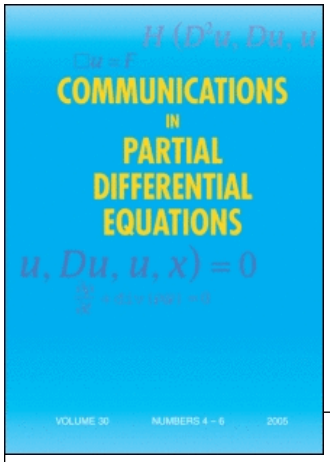

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Non—existence of positive solutions to semilinear elliptic equations on r" or r" through the method of moving planes

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NON-EXISTENCE OF POSITIVE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS ON $\mathbb{R}^n$ OR $\mathbb{R}^n_+$ THROUGH THE METHOD OF MOVING PLANES

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0. Introduction

In [6] Ding and Ni proved the following result for the problem

$$(0.1)\qquad \begin{cases} \Delta u + K(|x|)u^p = 0 \quad \text{in } \mathbb{R}^n\, n \geq 3 \\ u > 0\,, \quad u \in C^0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})\,. \end{cases}$$

Theorem A. *Let $p = \frac{n+2}{n-2}$ and $K(r) \in C^0[0, +\infty)$ be non decreasing and not identically constant. Then problem (0.1) has no radially symmetric solution.*

This result was later generalized to any $p > 1$ by T. Kusano and M. Naito [15] who proved that when $\frac{d}{dr}\left(r^{(n+2-p(n-2))/2}K(r)\right) \geq 0, \not\equiv 0$ problem (0.1) has no radially symmetric solution.

A natural question, asked by Li and Ni in [10], is whether under the hypotheses of Theorem A, (0.1) has non-radial solutions. We are able to give a negative answer to this question.

Theorem 1. *Let $K(r)$ be non negative and $C^1(0, +\infty)$ and let $\sigma = n + 2 - p(n-2)$. If either $p > \frac{n+2}{n-2}$ and $\frac{d}{dr}\left(r^\sigma K(r)\right) \geq 0$, or $1 < p \leq \frac{n+2}{n-2}$ and $\frac{d}{dr}\left(r^{\sigma/2}K(r)\right) \geq 0, \not\equiv 0$ then problem (0.1) has no solution.*

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When $0 < p < \frac{n+2}{n-2}$, $\frac{d}{dr}(r^\sigma K(r)) \geq 0$, $\not\equiv 0$ *and as* $r \to 0$ $K(r) = o(r^{-2})$, *then any solution to (0.1) is radially symmetric about the origin.*

It follows thus, for instance, that there is no nontrivial entire positive solution for $\Delta u + |x|^\gamma u^p = 0$ when $p > \frac{n+2}{n-2}$, $\gamma \geq -\sigma$ and also when $1 < p \leq \frac{n+2}{n-2}$, $\gamma > -\sigma/2$. Theorem 1 is a consequence of the following symmetry result.

Proposition 6. *Let* u *be a positive solution of* $\Delta u + g(|x|, u) = 0$ *in* $\mathbb{R}^n \setminus \{0\}$, *which belongs to* $C^0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$. *Let us assume that*

(i) $g(r,t) \geq 0$ *is locally Lipschitz in* $t > 0$ *and continuous in* $r > 0$, *and, for any compact set* $H \subset (0,\infty)$, $\sup_{t,s \in H, t \neq s} (g(r,t) - g(r,s))/(t - s) = o(r^{-2})$ *as* $r \to 0$,

(ii) $g(r,t)$ *is non decreasing in* t *for any* $r > 0$,

(iii) $g(1/r, r^{n-2}t)/r^{n+2}$ *is non increasing in* r *for any* $t > 0$,

Then $|x|^{2-n}u(x/|x|^2)$ *is radially symmetric about some point.*
Moreover if $|x|^{2-n}u(x/|x|^2)$ *is not continuous at the origin then* u *is radially symmetric about the origin. This last conclusion holds also if there exists* $\bar{r} \geq 0$ *such that for any* $t > 0$ *the function in (iii) is not constant in any neighborhood of* $\bar{r}$.

It is known that a solution u to (0.1) is radially symmetric if $K(r)$ is decreasing and u decays to 0 at infinity fast enough, see for instance Ni and Li [11]. The novelty of proposition 6 is that the nonlinearity g is not required to be monotone decreasing with respect to r and that no decay condition is required on u. The generalized Matukuma equation

$$\Delta u + \frac{|x|^{\lambda-2}}{(1+|x|^2)^{\lambda/2}} u^p = 0 \quad \text{in } \mathbb{R}^3,$$

($\lambda > 0$) proposed as a mathematical model in astrophysics, see [12], is an example of an equation where, for some λ and p, the coefficient of u^p is not monotone decreasing and proposition 6 applies. This equation also shows that the non existence part of theorem 1 for $1 < p < (n+2)/(n-2)$ is optimal: $r^{\sigma/2}K(r)$ is non decreasing if and only if $p \leq 1$ and it is known that for $\lambda = 2$, $p > 1$ there are solutions. Another symmetry result in the same spirit is theorem 2.

Theorem 2. *Let* u *be a positive solution of* $\Delta u + K(|x|)u^p = 0$, $p > 0$, *in* $\mathbb{R}^n \setminus \{0\}$, *in the class* $C^2(\mathbb{R}^n \setminus \{0\})$.

Let us assume that $K(r)$ *is non negative and continuous in* $(0,\infty)$ *and that there exists* $\bar{r} > 0$ *such that*

$$K(r)|r^2 - \bar{r}^2|^{\frac{n+2-(n-2)p}{2}} \tag{0.2}$$

is non-increasing in $(0, \bar{r}]$ *and non decreasing in* $[\bar{r}, \infty)$.

If the function in (0.2) is not identically constant then u *is radially symmetric about the origin.*

If the function in (0.2) is identically constant and if u*, or its Kelvin transform, is not continuous in the origin the same conclusion holds.*

Note that when $p \in (1, \frac{n+2}{n-2}]$ any K non-increasing in $(0, \bar{r}]$ and non decreasing in $[\bar{r}, \infty)$ satisfies (0.2), that this result applies also to solutions to (0.1) which are singular in the origin, and that it holds for any $p > 0$. Theorem 2 generalizes a result proved by the present author in [1], in the case $p = \frac{n+2}{n-2}$ and $u(x) = O(|x|^{2-n})$ at infinity.

The method of proof of theorem 2 and of proposition 6 is relatively simple: we use the moving plane method to prove symmetry of a suitable Kelvin transform of a solution. This Kelvin transform might be singular in some point but when the nonlinearity in the equation is increasing in u, this problem can be handled without much difficulty.

For the equation $\Delta u + g(u) = 0$, where

$$\text{(0.3)} \qquad \begin{array}{l} g \text{ is locally Lipschitz in } t > 0,\ g \text{ is non negative, and} \\ g(t)/t^{\frac{n+2}{n-2}} \text{ is non increasing,} \end{array}$$

the following result holds.

Theorem 3. *Let* u *be a* C^2 *positive solution of* $\Delta u + g(u) = 0$ *in* $\mathbb{R}^n$*, where* g *satisfies assumption (0.3).*

Then either u *is constant and* g *vanishes on that constant or*

$$\text{(0.4)} \qquad u(x) = \frac{k}{(|x - x_0|^2 + h^2)^{(n-2)/2}}$$

for some positive constants k*,* h *and some point* $x_0 \in \mathbb{R}^n$ *and* $g(t)$ *is a suitable positive multiple of* $t^{\frac{n+2}{n-2}}$ *for any* $t \in (0, max_{\mathbb{R}^n} u]$.

Note that in theorem 3 g is not required to be increasing.

In [3] Caffarelli, Gidas and Spruck proved the radial symmetry of all entire positive solutions of $\Delta u + g(u) = 0$ when g satisfies (0.3) and other extra assumptions. They could then prove, via ODE arguments, that all positive solutions of $\Delta u + u^p = 0$, $p = (n+2)/(n-2)$, have the form (0.4). Gidas and Spruck had already proved in [17] that the same equation does not have any positive entire solution for $1 \leq p < (n+2)/(n-2)$. A simpler proof of all this was later given by W. Chen and C. Li [4]. Theorem 3 generalizes and further simplifies these results.

The proof is the following:

– use the techniques introduced for proposition 6 to prove that any (i.e. centered at any point) Kelvin transform of a solution is radially symmetric,

– apply the following lemma (we state it in a simplified form):

Lemma 7. *If a function u, defined in $\mathbb{R}^n$, has the property that two suitable Kelvin transforms of u are radially symmetric, then either u is constant or (0.4) holds.*

The precise statement of Lemma 7 is in section 3 and its proof is absolutely elementary.

After we essentially completed our work we learned that C. C. Chen and C. S. Lin [5] proved theorem 3 under the extra assumptions g increasing, $g(t) = O(t^{\frac{n+2}{n-2}})$ as $t \to \infty$.

Under some extra assumptions theorem 3 can be generalized to allow g to depend also on $|x|$.

Theorem 4. *Let u be a C^2 positive solution of $\Delta u + g(|x|, u) = 0$ in $\mathbb{R}^n$, where $g(r,t)$ satisfies the assumptions of proposition 6 and it is non decreasing in r. Then either u is constant and $g \equiv 0$ or*

$$u(x) = \frac{k}{(|x - x_0|^2 + h^2)^{(n-2)/2}}$$

for some real constants $k > 0$, h and some point $x_0 \in \mathbb{R}^n$ and $g(r,t)$ is a suitable positive multiple of $t^{\frac{n+2}{n-2}}$ for any $t \in (0, max_{\mathbb{R}^n} u]$.

In the case $g(r,u) = K(r)u^p$ $p \geq \frac{n+2}{n-2}$ this result coincides with Theorem 1, while it is weaker for $1 < p < \frac{n+2}{n-2}$, in that it requires K be non decreasing. We remark that also the proof of this theorem, as well as that of theorem 3, does not depend on any ODE argument or on any Pohozaev identity and is based solely on repeated application of the moving plane method and on the previously mentioned Lemma 7. This provides a completely different proof of Theorem A.

These ideas, combined with some technical lemmas in [13] and in [16], can be used to prove a result similar to Theorem 3 for the corresponding nonlinear Neumann problem on the half space $\mathbb{R}^n_+ = \{x_n > 0\}$

$$\begin{cases} \Delta u + g(u) = 0 & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial u}{\partial x_n} = h(u) & \text{in } \partial\mathbb{R}^n_+ . \end{cases} \tag{0.5}$$

Theorem 5. *Let u be a $C^2(\mathbb{R}^n_+)\cap C^1(\overline{\mathbb{R}^n_+})$ positive solution of problem (0.5). Let us assume that g and h are locally Lipschitz in $(0,+\infty)$, $g \geq 0$ is non decreasing, $g(t)/t^{(n+2)/(n-2)}$ and $-h(t)/t^{n/(n-2)}$ are non increasing.*

Then one of the following assertions is true:

– there are some constants $a \geq 0$, $b > 0$ such that

$$u(x) = ax_n + b\,, \quad g \equiv 0\,, \quad \text{and} \quad h(b) = a\,;$$

– there are some constants $k > 0$, $d < 0$ and some point $x_0 \in \{x_n < 0\}$ such that

$$u(x) = \frac{k}{|x - x_0|^{n-2}}\,, \quad g \equiv 0\,, \quad \text{and} \quad h(t) = dt^{\frac{n}{n-2}}\ \forall t \in (0, \max_{\partial\mathbb{R}^n_+} u]\,;$$

moreover $dk^{2/(n-2)} = (2-n)|(x_0)_n|$, where $(x_0)_n$ is the n-th component of x_0;

– there are some constants $k > 0$, $\mu > 0$, $c > 0$, d and some point $x_0 \in \mathbb{R}^n$ such that

$$u(x) = \frac{k}{(|x - x_0|^2 + \mu^2)^{\frac{n-2}{2}}}\,,$$

$$g(t) = ct^{\frac{n+2}{n-2}}\ \forall t \in (0, \max_{\mathbb{R}^n_+} u]\,, \quad h(t) = dt^{\frac{n}{n-2}}\ \forall t \in (0, \max_{\partial\mathbb{R}^n_+} u]\,;$$

moreover $ck^{4/(n-2)} = (n-2)n\mu^2$ and $bk^{2/(n-2)} = (n-2)(x_0)_n$, where $(x_0)_n$ is the n-th component of x_0.

When $h(u) = du^{n/(n-2)}$ and either g is identically zero or or $g(u) = cu^p$, $1 < p \leq (n+2)/(n-2)$ this result is already known, it has been proved in part by B. Hu in [7] and in part by Y. Y. Li and M. Zhu in [13]. Here the method of proof is new, although similar in spirit.

After completing the manuscript the author was informed by Yan Yan Li that Y. Lou and M. Zhu proved in [14] that all non-negative solutions of $\Delta u = 0$ in $\mathbb{R}^{n+}$ with boundary condition $\partial u/\partial x_n = u^q$ are linear functions of x_n when $q > 1$. This is contained in theorem 5 only if $q \geq n/(n-2)$. In their proof the result for $q \in (1, n/(n-2))$ follows from the analogous result for $q \in [n/(n-2), \infty)$; when q is sub-critical they put the problem in an higher dimensional setting, making the power q super-critical in that setting (if u denotes a solution and $m > n$ is a positive integer the function $\bar{u}(y_1, \ldots, y_{(m-n)}, x_1, \ldots, x_n) := u(x_1, \ldots, x_n)$ solves the same problem in $\mathbb{R}^{m+}$). This idea implies that theorem 5 holds also if the hypothesis "$g(t)/t^{(n+2)/(n-2)}$ and $-h(t)/t^{n/(n-2)}$ are non increasing "is substituted by "for some $m \geq n$ $g(t)/t^{(m+2)/(m-2)}$ and $-h(t)/t^{m/(m-2)}$ are non increasing ".

Finally we mention that [2] contains some non existence results for the problem $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$ in $\{x_n > 0\}$, $du/dx_n + h(x)u^{n/(n-2)} = 0$ on $\{x_n = 0\}$.

1. Proofs of Theorems 1 and 2 and of Proposition 6

Proof of Theorem 2. To prove that u is radial it suffices to show that u is axisymmetric about any axis through the origin. Let l be such an axis: without loss of generality we can assume that l coincides with the x_n axis.

Let $\bar{x} = (0, \dots, 0, \bar{r})$ and let us consider the following conformal transformation of $\mathbb{R}^n$:

$$z = \bar{x} + \frac{\bar{r}^2(x - \bar{x})}{|x - \bar{x}|^2}.$$

The associated Kelvin transform of u

$$v(z) := u(\bar{x} + \frac{\bar{r}^2(z - \bar{x})}{|z - \bar{x}|^2}) \left(\frac{\bar{r}}{|z - \bar{x}|}\right)^{n-2}.$$

is a solution of

$$\Delta v + \tilde{K} v^p = 0 \quad \text{in } \mathbb{R}^n \setminus \{\bar{x}, 0\}$$

where

$$\tilde{K}(z) = K\left(\frac{\bar{r}|z|}{|z - \bar{x}|}\right)\left(\frac{|z - \bar{x}|}{\bar{r}}\right)^{(n-2)p-n-2}.$$

The function v is positive, $C^2(\mathbb{R}^n \setminus \{\bar{x}, 0\})$ (0 denotes the origin), decays to 0 at infinity as $|z|^{2-n}$ and may have a singularity at $\bar{x}$ and 0. Proving that u is axisymmetric with respect to the x_n axis is equivalent to prove that v is axisymmetric with respect to the z_n axis.

Let us write $z' = (z_1, \dots, z_{n-1})$. The function $\tilde{K}$ depends only on $|z'|$ and z_n: let us prove that it is non increasing as a function of $|z'|$. It suffices (by approximation) to prove it under the assumption that $K \in C^1$. The inequality $\partial\tilde{K}/\partial|z'| \leq 0$ is equivalent to

$$K'\left(\frac{\bar{r}|z|}{|z - \bar{x}|}\right)\frac{\bar{r}(|z - \bar{x}|^2 - |z|^2)}{|z||z - \bar{x}|} + ((n-2)p - n - 2)K\left(\frac{\bar{r}|z|}{|z - \bar{x}|}\right) \leq 0$$

Writing $r := \bar{r}|z|/|z - \bar{x}|$ the previous inequality becomes

$$K'(r)\frac{\bar{r}^2 - r^2}{r} + ((n-2)p - n - 2)K(r) \geq 0$$

for any $r > 0$, and this is equivalent to the assumption on $K(r)$ made in the statement.

To prove that v depends only on $|z'|$ and z_n we will use the moving plane method. This method is almost standard now, the only new difficulty here is due to the fact that v may be singular in some point. This part of the proof follows quite closely that of Theorem 4 in [8].

For $\lambda \in \mathbb{R}$ and for any $z \in \mathbb{R}^n$ let $z^\lambda = (2\lambda - z_1, z_2, \dots, z_n)$, $\Sigma(\lambda) = \{z \in \mathbb{R}^n : z_1 \le \lambda\} \setminus \{\bar{x}^\lambda, 0^\lambda\}$. Finally let $w(z,\lambda) = v(z^\lambda) - v(z)$. The function w satisfies

$$(1.1)\quad \Delta w(z) + \tilde{K}(z)p\xi^{p-1}w(z) = \left(\tilde{K}(z) - \tilde{K}(z^\lambda)\right) v(z,\lambda)^p \le 0 \quad \text{in } \Sigma(\lambda),$$

where ξ belongs to the interval whose endpoints are $v(z)$ and $v(z^\lambda)$. Let $g(z) = |z|^{-\alpha}$, $\alpha \in (0, n-2)$, and $\bar{w}(z,\lambda) = w(z,\lambda)/g(z)$. The function $\bar{w}$ satisfies

(1.2)

$$\Delta\bar{w}(z) + 2\frac{\nabla g}{g}(z)\cdot\nabla\bar{w}(z) + \left(\frac{\Delta g}{g}(z) + \tilde{K}(z)p\xi^{p-1}\right)\bar{w}(z) \le 0 \quad \text{in } \Sigma(\lambda).$$

First step. We prove that $\bar{w} \ge 0$ in $\Sigma(\lambda)$ for $\lambda < 0$, $|\lambda|$ large enough.

Suppose the contrary, then $\inf_{\Sigma(\lambda_i)} \bar{w}(z,\lambda_i) < 0$ on a sequence $\lambda_i \to -\infty$. If i is large this infimum can not be attained by a sequence converging to $\bar{x}^{\lambda_i}$ or to 0^{λ_i}. In fact, for instance, $\liminf_{z\to\bar{x}^\lambda} w(z,\lambda) = \liminf_{z\to\bar{x}} v(z) - v(\bar{x}^\lambda) > 0$ for $\lambda < 0$, $|\lambda|$ large, since v is non negative and superharmonic and $v(z) \to 0$ as $z \to \infty$.

Also $\bar{w}(z,\lambda) \to 0$ as $z \to \infty$ and thus, for each i, the previous infimum is attained at a point $q_i \in \Sigma(\lambda_i)$. In these points $\Delta\bar{w}(q_i,\lambda_i) \ge 0$ and $\nabla\bar{w}(q_i,\lambda_i) = 0$: in order to prove that (1.2) is violated in q_i and get a contradiction, it suffices to prove that the coefficient of $\bar{w}(\cdot,\lambda_i)$ in (1.2) is negative.

The sequence $(q_i^{\lambda_i})$ is not bounded (otherwise $\bar{w}(q_i^{\lambda_i},\lambda_i)$ would be eventually positive) and thus the number ξ appearing in (1.2), which belongs to $[v(q_i^{\lambda_i}), v(q_i)]$, is $O(|q_i|^{2-n})$. Therefore

$$\frac{\Delta g}{g}(q_i) + \tilde{K}(q_i)p\xi^{p-1} \le \frac{\alpha(\alpha+2-n)}{|q_i|^2} + O\left(\frac{1}{|q_i|^4}\right) < 0.$$

Second step. Let $\bar{\mu} = \sup\{\mu \le 0 : w(z,\lambda) \le 0 \text{ in } \Sigma(\lambda) \text{ for any } \lambda \le \mu\}$. Let us prove that if $\bar{\mu} < 0$ then $w(z,\bar{\mu}) \equiv 0$ (that is v is symmetric with respect to the hyperplane $z_1 = \bar{\mu}$). Suppose that $\bar{\mu} < 0$, then there exists a sequence of λ_i in $(\bar{\mu}, 0)$ converging to $\bar{\mu}$ such that for any i $\inf_{\Sigma(\lambda_i)} \bar{w}(z,\lambda_i) < 0$.

The function $w(z,\bar{\mu})$ is non negative in $\Sigma(\bar{\mu})$, therefore, by (1.1), it is superharmonic. If it is not identically 0, $\liminf_{z\to\bar{x}^{\bar{\mu}}} w(z,\bar{\mu}) > c > 0$ and thus also $\liminf_{z\to\bar{x}^{\lambda_i}} w(z,\lambda_i) > 0$ for i large enough. Similarly $\liminf_{z\to 0^{\lambda_i}} w(z,\lambda_i) > 0$ for i large enough.

The infimum of $\bar{w}(z,\lambda_i)$ in $\Sigma(\lambda_i)$ is thus attained at a point q_i. Repeating the arguments used in the first step one proves that the sequence (q_i) stays

bounded. A subsequence, that we still denote by (q_i), converges to some point $\bar{q} \in \Sigma(\bar{\mu})$ where $w(\bar{q}, \bar{\mu}) = 0$, $\nabla w(\bar{q}, \bar{\mu}) = 0$. This and the fact that $w(z, \bar{\mu}) \geq 0$ imply, via the strong maximum principle or the Hopf boundary lemma, that $w(z, \bar{\mu}) \equiv 0$.

Conclusion. Let us prove that

(a) if $K(r)|r^2 - \bar{r}^2|^{(n+2-(n-2)p)/2}$ is not identically constant,or

(b) if $u(x)$, or $|x|^{2-n}u(x/(|x|^2))$, are not continuous in the origin,

then $\bar{\mu} = 0$. Suppose the contrary, then $v(z)$ is symmetric about the hyperplane $\{z_1 = \bar{\mu}\}$ and thus $\tilde{K} = -\Delta v/v^p$ has the same symmetry. It is easy to see that, if $\bar{\mu} \neq 0$, this happens only if $K(r)|r^2 - \bar{r}^2|^{(n+2-(n-2)p)/2}$ is identically constant. This proves the assertion in case (a). Similarly if (b) holds v is not continuous at the origin or at $\bar{x}$ and thus it can not be symmetric with respect to $z_1 = \bar{\mu}$ when $\bar{\mu} \neq 0$.

Therefore $\bar{\mu} = 0$ and $v(z_1, z_2, \ldots, z_n) \leq v(-z_1, z_2, \ldots, z_n)$ for any z with $z_1 \leq 0$. Applying the same reasoning to the function $v(-z_1, z_2, \ldots, z_n)$ one concludes that v is symmetric with respect to $z_1 = 0$. Similarly one can prove that v is symmetric with respect to any hyperplane containing the z_n axis. □

Proof of proposition 6. This proof is similar to the previous one: in this case applying the moving plane method to the function v defined as the Kelvin transform of u about the origin one proves that v is radially symmetric about some point Q. We omit the details. We just remark that the assumption about the growth rate of $\sup_{t_1, t_2 \in H} (g(r, t_1) - g(r, t_2))/(t_1 - t_2)$ as $r \to 0$ is necessary to prove that in the equation which corresponds to (1.2) the coefficient of $\bar{w}$ is negative when z is large.

Clearly if Q is not the origin then v is continuous at the origin and moreover, for any $s > 0$, the function $g(1/r, r^{n-2}t)/r^{n+2}$, where t is the value of v on the sphere $\{|x - Q| = s\}$, is constant in $[||Q| - s|, |Q| + s]$. □

Proof of theorem 1. Let $g(r, t) = K(r)t^p$ and let u be a solution to (0.1). In this case $g(1/r, r^{n-2}t)/t^{n+2} = K(1/r)r^{-\sigma}$. Moreover the assumption about the behaviour of $r^\sigma K(r)$ or of $r^{\sigma/2}K(r)$ implies implicitly that, when $p > 1$, $K(r) = o(r^{-2})$ as $r \to 0$. Thus proposition 6 applies and u is radially symmetric about the origin, except possibly when $p > n + 2$, $K(r)r^\sigma$ is constant and the Kelvin transform of u is continuous at the origin. Except for this case the non-existence part of theorem 1 follows from the results proved by Ding and Ni and by Kusano and Naito quoted in the introduction.

To deal with the case which is still open, let v denote the Kelvin transform of u: by proposition 6 v is radially symmetric about some point Q. Moreover v is continuous at the origin and it satisfies problem (0.1) with $K(|x|)$

substituted by $K(1/|x|)|x|^{-\sigma}$, which is identically constant. This problem is thus radially symmetric about Q and the result of Kusano and Naito proves that it has no solution. $\square$

2. Proofs of Theorems 3 and 4

We start with two lemmas.

Lemma 6. *Let u be a positive solution of $\Delta u + g(u) = 0$ in $\mathbb{R}^n \setminus \{0\}$, which belongs to $C^0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\})$. Let us assume that g satisfies assumption (0.3). Then $|x|^{2-n}u(x/|x|^2)$ is radially symmetric about some point.*

Proof. This proof also is similar to that of Theorem 2. Since $g(t)$ is not assumed to be increasing in t, one new ingredient is needed to prove that, when $\bar{\mu} < 0$, either $w(x,\bar{\mu}) \equiv 0$ or $\liminf_{x\to 0^{\bar{\mu}}} w(x,\bar{\mu}) > c > 0$, (we use the notations introduced in the proof of theorem 2). We only show how to prove this and we omit all the other details. The function $w(x,\bar{\mu}) := v(x^{\bar{\mu}}) - v(x)$ satisfies

$$(2.1)\quad \Delta w(x,\bar{\mu}) = -\frac{g(|x^{\bar{\mu}}|^{n-2}v(x^{\bar{\mu}}))}{(|x^{\bar{\mu}}|^{n-2}v(x^{\bar{\mu}}))^{\frac{n+2}{n-2}}}v(x^{\bar{\mu}})^{\frac{n+2}{n-2}} + \frac{g(|x|^{n-2}v(x))}{(|x|^{n-2}v(x))^{\frac{n+2}{n-2}}}v(x)^{\frac{n+2}{n-2}}$$

and is non negative in $\Sigma(\bar{\mu})$. Suppose that $w(x,\bar{\mu}) \not\equiv 0$ and $\liminf_{x\to 0^{\bar{\mu}}} w(x,\bar{\mu}) = 0$. Then $w(x,\bar{\mu})$ is not superharmonic and therefore the right hand side of (2.1) is positive on a sequence $x_k \to 0^{\bar{\mu}}$. Since $v(x^{\bar{\mu}}) \geq v(x)$ in $\Sigma(\bar{\mu})$ and $g(t)/t^{\frac{n+2}{n-2}}$ is non increasing, it should be

$$(2.2)\qquad |x_k^{\bar{\mu}}|^{n-2}v(x_k^{\bar{\mu}}) > |x_k|^{n-2}v(x_k),$$

that is $v \to \infty$ on the sequence $x_k^{\bar{\mu}} \to 0$. A result proved by C. Li (Theorem 2 in [9]) implies that a non negative solution of $\Delta u(x) + g(|x|^{n-2}u(x))/|x|^{n+2} = 0$ in $\{0 < |x| < 1\}$ is asymptotically radial, i.e. $u(x) = \bar{u}(|x|)(1 + o(1))$ as $x \to 0$, where $\bar{u}(r)$ is the average of u on $\{|x| = r\}$. Thus, by (2.2), $v(x) \to \infty$ as $x \to 0$: this contradicts the assumption $\liminf_{x\to 0^{\bar{\mu}}} w(x,\bar{\mu}) = 0$. $\square$

Lemma 7. *Let u be a function defined on $\mathbb{R}^n$ and let us suppose that either*

(i) *u is radially symmetric about some point P and also its Kelvin transform centered in a point different from P is radially symmetric about some point Q or*

(ii) *the Kelvin transform of u centered in the origin is radially symmetric about some point P and also the Kelvin transform of u centered in a point T, $T \neq P$, $\|T\| = 1$, is radially symmetric about some point Q*

then either u is constant or $u(x) = k/(|x-x_0|^2+h^2)^{(n-2)/2}$ for some constants $k, h \in \mathbb{R}$ and some point $x_0 \in \mathbb{R}^n$.

Proof. Let us prove the result under assumption (i).

Let v_0 denote the Kelvin transform mentioned in the statement. To simplify the notations let us assume that the distance from P to the center of inversion of v_0 is 1, in the general case the proof works similarly. After a suitable rotation and translation we may suppose that $P = (0, \dots, 0, 1)$ and that the center of inversion of v_0 is the origin. The symmetry about P implies that $u(x) = f(|x - P|)$ for a suitable function f. The conformal transformation $z = x/|x|^2$ maps the sphere $|x - P|^2 = r^2$ in the sphere $S(r) = \{|z - (0, \dots, 0, 1/(1 - r^2))|^2 = r^2/(1 - r^2)^2\}$, when $r \neq 1$, and in the hyperplane $S(1) = \{z_n = 1/2\}$, when $r = 1$. Thus, for any $r \geq 0$,

$$v_0(z) = f(r)|z|^{2-n} \quad \text{for any } z \in S(r). \tag{2.3}$$

This relation implies that v_0 is axisymmetric about the x_n axis and that the center of symmetry Q of v_0 lies on the z_n axis unless $u \equiv 0$. Let $Q = (0, \dots, 0, q)$. If $q > 1$ or $q < 0$ then, for a suitable value of r, the sphere $S(r)$ is centered in Q and (2.3) contradicts the radial symmetry of v_0 about Q. Therefore $q \in [0, 1]$.

The constant value of v_0 on a sphere centered in Q is determined, through (2.3), by the value of v_0 in the intersection of that sphere with $S(r)$. More precisely for a given r let $I(r)$ be the interval $\{c > 0 : B(Q, c) \cap S(r) \neq \emptyset\}$ ($B(Q, c)$ denotes the sphere centered in Q and with radius c). Then for any z such that $|z - Q| \in I(r)$ we have

$$v_0(z) = f(r) \left(\frac{1 - q + qr^2}{|z - Q|^2 + q(1 - q)} \right)^{\frac{n-2}{2}}.$$

In particular this representation implies that if the two intervals $I(r_1)$ and $I(r_2)$ intersect for $r_1 \neq r_2$ then the value of $f(r)(1 - q + qr^2)^{\frac{n-2}{2}}$ at r_1 and r_2 must coincide. On the other hand we claim that given any two positive numbers $r_1 < r_2$ it is always possible to find a finite number of intermediate value $s_1 = r_1 < s_2 < \cdots < s_m = r_2$ such that $I(s_i) \cap I(s_{i+1}) \neq \emptyset$ (or, equivalently, such that for each i there is a sphere in the family $S(r)$ that intersect both $B(Q, s_i)$ and $B(Q, s_{i+1})$). This is an elementary fact which is due to the relative positions of the spheres centered in Q and of those in the family $S(r)$.

Then $f(r)(1 - q + qr^2)^{\frac{n-2}{2}}$ does not depend on r and

$$u(x) = \frac{k}{(1 - q + q|x - P|^2)^{\frac{n-2}{2}}}$$

for some constants k and $q \in [0, 1]$.

Let us now prove the lemma under assumption (ii).

We may suppose that $T = (0, \ldots, 0, 1)$. Let v_0 be as before and v_T be the Kelvin transform of u centered in T. The function v_T is also the Kelvin transform, centered in the origin, of $u_0(x_1, \ldots, x_{n-1}, 1 - x_n)$. This last function is centrally symmetric about some point different from the origin (more precisely about the point $(p_1, \ldots, p_{n-1}, 1-p_n)$, where $P = (p_1, \ldots, p_n)$). One can then apply part (i) of the Lemma to u_0. Writing u in terms of u_0 gives the desired result. ◻

Proof of Theorem 3. By Lemma 6 any Kelvin transform of u, centered at any point, is radially symmetric about some point. Then lemma 7(ii) implies the thesis. □

Proof of theorem 4. Let us argue by contradiction. Let u be an entire positive solution of $\Delta u + g(|x|, u) = 0$ and let $v(x)$ be a Kelvin transform of u centered at the origin. By proposition 6 v is radially symmetric about some point Q. **Let us prove that v satisfies the assumptions of lemma 7(i).** This will conclude the proof since the conclusions of Lemma 7 are invariant under Kelvin transformations and thus not only v but also u satisfies the conclusion of Lemma 7.

We may suppose that Q belongs to the non negative x_n semiaxis. Let us make a Kelvin transform w of v choosing a point T on the positive x_n semiaxis as the center of inversion. w is axisymmetric w.r.t. the x_n axis , to prove that it is radially symmetric it suffices then to prove that it is also symmetric w.r.t. some hyperplane whose normal θ forms an angle with the x_n axis which is an irrational multiple of π. *To prove this we use the moving plane method.*

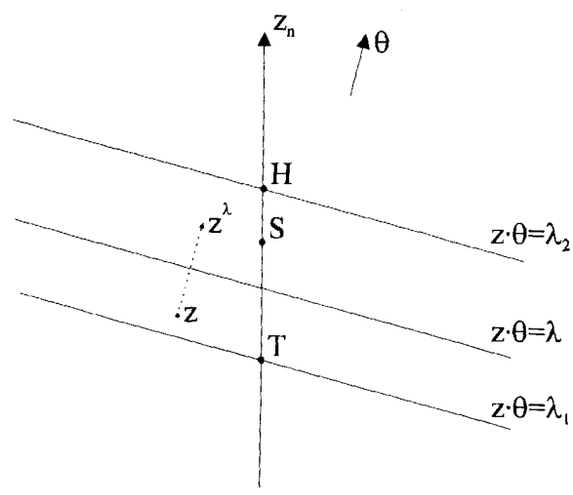

FIG 1

Let us introduce some notation. The conformal map $z(x)$ associated to the Kelvin transform centered in T maps the origin (where v may be singular) in $H := T - T/|T|$; it maps Q (the center of symmetry of v) in $S := T + (Q-T)/|Q-T|^2$, moreover $|x-Q| = d|z(x)-S|/|z(x)-T|$, $(d := |Q-T|)$.

The radial symmetry of v is thus equivalent to

$$w(z) = |z-T|^{2-n} v\left(\frac{d|z-S|}{|z-T|}\right). \tag{2.4}$$

It is important to remark that the point S lies in between T and H, see fig. 1. The function $w(z)$ is a $C^2(\mathbb{R}^n \setminus \{H\})$ solution of

$$\Delta w + (|z-H||T|)^{-n-2} f\left(\frac{|z-T|}{|z-H||T|}, (|z-H||T|)^{n-2} w\right) = 0 \quad \text{in } \mathbb{R}^n \setminus \{H\}.$$

Let us suppose also that θ points towards $\{z_n > 0\}$.

For $\lambda \in \mathbb{R}$ and for any $z \in \mathbb{R}^n$ let $z^\lambda = z + 2\theta(\lambda - z\cdot\theta)$ be the symmetric point of z with respect to the hyperplane $z\cdot\theta = \lambda$; let $\Sigma(\lambda) = \{z \in \mathbb{R}^n : z\cdot\theta \le \lambda\} \setminus \{H^\lambda\}$.

Let us start the moving plane procedure proving that $w(z) \le w(z^\lambda)$ in $\Sigma(\lambda)$ when $\lambda = \lambda_1 := \theta\cdot T$. Let now $z \in \Sigma(\lambda_1)$: the relative position of the points S and T with respect to the hyperplane $z\cdot\theta = \lambda_1$ implies that $|z-S|/d|z-T| < |z^{\lambda_1} - S|/d|z^{\lambda_1} - T|$. The function $v(r)$ is superharmonic and consequently non increasing: therefore $v(|z-S|/d|z-T|) \le v(|z^{\lambda_1} - S|/d|z^{\lambda_1} - T|)$. Since $|z-T| = |z^{\lambda_1} - T|$ (2.4) concludes the proof of the claim.

Let us now increase λ from λ_1 up to λ_2, where $\lambda_2 = H\cdot\theta$. Let $\bar\mu = \sup\{\mu \in [\lambda_1, \lambda_2) : w(z^\lambda) \ge w(z)$ in $\Sigma(\lambda)$ for any $\lambda \in [\lambda_1, \mu)\}$ and let us prove that $\bar\mu < S\cdot\theta \le \lambda_2$. To prove this we compute, using (2.4),

$$\lim_{z\cdot\theta=\lambda, |z|\to\infty} |z-T|^n \frac{\partial w}{\partial\theta} = (2-n)(\lambda - T\cdot\theta)v(d) - (S-T)\cdot\theta d v'(d).$$

The function $r^{2-n}v(1/r)$ is superharmonic and bounded from below, therefore either it is constant or its derivative is negative for r large. In the first case v already satisfies the conclusion of Lemma 7 (and there is nothing to prove), in the second case let us choose T in such a way that the previous derivative is negative in $d = |H-T|$. This implies that $(2-n)v(d) - dv'(d) < 0$ and therefore the previous limit is negative for any $\lambda < S\cdot\theta$, λ close to $S\cdot\theta$. Thus $\frac{\partial v}{\partial\theta} < 0$ in some point on the hyperplane $z\cdot\theta = \lambda$ and for those values of λ $w(z^\lambda) < w(z)$ in some point of $\Sigma(\lambda)$. This proves that $\bar\mu < S\cdot\theta$.

Let now $\lambda \in [\lambda_1, \lambda_2]$ and let $z \in S(\lambda)$. Since $|z - H| \geq |z^\lambda - H|$, $|z - T| \leq |z^\lambda - T|$ the assumptions on f imply that

$$\frac{f\left(\frac{|z-T|}{|z-H||T|}, (|z-H||T|)^{n-2}s\right)}{(|z-H||T|)^{n+2}} \leq \frac{f\left(\frac{|z-T|}{|z^\lambda-H||T|}, (|z^\lambda-H||T|)^{n-2}s\right)}{(|z^\lambda-H||T|)^{n+2}}$$
$$\leq \frac{f\left(\frac{|z^\lambda-T|}{|z^\lambda-H||T|}, (|z^\lambda-H||T|)^{n-2}s\right)}{(|z^\lambda-H||T|)^{n+2}},$$

for any $s > 0$. Via this inequality one can prove, as in the step 2 of the proof of Theorem 2, that v is symmetric with respect to the hyperplane $z \cdot \theta = \bar{\mu}$. $\square$

3. Proof of Theorem 5

The next lemma is the equivalent of lemma 7 in an half space. By l_Q we denote the line parallel to the x_n axis passing through a point $Q \in \partial\mathbb{R}^n_+$. Moreover for any point $P = (p_1, \ldots, p_n)$ let P' denote $(p_1, \ldots, p_{n-1})$.

Lemma 8. *Let u be a function bounded and positive on $\overline{\mathbb{R}^n_+}$ and let us suppose that either*

(i) *u is axisymmetric about some line l_P and also its Kelvin transform centered in a point which belongs to $\partial\mathbb{R}^n_+$ and which is different from P is axisymmetric about some line l_Q or*

(ii) *the Kelvin transform of u centered in the origin is axisymmetric about some line l_P and also the Kelvin transform of u centered in a point $T \in \partial\mathbb{R}^n_+$, $T \neq P$, $\|T\| = 1$, is axisymmetric about some line l_Q*

then either

- *u depends only on x_n or*
- *there exists a point $x_0 \in \partial\mathbb{R}^n_+$, a constant $\lambda > 0$ and a family of disjoint spheres $S(d) := \{|x' - (x_0)'|^2 + (x_n - d)^2 = d^2 - \lambda^2\}$ ($d \in (\lambda, +\infty)$) whose union is $\mathbb{R}^n_+$, such that on each sphere $S(d)$*

$$u(x) = \frac{k(d)}{(|x - x_0|^2 + \lambda^2)^{(n-2)/2}} \tag{3.1}$$

for some constants $k = k(d) > 0$ (see fig.2).

Remark. The last assertion of the previous lemma is equivalent to the radial symmetry about some point of a suitable Kelvin transform of u (see the proof of theorem 5, second case, for the details).

Proof. Let us prove the result under assumption (i).

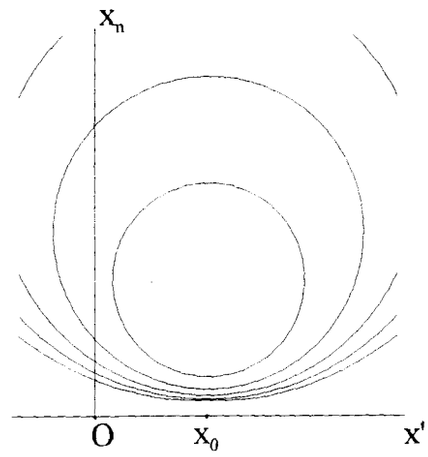

FIG 2: the spheres $S(d)$

Let v_0 denote the Kelvin transform in the statement. To simplify the notations let us assume that the distance from P to the center of inversion of v_0 is 1, in the general case the proof works similarly. After a suitable rotation and translation we may suppose that $P = (1, 0, \ldots, 0)$ and that the center of inversion of v_0 is the origin.

The conformal transformation $z = x/|x|^2$ maps the $(n-2)$-dimensional sphere $\{|x' - P'| = a, x_n = b\}$ (a, b non-negative constants) in a $(n-2)$-dimensional sphere, which we denote by $\Sigma(a, b)$.

The symmetry of u implies that $u(x) = f(|x' - P'|, x_n)$ for a suitable function f. Thus, for any $a, b \geq 0$,

$$v_0(z) = f(a, b)|z|^{2-n} \quad \text{for any } z \in \Sigma(a, b)\,. \tag{3.2}$$

By assumption v_0 is axisymmetric about l_Q. The set $\Sigma(a, b)$ is symmetric about the hyperplane containing the z_1 and the z_n axes. Thus (3.2) implies that Q lies on the z_1 axis. Let $Q = (q, 0, \ldots, 0)$. If $q > 1$ or $q < 0$ then, for a suitable value of a, the sphere $\Sigma(a, 0)$ lies on the hyperplane $\{x_n = 0\}$ and it is centered in Q: (3.2) would then contradict the radial symmetry of $v(z', 0)$ about Q. Therefore $q \in [0, 1]$.

The symmetry of v_0 implies that its value on each set $\{|z' - Q'| = e, z_n = p\}$ (e, p non negative constants) is constant: if such a set intersect $\Sigma(a, b)$ the value of v_0 on such a set it is determined, through (3.2), by the value of v_0 in the intersection of that set with $\Sigma(a, b)$. If for a given (a, b) we denote by

$I(a,b)$ the set of (e,p) such that $\{|z'-Q'|=e, z_n=p\}$ intersect $\Sigma(a,b)$ one easily computes that for any z such that $(|z'-Q'|, z_n) \in I(a,b)$ it is

$$v_0(z) = f(a,b)(b/z_n)^{\frac{n-2}{2}}.$$

In particular this representation implies that if the two sets $I(a_1,b_1)$ and $I(a_2,b_2)$ intersect for $(a_1,b_1) \neq (a_2,b_2)$ then $f(a_1,b_1)b_1^{(n-2)/2} = f(a_2,b_2)b_2^{(n-2)/2}$.

For $r > (1-q)/q$ if $q > 0$, $r > 0$ otherwise, let $B(r)$ denote the set $\{(a,b) : 2br = 1-q(1-a^2-b^2)\}$. We claim that if $(a_1,b_1),(a_2,b_2) \in B(r)$ it is possible to find a finite number of couples (α_i,β_i), $i=1,\dots,n$, such that $(\alpha_1,\beta_1)=(a_1,b_1)$, $(\alpha_n,\beta_n)=(a_2,b_2)$ and $I(\alpha_i,\beta_i)\cap I(\alpha_{i+1},\beta_{i+1}) \neq \emptyset$ (or, equivalently, such that for each i there is a set of the form $\{|z'-Q'|=e, z_n=p\}$ that intersect both $\Sigma(\alpha_i,\beta_i)$ and $\Sigma(a_{i+1},\beta_{i+1})$). This is an elementary fact which is due to the relative positions of the (n-2)-spheres $\Sigma(a,b)$ and of the sets of the form $\{|z'-Q'|=e, z_n=p\}$.

As a consequence

$$f(a,b)b^{(n-2)/2} \text{ is constant on } B(r). \tag{3.3}$$

We recall that $u(x) = f(|x'-P'|, x_n)$. When $q=0$ (3.3) implies that u depends only on x_n. When $q \in (0,1]$ (3.3) implies that on each sphere $\{|x'-P'|^2 + (x_n - r/q)^2 = (r/q)^2 - (1-q)/q\}$

$$u(x) = \frac{k(r)}{x_n^{\frac{n-2}{2}}}$$

for some constants $k = k(r)$. Finally let $d = r/q$ and $\lambda^2 = (1-q)/q$: to conclude the proof of lemma 8(i) it suffices to observe that on each of those spheres x_n is a multiple of $\lambda^2 + |x-P|^2$.

Lemma 8(ii) follows from lemma 8(i) in the same way that lemma 7(ii) follows from lemma 7(i). □

Lemma 9. *Let u, g and h be as in theorem 5. Then $|x|^{2-n}u(x/|x|^2)$ is axisymmetric about some line parallel to the x_n axis.*

Proof. The function $v(z) = |z|^{2-n}u(z/|z|^2)$ is a solution of the following problem

$$\begin{cases} \Delta v + \dfrac{g(|z|^{n-2}v)}{|z|^{n+2}} = 0 & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial v}{\partial z_n} = \dfrac{h(|z|^{n-2}v)}{|z|^n} & \text{in } \partial\mathbb{R}^n_+ \setminus \{0\}, \end{cases} \tag{3.4}$$

which decays to 0 at infinity as $|z|^{2-n}$ and which might be singular in the origin. Theorem 1.1 in [16] implies that a solution to problem (3.4) *which is regular at the origin* is axisymmetric about some line parallel to the z_n axis. This is proved through the moving plane method. Lemma 9 can be proved as Theorem 1.1 in [16] with the help of the following lemma. We omit the details.

For $\lambda > 0$ and for any $z \in \mathbb{R}^n_+$ let $z^\lambda = (2\lambda - z_1, z_2, \dots, z_n)$, $\Sigma(\lambda) = \{z \in \mathbb{R}^n_+ : z_1 \le \lambda\}$. Finally let $w(z, \lambda) = v(z^\lambda) - v(z)$.

Lemma 10.

(i) *The function v is larger than a positive constant in some neighborhood of the origin.*

(ii) *Let us suppose that for some $\lambda < 0$ the function $w(\cdot, \lambda)$ is non negative in $\Sigma(\lambda)$. Then either $w \equiv 0$ or w is larger than a positive constant in some neighborhood of 0^λ.*

Proof of Lemma 10. This proof is based on ideas presented in the proof of Lemma 2.1 in [13].

Let us prove the first claim. First let us remark that the assumptions on h imply that $h(0) \le 0$. The constant $\varepsilon := \min_{|z|=1, z_n \ge 0} v$ is positive, since v is positive in $\mathbb{R}^n_+$ and v can not vanish on $\partial\mathbb{R}^n_+ \setminus \{0\}$, due to the Hopf boundary Lemma and to the sign of $h(0)$.

Let $c = \max\{h(\varepsilon/2)/(\varepsilon/2), 0\}$ and, for $\delta \in (0, 1)$, let

$$\phi(z) = \frac{\varepsilon}{2(c+1)} + \frac{\varepsilon}{2} z_n - \varepsilon \left(\frac{\delta}{|z|}\right)^{2-n}.$$

Let us prove that $v \ge \phi$ in $\{\delta \le |z| \le 1, z_n \ge 0\}$. Since $v - \phi$ is superharmonic it suffices to prove that $v \ge 0$ on the boundary of that set. It can be easily checked that this is true on $\{|z| = 1, z_n \ge 0\}$ (where $v \ge \varepsilon \ge \phi$) and also on $\{|z| = \delta, z_n \ge 0\}$ (where $\phi \le 0$, if δ is chosen small enough). Let us assume, by contradiction, that $\min_{\delta < |z| < 1, z_n = 0}(v - \phi)$ is negative. Let z_0 be a point where that minimum is attained. Then

$$v(z_0) < \phi(z_0) \le \frac{\varepsilon}{2(c+1)}. \tag{3.5}$$

On the other hand the derivative of $(v - \phi)$ with respect to z_n should be non negative in z_0, that is $h(|z_0|^{n-2} v(z_0))/|z_0|^n \ge \varepsilon/2$. Since $h(t)/t^{n/(n-2)}$ is non decreasing this implies

$$\frac{h(\varepsilon/2)}{\varepsilon/2} v(z_0) \ge \frac{h(|z_0|^{n-2} v(z_0))}{(|z_0|^{n-2} v(z_0))^{\frac{n}{n-2}}} v(z_0)^{\frac{n}{n-2}} \ge \varepsilon/2.$$

Due to the definition of c this inequality contradicts (3.5). We have thus proved that $v \geq \phi$ in $\{\delta \leq |z| \leq 1, z_n \geq 0\}$. Letting $\delta \to 0$ we obtain

$$v(z) \geq \frac{\varepsilon}{2(c+1)} + \frac{\varepsilon}{2} z_n$$

which concludes the proof of the first claim.

The second claim can be proved similarly. Since $v(z^\lambda) \geq v(z)$ in $\Sigma(\lambda)$ and g is non-decreasing the function $w(z) := v(z^\lambda) - v(z)$ is superharmonic. Moreover it satisfies the following boundary condition on $\overline{\Sigma(\lambda)} \cap \{z_n = 0\} \setminus \{0^\lambda\}$

$$\frac{\partial w}{\partial z_n} = \frac{h(|z^\lambda|^{n-2} v(z^\lambda))}{|z^\lambda|^n} - \frac{h(|z|^{n-2} v(z))}{|z|^n}. \tag{3.6}$$

Since $h(t)/t^{n/(n-2)}$ is non decreasing, (3.6) implies that

$$\frac{\partial w}{\partial z_n} \leq \frac{h(|z|^{n-2} v(z^\lambda)) - h(|z|^{n-2} v(z))}{|z|^n (v(z^\lambda) - v(z))} w,$$

at any point $z \in \overline{\Sigma(\lambda)} \cap \{z_n = 0\} \setminus \{0^\lambda\}$ where $v(z^\lambda) > v(z)$, and

$$\frac{\partial w}{\partial z_n} \leq 0 \tag{3.7}$$

at any point $z \in \overline{\Sigma(\lambda)} \cap \{z_n = 0\} \setminus \{0^\lambda\}$ where $v(z^\lambda) = v(z)$

To simplify the notations let us assume that $\lambda < -1$: in this case the distance from 0^λ to the hyperplane $\{z_1 = \lambda\}$ is larger than 1 and $B^+(0^\lambda, 1) := \{0 < |z - 0^\lambda| \leq 1, z_n \geq 0\}$ is contained in $\Sigma(\lambda)$.

Let $\varepsilon := \min_{|z-0^\lambda|=1, z_n \geq 0} w$: if $w \not\equiv 0$ then $\varepsilon > 0$. This is true since w can not vanish in any point in the interior of $\Sigma(\lambda)$ due to the strong maximum principle and w can not vanish on $\partial \mathbb{R}^n_+ \setminus \{0^\lambda\}$, due to the Hopf boundary Lemma and to (3.7).

We have already observed that v can not vanish on $\partial \mathbb{R}^n_+ \setminus \{0\}$: let $m = \min_{B^+(0^\lambda,1) \cap \{z_n = 0\}} v$. The function h is locally lipschitz in $(0, +\infty)$ and therefore the supremum of

$$\frac{h(|z|^{n-2} t) - h(|z|^{n-2} s)}{|z|^n (t - s)}$$

where $t, s \in [m, m + \varepsilon/2]$, $t > s$, $r \in B^+(0^\lambda, 1) \cap \{z_n = 0\}$ is bounded. Let c be that supremum and let ϕ be defined as above (with $|z|$ replaced by $|z - 0^\lambda|$). The proof of the second claim of lemma 10 can now be concluded exactly as that of the first one. ∎

Proof of Theorem 5. Let u be a solution to problem (0.5). Due to Lemma 9, u satisfies the assumptions of Lemma 8(ii). We can thus distinguish two cases: u depends only on x_n or u satisfies (3.1).

First case If $u = u(x_n)$ then it satisfies

$$\begin{cases} \dfrac{d^2u}{dx_n^2} + g(u) = 0 \quad \text{in } (0, +\infty) \\ \dfrac{du}{dx_n}(0) = h(u)\,. \end{cases}$$

If $g \equiv 0$ then $u(x_n) = ax_n + b$ with $a \geq 0$, $b > 0$ and $a = h(b)$.

If $g \not\equiv 0$ the previous problem has no solution. This is true since u is concave and to be positive on $(0, +\infty)$ it should be non decreasing. If this happens $g(u)$ is larger than a positive constant (g is non decreasing) and the equation implies that d^2u/dx_n^2 is less than a negative constant. Therefore u can not be non decreasing in $[0, +\infty)$.

Second case In this case let us consider the conformal transformation

$$z = T + \frac{x - T}{|x - T|^2} 4\lambda^2,$$

where $T = (P', -\lambda)$. This transformation maps $\mathbb{R}^n_+$ in the ball $|z - Q| \leq 2\lambda$, where $Q = (P', \lambda)$, and it maps each sphere $S(d)$ in a sphere centered in Q (see (3.1) for the definition of $S(d)$). Moreover the identities $u(x) = k(d)/(|x - P|^2 + \lambda^2)^{(n-2)/2}$ are equivalent to the radial symmetry about Q of the function

$$v(z) := \left(\frac{|z - T|}{2\lambda}\right)^{\frac{n-2}{2}} u(x(z))$$

The function v satisfies the following nonlinear Neumann problem:

$$\begin{cases} \Delta v + \left(\dfrac{|z - T|}{2\lambda}\right)^{-n-2} g\left(\left(\dfrac{|z - T|}{2\lambda}\right)^{n-2} v\right) = 0 \quad \text{on } |z - Q| \leq 2\lambda, \\ \dfrac{(2-n)v}{4\lambda} - \dfrac{dv}{d\nu} = \left(\dfrac{|z - T|}{2\lambda}\right)^{-n} h\left(\left(\dfrac{|z - T|}{2\lambda}\right)^{n-2} v\right) \quad \text{on } |z - Q| = 2\lambda, \end{cases}$$

(ν denote the unit exterior normal to the boundary of $|z - Q| \leq 2\lambda$). Since v is radially symmetric about Q also the nonlinear terms in the equation and in the boundary condition should be radially symmetric. Since $T \neq Q$ and the value of v on the boundary is different from zero (since $h(0) \leq 0$) it should be

$$h(t) = dt^{\frac{n}{n-2}}$$

for any t in the range $\{(|z-T|/2\lambda)^{n-2}v(z) : |z-Q| = 2\lambda\}$ that is for any $t \in (0, \max_{\partial\mathbb{R}^n_+} u]$. Similarly

$$g(t) = ct^{\frac{n+2}{n-2}}$$

for some $c \geq 0$, and for any $t \in (0, \max_{\mathbb{R}^n_+} u]$.

If $c = 0$ then v is constant and

$$u(x) = \frac{k}{|x-T|^{n-2}}$$

for some $k > 0$. A posteriori one computes that this satisfies the boundary condition of problem (0.5) if and only if $dk^{2/(n-2)} = (2-n)|T_n|$, where T_n is the n-th component of T. Note that this forces d to be negative.

If $c > 0$ then, by the uniqueness for the ODE problem, v is a multiple of $(|z-Q|^2+\sigma^2)^{-(n-2)/2}$ for some $\sigma > 0$. Expressing u in terms of v, using the relation $|z-Q| = 2\lambda|x-Q|/|x-T|$ and by a simple computation, one gets

$$u(x) = \frac{k}{(|x-x_0|^2+\mu^2)^{\frac{n-2}{2}}}$$

for some positive constants k and μ and some point $x_0 \in \mathbb{R}^n$. A posteriori one computes that this satisfies the equation of problem (0.5) if and only if $k^{4/(n-2)}c = (n-2)n\mu^2$ and the boundary condition if and only if $bk^{2/(n-2)} = (n-2)(x_0)_n$, where $(x_0)_n$ is the n-th component of x_0. □

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