NON-EXISTENCE AND SYMMETRY OF SOLUTIONS TO THE SCALAR CURVATURE EQUATION

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0. Introduction

The problem of understanding whether any entire positive solution of $\Delta u + f(|x|, u) = 0$, which decays to 0 at infinity, is radial has received much attention. B. Gidas, W.M. Ni and L. Nirenberg proved in [6] that when K is radial and strictly decreasing and $p > \frac{n+1}{n-2}$ then all C^2 positive solutions of $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$ which decay at infinity like $|x|^{2-n}$ are radial (see also [9], [10] and [4] for related results).

In this paper we consider the scalar curvature problem on $\mathbb{R}^n$,

$$\begin{cases} \Delta u + K(x)u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n \\ u > 0, \quad u(x) = O(|x|^{2-n}) \text{ as } x \to \infty\,, \end{cases} \tag{0.1}$$

where $n \geq 3$ and K is a continuous positive function. We prove the following result.

Theorem 1. *Let us assume that $K(x) = K(|x|)$ and that $K(r) \in C^0([0,\infty])$. Let us suppose that there exist $\bar{r} > 0$ such that $K(r)$ is non-increasing in $(0, \bar{r}]$ and non decreasing in $[\bar{r}, \infty)$. Then any C^2 solution of (0.1) is radial.*

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This result is very sensitive to small perturbations of K. In [B, Theorem 3] the author displayed a radial function $\tilde{K}$ such that (0.1) has non-radial solutions but no radial solution. This $\tilde{K}$ results from adding a small bump to a certain radial function, which is decreasing in $(0,1]$ and increasing in $[1,\infty)$. The C^1-distance of $\tilde{K}$ to a function which satisfies the assumptions of theorem 1 can be made arbitrarily small.

Since $K(r)$ is not decreasing, the moving plane method seems not suitable for proving theorem 1. We will use, as an essential tool, the invariance of problem (1) under conformal transformations.

A natural question is whether the same symmetry result is true when K is non decreasing in $(0,\bar{r}]$ and non increasing in $[\bar{r},\infty)$. The author does not have an answer to this question.

Theorem 1 allows us to prove a non-existence result for problem (0.1), and for the corresponding problem on S^n, that is

$$\text{(0.2)} \qquad \begin{cases} \Delta u - \dfrac{n(n-2)}{4} u + R(x) u^{\frac{n+2}{n-2}} = 0 \quad \text{in } S^n \,, \\ u > 0 \,, \quad u \in H^1(S^n) \,. \end{cases}$$

Let $\mathcal{D}^{1,2}(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with the norm $\|\nabla \cdot\|_{L^2}$.

Theorem 2. *Take two numbers ρ_1, ρ_2 larger than $\frac{n(n-2)}{n+2}$ and such that $\frac{1}{\rho_1} + \frac{1}{\rho_2} \geq \frac{2}{n-2}$. Take two positive numbers K_0, K_∞. Then there exists a positive radial function K, which is smooth in $(0,+\infty)$ and satisfies*

$$K(x) = K_0 - \varepsilon |x|^{\rho_1} \quad \text{near } 0,$$
$$K(x) = K_\infty - \varepsilon |x|^{-\rho_2} \quad \text{near } \infty,$$

(for some $\varepsilon > 0$), such that (0.1) has no solution in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. The function $K(r)$ is decreasing in $(0,1]$ and increasing in $[1,\infty)$, and $K(1)$ can be made arbitrarily close to 0.

The author and H. Egnell were able to prove Theorem 2 (see [2, Theorem 0.3]) only in the class of radial solutions, (i.e. to prove the non-existence of radial solutions for such a K). Due to Theorem 1 we can now remove that restriction. We remark that one can apply theorem 1 since any positive solution of $\Delta u + K(x) u^{(n+2)/(n-2)} = 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ is a C^2 solution.

If R is the function on S^n corresponding to K via the stereographic projection, the existence of a solution for (0.1) is equivalent to the existence of one for (0.2).

Little is known about the non-existence of solutions to problem (0.2). Kazdan and Warner in [7] proved that every solution of (0.2) must satisfy $n+1$ integral conditions. These imply that when R is a monotonic function of the distance to a point on the sphere this problem has no solution. See also [3] for a result in the same spirit.

Escobar and Schoen, in [5], proved that when R is symmetric with respect to the antipodal map and it has an absolute maximum point where the first $n-2$ derivatives of R vanish, then (0.2) has a solution. Theorem 2, rephrased for problem (0.2), shows that the assumption about the maximum point is really needed. Infact when $\rho_1 = \rho_2 = n-2$ and $K_0 = K_\infty$ the function K in theorem 2 is such that, when transformed on S^n, it is symmetric with respect to the antipodal map.Very recently Yan Yan Li [8] has proved an existence result for (0.2) based on a counting index criteria. He assumes that R is very flat near all its critical points. The flatness condition in each critical point is closely related to the condition which appears in the result of Escobar and Schoen and also to the conditions in theorem 2.

In [1] the author has proved the following existence result.

Theorem A ([1]). *Let $R \in C^2(S^n)$ and let us suppose that $R_{max} := \max_{S^n} R$ is achieved in two isolated points y, z at least. Moreover let us assume that R has only finitely many critical points and that if w is a critical point for R with $R(w) \in (2^{-\frac{2}{n-2}} R_{max}, R_{max})$ then either $\Delta R(w) < 0$ and w is a strict local maximum or $\Delta R(w) > 0$. If either*

i) there exist two constants $c > 0$ and $\rho > n-2$ such that

$$R(x) \geq R_{max} - c|x-y|^\rho \quad \textit{for any } x \textit{ in a neighborhood of } y,$$
$$R(x) \geq R_{max} - c|x-z|^\rho \quad \textit{for any } x \textit{ in a neighborhood of } z,$$

or

ii) there exists a continuous path $x(t)$ in S^n, connecting y and z, with $\min_t R(x(t)) > 2^{-\frac{2}{n-2}} R_{max}$,

then there exists a solution to problem (0.2).

This theorem still holds if condition i) is substituted by the slightly more general one:

i') there exist two constants ρ_1, ρ_2 larger than $\frac{n(n-2)}{n+2}$ which satisfy $\frac{1}{\rho_1} + \frac{1}{\rho_2} < \frac{2}{n-2}$ and such that

$$R(x) \geq R_{max} - c|x-y|^{\rho_1} \quad \text{for any } x \text{ in a neighborhood of } y,$$
$$R(x) \geq R_{max} - c|x-z|^{\rho_2} \quad \text{for any } x \text{ in a neighborhood of } z,$$

for some $c > 0$.

Note that when $n = 3$ condition i) is automatically satisfied by any regular R. The same is true also for the flatness conditions appearing in [5] and [8].

The next theorem implies that when $n > 3$ one of the assumptions i') or ii) is really needed and it also confirms the fact that further complications arise when passing from the case $n = 3$ to the case $n > 3$.

Corollary 3. *Let $n \geq 3$. Take two distinct points y and z in S^n and two numbers ρ_1, ρ_2 larger that $\frac{n(n-2)}{n+2}$, such that $\frac{1}{\rho_1} + \frac{1}{\rho_2} \geq \frac{2}{n-2}$. Take two positive numbers R_y, R_z. Then there exists a positive function R, with $R \in C^\infty(S^n \setminus \{y, z\})$ and*

$$R(x) = R_y - \varepsilon|x - y|^{\rho_1} \quad \text{for } x \text{ in a neighborhood of } y,$$
$$R(x) = R_z - \varepsilon|x - z|^{\rho_2} \quad \text{for } x \text{ in a neighborhood of } z,$$

(for some $\epsilon > 0$), such that (0.2) has no solution.

The value of such a R on its critical points different from y and z can be made arbitrarily close to 0.

Clearly such a function R cannot satisfy condition ii) in theorem A.

If $n \geq 4$ and $\rho_1 = \rho_2 = n - 2$ the function $R \in C^2(S^n)$, while if $n = 3$, for any choice of ρ_1 and ρ_2, the derivatives of R are not continuous in z or in y.

Corollary 3 follows immediately from Theorem 2 via a conformal map.

1. Proof

Proof of theorem 1. If K is identically constant then all C^2 solutions of (0.1) are known to be radial (see [6]).

Let us assume that K is not identically constant.

Let $x_1, \ldots, x_{n+1}$ be a coordinate system in $\mathbb{R}^{n+1}$ and let S^n be embedded in $\mathbb{R}^{n+1}$ in such a way that the north pole N has coordinate $(0, \ldots, 0, 1)$ and the south pole $S = (0, \ldots, 0, -1)$. Let $R(x)$ be the function on S^n corresponding to K via the stereographic projection $s : S^n \setminus \{N\} \to \mathbb{R}^n$. Since K is radially symmetric R is invariant under the group of rotations about the $N - S$ axis, that is $R(x) = R(x_{n+1})$.

Without any loss of generality we can suppose $\bar{r} = 1$. Therefore R is non decreasing for $x_{n+1} \leq 0$ and non increasing for $x_{n+1} \geq 0$.

Let u be a C^2 solution of (0.1). It is well known that

$$v(x) = u(s(x))\left(\frac{1 + |s(x)|^2}{2}\right)^{\frac{n-2}{2}}$$

is a positive C^2 solution to

$$\Delta v - \frac{n(n-2)}{4}v + R(x)v^{\frac{n+2}{n-2}} = 0\,.$$

We will prove that $v(x) = v(x_{n+1})$. To prove this it suffices to show that v is invariant under the reflection about any hyperplane containing N and S.

Let Π be such a hyperplane. We can suppose without any loss of generality that $\Pi = \{x_2 = 0\}$: we will prove that

$$v(x_1, x_2, x_3, \dots, x_{n+1}) = v(x_1, -x_2, x_3, \dots, x_{n+1}) \,. \tag{1.1}$$

Let $\tilde{N} = (1, 0, \dots, 0)$. Let $\tilde{s}$ be the stereographic projection defined choosing the point $\tilde{N}$ as the north pole. If $y_1, y_2, \dots, y_n$ are euclidean coordinates in $\mathbb{R}^n$ we have

$$\begin{cases} y_i = \dfrac{x_{i+1}}{1 - x_1} \quad i = 1, \dots, n, \end{cases} \qquad \begin{cases} x_1 = \dfrac{|y|^2 - 1}{1 + |y|^2} \\ x_i = \dfrac{2y_{i-1}}{1 + |y|^2} \quad i = 2, \dots, n+1 \end{cases}$$

Let $\tilde{K}$ denote the function on $\mathbb{R}^n$ corresponding to R via $\tilde{s}$: we have

$$\tilde{K}(y) = R(\frac{2y_n}{1 + |y|^2}).$$

The function $\tilde{K}$ is clearly symmetric with respect to the hyperplane $\{y_1 = 0\}$: let us prove that, as a function of y_1, it is non increasing for $y_1 \geq 0$ and non decreasing for $y_1 \leq 0$. It suffices (by approximation) to prove it under the assumption that $R \in C^1$. It is

$$\frac{\partial \tilde{K}}{\partial y_1} y_1 = -4R'(\frac{2y_n}{1 + |y|^2}) \frac{y_n y_1^2}{(1 + |y|^2)^2},$$

and the right-hand side is non-positive, since $R'(x_{n+1}) x_{n+1} \geq 0$. Let

$$\tilde{u}(y) = v(\tilde{s}^{-1}(y)) (\frac{2}{1 + |y|^2})^{\frac{n-2}{2}}.$$

The function $\tilde{u}$ is a C^2 positive solution of $\Delta \tilde{u} + \tilde{K} \tilde{u}^{\frac{n+2}{n-2}} = 0$ in $\mathbb{R}^n$ which decays at infinity as $|y|^{2-n}$. By a result proved by Congming Li [4, Theorem 5], $\tilde{u}(y_1, \cdot, \dots, y_n)$ is symmetric about some hyperplane $\{y_1 = \mu\}$. Then $\tilde{K} = -\Delta \tilde{u} / \tilde{u}^{(n+2)/(n-2)}$ has the same symmetry and this, together with the assumptions on K (in particular the fact that K is not identically constant) implies that $\mu = 0$. Therefore

$$\tilde{u}(y_1, y_2, \dots, y_n) = \tilde{u}(-y_1, y_2, \dots, y_n),$$

that is v satisfies (1.1). Since Π can be chosen arbitrarily, we have in fact that $v(x) = v(x_{n+1})$. Hence u is radial. $\square$

We remark that when $n = 2$ the composition $\tilde{s} \circ s$ is the Moebius conformal map $\zeta = (\bar{z} + 1)/(\bar{z} - 1)$.

ACKNOWLEDGMENTS

The author thanks his friend Henrik Egnell for all the things he learned working with him.

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