(\rho\cos\theta, \rho\sin\theta) = \rho^2 g(\theta)\,, \tag{5}$$

where $g(\theta)$ is a continuous periodic function of period π. Furthermore, by the convexity of w_x, g has left and right derivatives for any θ, which we denote by $g_l(\theta)$ and $g_r(\theta)$ respectively.

In the following α will denote a pair of angles (α_1, α_2) in $[0, 2\pi] \times [0, 2\pi]$, with $\alpha_1 < \alpha_2$. For positive h set $S(\alpha, h) := \{(\rho, \theta) : \alpha_1 \le \theta \le \alpha_2,\ 0 \le \rho \le h\}$. The above differentiability property of μ implies

$$\lim_{h\to 0} \frac{m(\partial u(S(\alpha, h)))}{m(S(\alpha, h))} = c \quad ,$$

where c is a number independent of α. Note that, due to the convexity of $u - |x|^2$, the constant c is strictly positive. By a simple computation: $m(\partial u(S(\alpha, h))) = h^2 m(\partial w_{x,h}(S(\alpha, 1)))$, and $m(S(\alpha, h)) = h^2 m(S(\alpha, 1))$, hence

$$\lim_{h\to 0} \frac{m(\partial w_{x,h}(S(\alpha, 1)))}{m(S(\alpha, 1))} = c \quad . \tag{6}$$

As we are going to show, (6) implies that w_x is a symmetric quadratic form. We assume initially that $g_l(\theta_0) \ne g_r(\theta_0)$ for some θ_0, and seek a contradiction. If $R := \{(\rho, \theta) : 0 < \rho \le 1,\ \theta = \theta_0\}$, $\partial w_{x,h}(R)$ would have positive measure by (5). But

$$\lim_{h\to 0} m(\partial w_{x,h}(S(\alpha, 1)) \ge m(\partial w_x(R))$$

for any fixed α such that $\alpha_1 < \theta_0 < \alpha_2$. Then, for sufficiently small $\alpha_2 - \alpha_1$, the left side of (6) would be larger than any positive quantity and this conflicts with (6). So g has first derivative at each θ and then w_x is differentiable for $\rho > 0$. Furthermore it is easily seen that $m(\partial w_x(\partial S(\alpha, 1))) = 0$. Then from the Corollary quoted in the Appendix, it follows that

$$\frac{m(\partial w_x(S(\alpha, 1)))}{m(S(\alpha, 1))} = c\,. \tag{7}$$

For future use, let us observe that the above argument works also if only circular sectors $S(\alpha, h)$ with α_i rational are considered. In fact we can prove (7) for such sectors in the above way, then by the regularity of w_x and the density property of rational numbers, we deduce (7) for all α.

The set $\partial w_x(S(\alpha, 1))$ is the union of the line segments joining the origin with the points $\nabla w_x(\cos\theta, \sin\theta)$, where $\alpha_1 \le \theta \le \alpha_2$; this is because w_x has the form (5). Hence as $\alpha_2 - \alpha_1$ tends to zero, the two dimensional measure of $\partial w_x(S(\alpha, 1))$ behaves like the area of the triangle with vertices in the origin, in $\nabla w_x(\alpha_1, 1)$ and $\nabla w_x(\alpha_2, 1)$. Then passing to the limit in (7) as θ is fixed and $\alpha_2 - \alpha_1$ tends to zero, with $\alpha_1 \leq \theta \leq \alpha_2$, we see that $g''(\theta)$ exists and

$$4g^2(\theta) - g'^2(\theta) + 2g(\theta)g''(\theta) = c\,. \tag{8}$$

Since $c > 0$, the only solutions of (8) are the quadratic polynomials $A\cos^2\theta + 2B\cos\theta\sin\theta + C\sin^2\theta$ (with $AC - B^2 = c/4$), and this is equivalent to saying that w_x is a symmetric quadratic form in orthogonal coordinates; this concludes the proof of Lemma 2.

Now we come back to the n–dimensional case. Let π be a two dimensional linear subspace of $I\!\!R^n$ and for any x let π_x be the plane through x parallel to π; we denote by u_{π_x} the restriction of u to π_x. Let k be an integer, α be defined as before with rational components, and $S_\pi(x, \frac{1}{k}, \alpha)$ be the circular sector with vertex in x, radius $\frac{1}{k}$ and whose sides form angles α_1 and α_2 with some fixed axis in π. Let

$$f_{k,\alpha}(x) := \frac{m(\partial u_{\pi_x}(S_\pi(x, \frac{1}{k}, \alpha)))}{m(S_\pi(x, \frac{1}{k}, \alpha))}\,.$$

where m is the two dimensional Lebesgue measure as before. From Proposition 1 in the Appendix we know that $f_{k,\alpha}$ are upper semicontinuous and hence measurable. Therefore

$$\overline{\overline{f}}_\alpha(x) := \limsup_{k\to\infty} f_{k,\alpha}(x)\,,\ \overline{f}_\alpha(x) := \liminf_{k\to\infty} f_{k,\alpha}(x)\,,$$

are measurable too and by the previous two dimensional result and Fubini's theorem they coincide a.e. Let f_α be the common value of these two functions. By the same reasons, for any two pairs of rational angles α and β, f_α and f_β coincide a.e. We deduce that the set $E_\pi := \{x : f_\alpha(x) = f_\beta(x),\ \alpha, \beta \text{ rational}\}$ is of full measure. Repeating the same argument for each π in a countable dense subset $\mathcal{F}$ of the set of all two dimensional linear subspaces of $I\!\!R^n$, we see that $E := \cap_{\pi\in\mathcal{F}} E_\pi$ is a set of full measure. Now let x be in E, and such that $w_{x,h}$ converge. Fix π in $\mathcal{F}$, let w_{π_x} denote the restriction of w_x to π_x; w_{π_x} is the limit of the function obtained by rescaling u_{π_x} in π_x, then, since $x \in E \subset E_\pi$, by Lemma 2 w_{π_x} is a symmetric quadratic form. Then any restriction of w_x to a plane in $\mathcal{F}$ is a symmetric quadratic form so that w_x is itself a symmetric quadratic form.

We now prove that the gradient of u is differentiable a.e.. Let x be a point where $w_{x,h}$ converges to w_x and where u is twice differentiable, we show that

$$\lim_{\|z-x\|\to 0} \frac{\|y - \nabla u(x) - (z-x)H_u(x)\|}{\|z-x\|} = 0\,, \quad \forall y \in \partial u(z) \quad , \tag{9}$$

where $H_u(x)$ is the hessian matrix of u in x. By definition (2) the left side of (9) equals

$$\lim_{h\to 0} \|z' - \nabla w_x(x+\nu)\|\,, \quad \forall z' \in \partial w_{x,h}(x+\nu)\,,$$

where $\nu = \frac{z-x}{\|z-x\|}$ and $h = \|z-x\|$. Then (9) follows from the fact that w_x is differentiable at each point and from Proposition 2 in the Appendix.

Remark 1 The proof given above implies the following property of convex functions: *A convex function u is twice differentiable at a point x, where ∇u exists, if and only if ∇u is differentiable at x.*

In fact the second differentiability of u and the differentiability of ∇u are both equivalent to the fact that $w_{x,h}$ converges to a symmetric quadratic function as h tends to zero.

Remark 2 At a point where $w_{x,h}$ does *not* converge to a symmetric quadratic function, two events may occur. The function $w_{x,h}$ may tend to a function w_x which is not a quadratic form; an example is given by $u(x_1, x_2) = \max\{{x_1}^2, {x_2}^2\}$, with $x = (0,0)$. On the other hand, $w_{x,h}$ may have no limit. In particular, given two arbitrary convex sets C_1 and C_2, it is possible to find a function u such that at a point x two subsequences of $w_{x,h}$ tend to distinct limits, whose level sets are homothetic to C_1 and C_2 respectively.

Remark 3 The property of w_x expressed by (7) has the following generalization (the proof works identically) to the n–dimensional case, where circular sectors $S(\alpha, 1)$ are replaced by intersections of circular cones with the level set $S_x = \{y \in I\!R^n : w_x(y) \le 1\}$:

If T is any circular cone with nonempty interior and vertex in x, then

$$\frac{m(\partial w_x(S_x \cap T))}{m(S_x \cap T)} = \text{constant independent of } T\,. \tag{10}$$

The left hand side of (10) can be expressed in terms of the support function σ (with respect to x) and the curvature function f of S_x; f and σ are both defined on the unit sphere of $I\!R^n$. Then one finds the following equation, equivalent to (10)

$$f(\xi)\sigma^{n+1}(\xi) = \text{constant}, \; \|\xi\| = 1\,. \tag{11}$$

As we have proved, in dimension two (11) characterizes ellipses. Condition (12) is closely related to some extremal problems for convex bodies in $\mathbb{R}^n$. For instance Petty in [9] uses condition (11), together with some extra information, to prove that ellipsoids are the only bodies for which equality hold in the Blaschke–Santaló inequality.

3 Appendix

In the previous section we have used some results on sequences of convex functions that are perhaps well known. For the sake of completeness, here we state and prove them rigorously.

In what follows u_i is a sequence of real–valued strictly convex functions on $\mathbb{R}^n$, which converges pointwise to a function u strictly convex.

Proposition 1 *Let C be a compact set and A be an open bounded set; then*

$$\begin{aligned}\limsup_{i\to\infty} m(\partial u_i(C)) &\leq m(\partial u(C))\,,\\ \liminf_{i\to\infty} m(\partial u_i(A)) &\geq m(\partial u(A))\,.\end{aligned}$$

Proof For any subset I of $\mathbb{R}^n$, let X_I denote the characteristic function of I. Let $v \in \mathbb{R}^n$ such that $\limsup_{i\to\infty} X_{\partial u_i(C)}(v) = 1$; then a subsequence i_k exists such that $v \in \partial u_{i_k}(C)$ for all k. Thus for any k a point $x_k \in C$ exists such that

$$u_{i_k}(y) \geq u_{i_k}(x_k) + v(y - x_k)\,.$$

Up to subsequence, x_k converge to $x \in C$. Passing to the limit in the previous inequality we get

$$u(y) \geq u(x) + v(y - x)$$

(we recall that u_i converge uniformly in C). As y is arbitrary, $v \in \partial u(C)$; thus $\limsup_{i\to\infty} X_{\partial u_i(C)} \leq X_{\partial u(C)}$. The first inequality of Proposition 1 follows from Fatou's lemma.

Now let v belong to $\partial u(x)$ with $x \in A$; without loss of generality, we may assume $v = 0$. Then x is the absolute minimum of u. Let B be a ball in A centered in x; $u(y)$ is strictly greater than $u(x)$ for any y on ∂B. By the uniform convergence on compact sets, u_i has its minimum in B from a certain i_0 on, i.e. $0 \in \partial u_i(A)$ definitely. Then $\liminf_{i\to\infty} X_{\partial u_i(A)} \geq X_{\partial u(A)}$. Using Fatou's lemma the second inequality follows .

Corollary *If C is a compact set such that $m(\partial u(\partial C)) = 0$,*

$$\lim_{i\to\infty} m(\partial u_i(C)) = m(\partial u(C)).$$

Proposition 2 *Let u be differentiable at each point and let $v_i(x)$ be any vector in $\partial u_i(x)$; then v_i converges uniformly to ∇u on compact sets.*

Proof We argue by contradiction. Let K be compact, and $\varepsilon > 0$ such that for a certain subsequence i_k

$$\|v_{i_k}(x_k) - \nabla u(x_k)\| \geq \varepsilon \tag{12}$$

for some x_k in K. For any $y \in I\!R^n$ we have

$$u_{i_k}(y) \geq u_{i_k}(x_k) + v_{i_k}(x_k)(y - x_k)\,;$$

then, if x is a limit point for x_k in K, both $v_{i_k}(x_k)$ and $\nabla u(x_k)$ tend to $\nabla u(x)$, and this conflicts with (12).

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