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CUP reference:

Date of delivery:

Journal and Article number: JLM 603

Volume and Issue Number: 00(0)

Colour Figures: Nil

Number of pages (not including this page): 18

Author queries:

Typesetter queries:

Non-printed material:
MATHERON’S CONJECTURE
FOR THE COVARIogram PROBLEM

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Dedicated to my teacher, Carlo Pucci

Abstract

The covariogram of a convex body \( K \) provides the volumes of the intersections of \( K \) with all its possible translates. Matheron conjectured in 1986 that this information determines \( K \) among all convex bodies, up to certain known ambiguities. It is proved that this is the case if \( K \subset\mathbb{R}^2 \) is not \( C^1 \), or it is not strictly convex, or its boundary contains two arbitrarily small \( C^2 \) open portions ‘on opposite sides’. Examples are also constructed that show that this conjecture is false in \( \mathbb{R}^n \) for any \( n \geq 4 \).

1. Introduction

Let \( K \) be a convex body in \( \mathbb{R}^n \). (See Section 2 for all unexplained definitions and notation.) The covariogram \( g_K(x) \) of \( K \) is the function

\[
g_K(x) = V(K \cap (K + x)),
\]

where \( x \in \mathbb{R}^n \) and \( V \) denotes volume in \( \mathbb{R}^n \). The covariogram is clearly unchanged by a translation or a reflection of \( K \). (The term reflection will always mean reflection in a point.) The author of this paper was motivated by the following problem, posed by Matheron in 1986 (compare [10, p. 86; 11]). Does the covariogram determine a convex body, among all convex bodies, up to translation and reflection?

The same problem was posed independently in the context of probability theory. Adler and Pyke [1] asked whether the distribution of the difference \( X - Y \) of two independent random variables \( X, Y \) that are uniformly distributed over \( K \) determines \( K \) up to translation and reflection. Since it is easily proved that

\[
g_K = 1_K * 1_{-K}, \tag{1.1}
\]

and this convolution is, up to a multiplicative factor, the probability density of \( X - Y \), this problem is equivalent to the previous one. The same authors [2] also found the covariogram problem relevant to the study of certain Brownian processes (scanning Brownian processes).

The problem also arises in Fourier analysis. The phase retrieval problem involves determining a function \( f \) with compact support in \( \mathbb{R}^n \) from the modulus of its Fourier transform \( \hat{f} \). It is of great relevance in many applications, and there is a vast literature (see [9] for a survey). Taking Fourier transforms in (1.1) and using the relation

\[
\overline{1_{-K}(\xi)} = \overline{1_K(\xi)},
\]

Received 27 April 2003.

2000 Mathematics Subject Classification 52A20 (primary), 60D05, 52A22, 52A38 (secondary).
we obtain
\[
\hat{g}_K(\xi) = \hat{1}_K(\xi) \hat{1}_{-K}(\xi) = |\hat{1}_K(\xi)|^2.
\] (1.2)

Thus the phase retrieval problem reduces to the covariogram problem when \(f\) is the characteristic function of a convex body.

See [4] for a more detailed introduction to the covariogram problem and related problems. In the update of the book [6] (which can be found at http://www.ac.wwu.edu/~gardner), there is a reference to the covariogram problem, and more can be found about it in [7].

A first contribution to the question posed by Matheron was made in 1993 by Nagel [12, 13], who gave a positive answer when \(K\) is a planar convex polygon (see also [3] for a much shorter proof). Schmitt [15] proved the same result for a class of planar polygons which also include some nonconvex ones.

More recently, the present author, Segala and Volčič [4] confirmed that if \(K\) is a planar convex body whose boundary is \(C^2\) (that is, \(C^2\) with strictly positive curvature), then \(K\) is determined, among all convex bodies, by its covariogram (up to translation and reflection).

In this paper we present another step towards the confirmation of Matheron’s conjecture in the plane. We say that two open arcs on the boundary of a convex body \(K\) are opposite if there is a point in one arc and a point in the other arc which have opposite outer normals.

**Theorem 1.1.** Let \(K\) be a plane convex body such that one of the following holds.

1. \(K\) is not strictly convex.
2. \(K\) is not \(C^1\).
3. \(\partial K\) contains two opposite open arcs which are \(C^2\).

The covariogram determines \(K\), up to translations or reflections.

Very little is known regarding the covariogram problem when the dimension is larger than 2. It is known that centrally symmetric convex bodies, in any dimension, are determined by their covariogram, up to translations. This is a consequence of the fact that \(g_K\) determines the volume of \(K\) (=\(g_K(0)\)) and its difference body \(D_K\) (it coincides with the support of \(g_K\)) and of the Brunn–Minkowski inequality. This inequality implies that among all convex bodies with the same difference body, the centrally symmetric one is the only set of maximal volume (see [6, Theorem 3.2.3]). Goodey, Schneider and Weil [8] proved that the covariogram also determines each member in a certain dense subclass of the class of convex polytopes.

We are able to construct examples that show that Matheron’s conjecture is false in \(\mathbb{R}^n\), for any \(n \geq 4\).

**Theorem 1.2.** Let \(K \subset \mathbb{R}^n\) and \(H \subset \mathbb{R}^k\) be convex bodies and let \(\mathcal{L}\) be a nondegenerate linear transformation. Then the two convex bodies \(\mathcal{L}(K \times H)\) and \(\mathcal{L}(K \times (-H))\) in \(\mathbb{R}^{n+k}\) have the same covariogram.

If neither \(K\) nor \(H\) are centrally symmetric, then \(\mathcal{L}(K \times H)\) cannot be obtained from \(\mathcal{L}(K \times (-H))\) through a translation or a reflection.
Corollary 1.3. For each \( n \geq 4 \), there exist pairs of convex bodies in \( \mathbb{R}^n \) which have the same covariogram and which are not a translation or a reflection of each other.

Two finite multisets (sets with repetitions allowed) \( E \) and \( F \) in \( \mathbb{R}^n \) are said to be homometric if the multisets of the vector differences \( \{x - y : x, y \in E\} \) and \( \{x - y : x, y \in F\} \) are identical, counting multiplicities. The problem of determining all the multisets which are homometric to a given multiset arises naturally in the phase retrieval problem for certain classes of unknown functions (see \([14]\)). It is known that if \( U \) and \( V \) are any two finite sets, then the two multisets \( U + V \) and \( U + (-V) \) are homometric. This construction suggested the examples in Theorem 1.2 to us.

To conclude, we mention two results which are intermediate steps in the proof of Theorem 1.1 and which are, in our opinion, in the context of this problem, of interest in themselves.

Proposition 1.4. Let \( H, K \) be plane convex bodies with equal covariograms, and let us assume that \( \partial K \cap \partial H \) contains an open arc. Then \( H \) is a translation or a reflection of \( K \).

Lemma 1.5. Let \( K \) and \( H \) be plane convex bodies with equal covariograms. Any parallelogram \( P \) inscribed in \( K \) has a translate which is inscribed in \( H \).

The paper is organized as follows. Section 3 contains some preliminary lemmas and the proof of Lemma 1.5. Sections 4, 5 and 6 contain the proof of Theorem 1.1 under assumptions (1), (2) and (3) respectively. Proposition 1.4 is proved in Section 5, and Theorem 1.2 in Section 7.

2. Definitions, notation and preliminaries

As usual, \( S^{n-1} \) denotes the unit sphere and \( o \) the origin in Euclidean \( n \)-space \( \mathbb{R}^n \). The Euclidean norm is written \( \| \cdot \| \). If \( u \in S^{n-1} \), we denote by \( u^{\perp} \) the \((n - 1)\)-dimensional subspace orthogonal to \( u \), and we denote by \( l_u \) the line through the origin parallel to \( u \). We write \([x, y] \) for the line segment with endpoints \( x \) and \( y \).

If \( A \) is a set, we denote by \( \text{int} \, A \) and \( \partial A \) the interior and boundary of \( A \), respectively. The characteristic function of \( A \) is denoted by \( 1_A \). The reflection of \( A \) in the origin is \( -A \). The convex hull \( \text{conv} \, A \) is the smallest convex set containing \( A \).

A convex body is a compact convex set with nonempty interior. If \( K \) is a \( k \)-dimensional convex body in \( \mathbb{R}^n \), we denote its volume, that is, its \( k \)-dimensional Lebesgue measure, by \( V(K) \). The difference body \( DK \) of \( K \) is defined by

\[
DK = K + (-K) = \{x + y : x \in K \text{ and } y \in -K\}.
\]

If \( x \in \partial K \), the normal cone of \( K \) at \( x \) is denoted by \( N_K(x) \) and is the set of all outer normal vectors to \( K \) at \( x \), together with \( o \). For \( u \in S^{n-1} \), the face \( F_K(u) \) is the set of points in \( \partial K \) with \( u \) as a unit outer normal vector. In Section 6, where \( K \) is assumed to be strictly convex and \( F_K(u) \) is always a point, we will write \( p_K(u) \) instead of \( F_K(u) \). When \( K \subset \mathbb{R}^2 \), by \( \tau_K(u) \) we denote the curvature of \( \partial K \) in \( p_K(u) \).
It has long been known, and is easy to see, that $D_K$ is the support of $g_K$. Moreover, if $K \subset \mathbb{R}^2$ and $u \in S^1$, we have

$$V(F_{DK}(u)) = V(F_K(u)) + V(F_K(-u)).$$

(2.1)

This fact was also observed and used by Nagel.

Matheron [10, p. 86] himself noted that

$$\frac{\partial g_K(ru)}{\partial r} = -a_K(r, u),$$

(2.2)

where $r > 0$, $u \in S^{n-1}$ and

$$a_K(r, u) = V(\{y \in u^\perp : V(K \cap (l_u + y)) \geq r\}).$$

(2.3)

Zhang [16] called the function $a_K$ the restricted chord projection function and observed that the $1/(n - 1)$ power of $a_K(\cdot, u)$ is concave on its support.

These identities suggest another interpretation of the covariogram: the information that it provides is equivalent to knowing, for each $u \in S^{n-1}$, the distribution of the lengths of the chords of $K$ which are parallel to $u$ (or, using different terms, the rearrangement of the X-rays of $K$ in the direction $u$; see the online update of [6] mentioned above).

When $K \subset \mathbb{R}^2$, $a_K$ is simply the maximal distance between two parallel chords of $K$ of length $r$ (if these exist).

3. Some lemmas

The next lemma exploits the fact that $D_K$ is the support of $g_K$.

**Lemma 3.1.** Let $K, H$ be plane convex bodies with equal covariograms, and let $u \in S^1$.

1. If $F_K(u) = F_H(u)$, then $F_K(-u) = F_H(-u)$.
2. If $\partial K \cap \partial H$ contains an open arc $A$ and $\text{int } H \cap \text{int } K \neq \emptyset$, then

$$F_K(v) = F_H(v) \quad \text{and} \quad F_K(-v) = F_H(-v)$$

for each unit outer normal $v$ to $K$ in some point of $A$.

**Proof.** The relative position of $F_K(u)$ and $F_K(-u)$ is determined, up to a reflection, by $D_K$ via the identity

$$F_K(u) - F_K(-u) = F_{DK}(u).$$

(3.1)

The presence of a segment on $\partial K$ can be detected from $D_K$ and the behavior of $g_K$ near the origin; the shape of the portion of $\partial K$ which is ‘opposite’ to the segment can be detected from the form of some level lines of $g_K$.

**Lemma 3.2.** Let $K, H$ be plane convex bodies with equal covariograms, let $u \in S^1$, and assume that $F_K(u)$ is a segment. After, possibly, a translation or a reflection,

$$F_K(u) = F_H(u) \quad \text{and} \quad F_K(-u) = F_H(-u).$$

(3.2)
Let us assume, moreover, that (3.2) holds and that \( V(F_K(u)) > V(F_K(-u)) \). Let \( \pi^+ = \{ q \in \mathbb{R}^2 : \langle q, -u \rangle \geq c \} \), with \( c \) chosen in such a way that the chord \( K \cap \partial \pi^+ \) intersects \( \text{int} \ K \) and has length \( V(F_K(u)) \). Then \( \partial K \cap \pi^+ = \partial H \cap \pi^+ \).

**Proof.** If \( \partial K \) contains at least one segment orthogonal to \( u \), then \( \partial DK \) also contains a segment orthogonal to \( u \), and vice versa. If \( F_K(u) \) and \( F_K(-u) \) are both segments, then \( g_K(rv) \), where \( v \) is a unit vector orthogonal to \( u \), is a linear function of the real parameter \( r \) in some neighborhood \( I \) of 0 (and vice versa). Moreover

\[
V(I) = \min(V(F_K(u)), V(F_K(-u))).
\]

This relation, together with (2.1), determines the lengths of \( F_K(u) \) and \( F_K(-u) \). Therefore, after, possibly, a translation and a reflection,

\[
F_H(u) = F_K(u);
\]

Lemma 3.1(1) proves that \( F_H(-u) = F_K(-u) \) too.

Let us prove the second claim. Choosing a suitable reference system, we may assume that \( u = (-1, 0) \) and

\[
K = \{(x, y) : 0 \leq x \leq b, \ f(x) \leq y \leq g(x)\}
\]

for suitable \( b > 0 \) and functions \( f \) and \( g \) defined in \( [0, b] \). If \( r = V(F_K(u)) \), we may also assume that

\[
-f(0) = g(0) = r/2.
\]

By definition of \( \pi^+ \), \( g(c) - f(c) = r \) and

\[
g(x) - f(x) \leq r \quad \text{for any } x \in (c, b].
\]

Let \( x_0 \in (c, b] \); if \( y_0 \in [g(x_0) - r/2, \ f(x_0) + r/2] \), then

\[
K \cap (K + (x_0, y_0)) = K \cap \{(x, y) : x \geq x_0\}
\]

and

\[
g_K(x_0, y_0) = V(K \cap \{(x, y) : x \geq x_0\}),
\]

while if \( y_0 \notin [g(x_0) - r/2, \ f(x_0) + r/2] \), then

\[
K \cap (K + (x_0, y_0)) \subsetneq K \cap \{(x, y) : x \geq x_0\}
\]

and

\[
g_K(x_0, y_0) < V(K \cap \{(x, y) : x \geq x_0\}).
\]

Therefore the covariogram \( g_K(x_0, y_0) \) is constant, as a function of \( y_0 \), exactly in \([g(x_0) - r/2, \ f(x_0) + r/2] \). This argument determines the functions \( f(x) + r/2 \) and \( g(x) - r/2 \) in \([c, b] \) and, up to translations, \( \partial K \cap \pi^+ \). \( \square \)

Gardner and Zhang \([7]\) proved that \( g_K \) is a log concave function on its support. It is elementary to prove that the gradient of \( g_K \) exists in each point \( x \) such that \( \partial K \cap \partial (K + x) \) consists of two points \( p_1, p_2 \) where the two boundaries intersect transversally. Matheron \([11]\) computed \( \nabla g_K \) as

\[
\nabla g_K(x) = -\int_{\partial K \cap (K + x)} n(y) dS(y), \tag{3.3}
\]
where $dS$ is the arc length and $n(y)$ is the outer normal at the point $y \in \partial K$ (which exists $dS$ almost everywhere). An integration by parts proves that $\nabla g_K(x)$ is equal to a rotation of $p_2 - p_1$ by an angle of $\pm \pi/2$, where the $\pm$ is chosen in such a way that $\langle \nabla g_K(x), x \rangle \leq 0$.

The proof of Lemma 1.5 is based essentially on this expression of the gradient.

\textbf{Proof of Lemma 1.5.} Let $S_1$ and $S_2$ be two parallel edges of $P$, and let $r \in \mathbb{R}$ and $v \in S^1$ be the length and direction respectively of $S_1$. If $S_1$ or $S_2$ are on $\partial K$, then Lemma 3.2 concludes the proof. Similar arguments apply if one or both of the other two edges of $P$ are on $\partial K$.

If no side of $P$ is contained in $\partial K$, then $\partial K \cap (\partial K + rv)$ consists of an endpoint $p_1$ of $S_1$ and of an endpoint $p_2$ of $S_2$, and moreover $\partial K$ and $\partial K + rv$ intersect transversally in $p_1$ and $p_2$. The gradient $\nabla g_K(rv)$ exists and it determines the vector $p_2 - p_1$, up to the sign, and the relative position of $S_2$ and $S_1$. This is equivalent to saying that the relative position of the pair of chords of $K$ parallel to $v$ and of length $r$ and the relative position of the corresponding chords of $H$ are the same and thus a translation maps the first pair in the second one. \hfill \Box

\textbf{Lemma 3.3.} Let $H, K$ be different plane convex bodies with equal covariograms. Let us assume that there exists a nonempty interval $U \subset S^1$ such that

$$F_K(u) = F_H(u) \quad \text{and} \quad F_K(-u) = F_H(-u) \quad \forall u \in U,$$

and let us assume that $U$ is a maximal interval (with respect to inclusion) with this property. Let $A$ (and $B$) be the maximal closed arc contained in $\partial K \cap \partial H$ that contains $\bigcup_{u \in U} F_K(u)$ (and $\bigcup_{u \in U} F_K(-u)$, respectively).

Then $A$ or $B$ are points, or $A$ and $B$ are parallel line segments of equal length, or $A$ is a reflection of $B$.

\textbf{Proof.} Since $H \neq K$, neither $A$ nor $B$ coincides with $\partial K$. Let us first suppose that these arcs are nondegenerate and let $a_1$ and $a_2$ (and $b_1$ and $b_2$) be, respectively, the left and right endpoint of the arc $A$ (and $B$, respectively) in counterclockwise order in $\partial K$ (see Figure 1).

We claim that if $u_1$ denotes the left endpoint of $U$ in counterclockwise order on $S^1$, then

$$u_1 \in N_K(a_1) \cap N_H(a_1) \quad \text{and} \quad -u_1 \in N_K(b_1) \cap N_H(b_1).$$

Let us prove for instance that $u_1 \in N_K(a_1)$. We do this by arguing by contradiction. Since $A$ is closed, $F_K(u_1) \cap A \neq \emptyset$. If $a_1 \notin F_K(u_1)$, then $F_K(u_1) \subset \text{relint } A$. This implies that the set of the $w \in S^1$ which, in counterclockwise order, strictly precede $u_1$ and strictly follow any $v \in N_K(a_1)$ is nonempty. Let $w$ be in that set or let $w = u_1$. Since $w \notin N_K(a_1)$, then $F_K(w) \subset \text{relint } A$. Since $\partial H$ and $\partial K$ coincide in $A$, then $F_K(w) = F_H(w)$. Lemma 3.1 implies that the same equality holds for $-w$. This contradicts the maximality of $U$ and concludes the proof of the claim.

Now let

$$v = \frac{a_1 - b_1}{\|a_1 - b_1\|}.$$ 

There is an open arc in $S^1$ with one endpoint at $v$ such that, for each $u$ in this arc, the line $l_u + a_1$ intersects the relative interior of $B$ and $l_u + b_1$ intersects the
relative interior of $A$, with the latter at $b'_1$, say (see Figure 1). Let

$$r_1 = V(K \cap (l_u + a_1)) \quad \text{and} \quad r_2 = V(K \cap (l_u + b_1)).$$

We shall prove that $r_1 = r_2$.

Suppose that $r_1 \neq r_2$, that is without loss of generality, that $r_1 < r_2$. Consider a chord $Q$ of $K$, parallel to $u$ and outside the strip bounded by $l_u + a_1$ and $l_u + b_1$, and close enough to $l_u + b_1$ to ensure that $r = V(Q) > r_1$. By our assumption concerning the maximality of $B$, we can (interchanging $K$ and $H$, if necessary) assume that an endpoint $q$ of $Q$ belongs to $H \setminus \partial H$ and the other endpoint $q'$ belongs to $A$.

By (3.5), there are common supporting lines $l'$ and $l$ to $K$ and $H$ at $a_1$ and $b_1$, respectively, orthogonal to $u_1$. Let $m'$ be any supporting line to $K$ at $b'_1$ and note that $Q$ lies between $l$ and $m'$, which are either parallel or meet in the half-plane bounded by $l_u + b_1$ not containing $a_1$. Since $Q$ is parallel to $K \cap (l_u + b_1)$, we have $r \leq r_2$, with equality if and only if $q \in l$ and $b'_1, q' \in l'$ ($= m'$). However, if $q \in l$, $q$ is a point of $H$ which belongs to a line which supports $H$, and thus it belongs to $\partial H$, which contradicts the assumption $q \in H \setminus \partial H$. Therefore $r < r_2$.

Since $r_1 < r < r_2$, there is a common chord $R$ of $K$ and $H$ of length $r$, parallel to $u$, contained in the strip bounded by $l_u + a_1$ and $l_u + b_1$, and with endpoints on the arcs $A$ and $B$ (see Figure 1). By Lemma 1.5, a translate of the parallelogram whose opposite edges are $R$ and $Q$ is inscribed in $H$. Since $R$ is a chord of $H$, this may happen only if $Q$ is also a chord of $H$. However, this is a contradiction, since the endpoint $q$ of $Q$ does not belong to $\partial H$.

It follows that $r_1 = r_2$. Therefore the two endpoints of $K \cap (l_u + a_1)$ and $K \cap (l_u + b_1)$ different from $a_1$ and $b_1$ are symmetric with respect to $(a_1 + b_1)/2$. Let

$$v_1 = \frac{a_1 - b_2}{\|a_1 - b_2\|} \quad \text{and} \quad v_2 = \frac{a_2 - b_1}{\|a_2 - b_1\|}.$$  (3.6)
Suppose, without loss of generality, that \( v, v_1 \) and \( v_2 \) lie in that counterclockwise order on \( S^1 \). Then we may repeat the above argument for any \( u \) contained in the open arc of \( S^1 \) bounded by \( v \) and \( v_1 \) and conclude that \( B \) is the reflection in the point \((a_1 + b_1)/2 \) of a subarc of \( A \) with one endpoint at \( a_1 \). Repeating the whole argument with the roles of \( a_1 \) and \( a_2 \) (and \( b_1 \) and \( b_2 \)) reversed, we see that \( B \) is also the reflection in the point \((a_2 + b_2)/2 \) of a subarc of \( A \) with one endpoint in \( a_2 \). It is easy to see that this implies that either these subarcs of \( A \) are equal to \( A \) or \( A \) and \( B \) are parallel segments.

In the first case \( A \) is a reflection of \( B \).

To prove that if \( A \) and \( B \) are parallel segments then they have the same length, we argue by contradiction, assuming, without loss of generality, that \( A \) is longer than \( B \). The second part of the claim of Lemma 3.2 implies that \( \partial K \cap \partial H \) contains an arc which strictly contains \( B \), contradicting the maximality of \( B \).

To conclude the proof of the lemma, we have to consider the possibility that either \( A \) or \( B \), or both, are points.

If both \( A \) and \( B \) are points, then there is nothing to prove. Let us assume, without loss of generality, that \( B \) is a point while \( A \) is nondegenerate.

If \( A \) is a segment, one gets a contradiction, as in the case when \( A, B \) are parallel segments; we may thus suppose that \( A \) is not a segment.

Let \( u_0 \in S^1 \) be such that \( F_K(u_0) \) is a point in relint \( A \).

For \( r > 0 \), small let us consider the two segments \( S_1 \) and \( S_2 \) which are inscribed in \( K \), of length \( r \) and orthogonal to \( u_0 \). The endpoints of one of the two segments, let us say \( S_1 \), are in \( A \), and \( S_1 \) is inscribed also in \( H \). It follows by Lemma 1.5 that the parallelogram whose opposite edges are \( S_1 \) and \( S_2 \) is inscribed in \( H \) too.

Repeating this argument for any \( r \) small proves that the boundaries of \( H \) and \( K \) coincide in a neighborhood of the point \( B \), contradicting the maximality of \( B \). \( \Box \)

4. Proof of Theorem 1.1: non-strictly convex bodies

Let \( H \neq K \) be as in Lemma 3.3. If the symmetric maximal arcs \( A \) and \( B \) of \( \partial H \cap \partial K \) contain two parallel segments of equal length, we concentrate on some parallelograms which are inscribed in \( K \) and have two opposite vertices on those segments. Using Lemma 1.5, we are able to prove that the reflection that maps one segment into the other maps \( H \) in \( K \).

**Lemma 4.1.** Let \( H, K, U, A \) and \( B \) be defined as in Lemma 3.3. If \( F_K(u) \) is a segment, for some \( u \in U \), then \( H \) is a reflection of \( K \).

**Proof.** Choosing a suitable reference system, we may assume that \( u = (-1,0) \) and

\[
K = \{(x,y) : -a \leq x \leq a, f_K(x) \leq y \leq g_K(x)\}
\]

\[
H = \{(x,y) : -a \leq x \leq a, f_H(x) \leq y \leq g_H(x)\}
\]

for suitable \( a > 0 \) and functions \( f_H, f_K, g_H \) and \( g_K \) defined in \([-a,a]\). We may also assume that \( f_K(a) = -g_K(-a) \).

\[
A \supset F_K(u) = \{(-a,y) : f_K(-a) \leq y \leq g_K(-a)\},
\]

\[
B \supset F_K(-u) = \{(a,y) : f_K(a) \leq y \leq g_K(a)\}.
\]
The identities $F_K(u) = F_H(u)$ and $F_K(-u) = F_H(-u)$ are equivalent to

$$f_H(a) = f_K(a), \quad g_H(a) = g_K(a),$$

$$f_H(-a) = f_K(-a), \quad g_H(-a) = g_K(-a).$$

(4.1)

$A = -B$, by Lemma 3.3 and the choice of the origin, and consequently $F_K(u) = -F_K(-u)$. This implies that

$$f_H(a) = -g_H(-a), \quad f_H(-a) = -g_H(a),$$

$$f_K(a) = -g_K(-a), \quad f_K(-a) = -g_K(a).$$

(4.2)

First we prove that if

$$p \in \partial K \setminus \partial H$$

(4.3)

is close to $A$, then $-p \in \partial H$.

Let $p = (x, g_K(x))$. We identify three cases according to the sign of $g_K(x) + f_K(-x)$.

Case 1 ($g_K(x) + f_K(-x) = 0$): This is equivalent to $-p \in \partial K$. In this case

$$(a, g_K(a)) - (x, g_K(x)) = (-x, f_K(-x)) - (-a, f_K(-a))$$

and

$$P := \text{conv}(p, (-x, f_K(-x)), (a, g_K(a)), (-a, f_K(-a)))$$

is a parallelogram inscribed in $K$. Let $Q$ be a translate of $P$ inscribed in $H$, which exists by Lemma 1.5. We claim that

$$P = Q.$$  

(4.4)

This assertion is a consequence of the fact that the two opposite vertices $(a, g_K(a)), (-a, f_K(-a))$ of $P$ are suitable endpoints of parallel faces of $\partial H$. Identity (4.4) implies that the vertices of $P$ belong to $H$ and in particular that $p \in \partial H$. This contradicts (4.3) and proves that this case never happens.

Case 2 ($g_K(x) + f_K(-x) < 0$): Let $y_0 = g_K(x) + f_K(-x) - f_K(-a)$ and let $b \in \mathbb{R}$ be such that $(b, g_K(b))$ is an endpoint of $A$ (see Figure 2).

If $p$ is close enough to $A$ (that is, $x$ is close to $b$), then $g_K(x) + f_K(-x)$ is close to 0, by continuity of $g_K$ and $f_K$ and the identity $A = -B$. We may thus assume that $g_K(x) + f_K(-x) > f_K(a) - g_K(a)$, since $f_K(a) - g_K(a) = -V(F_K(-u)) < 0$. This implies, by (4.2), that $f_K(a) < y_0 < g_K(a)$ and the point $(a, y_0) \in \text{relint } F_K(-u)$. The parallelogram

$$P := \text{conv}(p, (-x, f_K(-x)), (a, y_0), (-a, f_K(-a)))$$

is inscribed in $K$. Let $Q$ be a translate of $P$ inscribed in $H$. We claim that

$$Q = P + (0, v) \quad \text{with } 0 < v \leq g_K(a) - y_0.$$  

(4.5)

This assertion is a consequence of the fact that the two opposite vertices $(a, y_0), (-a, f_K(-a))$ of $P$ are on parallel vertical faces of $\partial H$ and that the translation has to keep those vertices inside the two faces (see Figure 2). Moreover it cannot be $v = 0$ by (4.3).

Let us prove that

$$Q = P + (0, v) \quad \text{with } v = g_K(a) - y_0.$$  

(4.6)

We argue by contradiction, assuming that $0 < v < g_K(a) - y_0$, that is, assuming that the vertex $(a, y_0 + v)$ of $Q$ belongs to $\text{relint } F_H(-u)$. For $\varepsilon > 0$, let $P_\varepsilon$ be a
Figure 2. $H$, $K$, $P$ and $P_\varepsilon$ (broken line).

parallelogram inscribed in $K$ with an edge equal to $[p,(a,y_0+\varepsilon)]$, and let $Q_\varepsilon$ be the corresponding translate which is inscribed in $H$.

For small $\varepsilon$, the point $(a,y_0+v+\varepsilon) \in \text{relint } F_H(-u)$ and the segment $[p,(a,y_0+\varepsilon)]+(0,v)$ is a chord of $H$. Therefore

$$Q_\varepsilon = P_\varepsilon + (0,v). \quad (4.7)$$

Let $(x_{1,\varepsilon},f_K(x_{1,\varepsilon}))$ and $(x_{2,\varepsilon},f_K(x_{2,\varepsilon}))$, with $x_{1,\varepsilon} < x_{2,\varepsilon}$, be the vertices of the edge of $P_\varepsilon$ opposite to $[p,(a,y_0+\varepsilon)]$. The point $(x_{1,\varepsilon},f_K(x_{1,\varepsilon})+(0,v))$ is a vertex of $Q_\varepsilon$ and belongs to $\partial H$, by (4.7). Therefore

$$f_H(x_{1,\varepsilon}) = f_K(x_{1,\varepsilon}) + v. \quad (4.8)$$

Let us prove that $f_K$ is not an affine function in $[-a,-b]$. Since $A = -B$, the point $(-b,f_K(-b))$ is an endpoint of $B$ and $g_K(-b) = -f_K(-b)$.

$$g_K(x) \geq -f_K(-b) \frac{x-b}{a-b} + g_K(a) \frac{a-x}{a-b} \quad \text{by concavity of } g_K,$$

$$f_K(x) = f_K(-b) \frac{x-b}{a-b} + f_K(-a) \frac{a-x}{a-b} \quad \text{if } f_K \text{ is affine.}$$

These formulas and (4.2) violate the assumption that $g_K(x) + f_K(-x) < 0$. We claim now that $P_\varepsilon$ depends continuously on $\varepsilon$ in $\varepsilon = 0$. If the continuity is not true, then there is a sequence $\varepsilon_n \to 0$ such that $(P_{\varepsilon_n})_n$ converges to a parallelogram $P' \neq P$ inscribed in $K$, with an edge equal to $[p,(a,y_0)]$. This may happen only if $\partial K$ contains a segment which contains the edges of $P$ and of $P'$ which are opposite to $[p,(a,y_0)]$. If this happens, then the function $f_K$ is affine in $[-a,-x]$. If $x$ is close enough to $b$, then this is not possible, since $f_K$ is not an affine function in $[-a,-b]$.

This continuity implies that as $\varepsilon \to 0$, $x_{1,\varepsilon} \to -a$; (4.8) and (4.1) thus contradict the continuity of $f_H$ and $g_H$. The proof of (4.6) is concluded.
Identity (4.6) implies that the vertices of \( P + (0,v) \) are in \( \partial H \) and in particular that \((-x, f_K(-x)+v) \in \partial H.\) This point, by the definition of \( y_0 \) and (4.2), coincides with \(-p.\)

Case 3 \((f_K(-x)+g_K(x) > 0):\) The proof goes similarly to the second case and we omit it.

Let us now prove that
\[ H = -K. \tag{4.9} \]

Let \( A^- \) (and \( B^- \)) be the connected component of \( \partial K \cap \partial (-H) \) which contains \( A \) (and \( B, \) respectively). If (4.9) is not true, then \( A^- = -B^- \), by Lemma 3.3, applied to \( K \) and \(-H.\) As a consequence, \( A^- \subset \partial K \cap \partial H, \) and thus \( A^- \subset A. \) We conclude that \( A^- = A. \)

On the other hand, the first part of the proof of this lemma implies that \( A \) is strictly contained in \( A^- \). This contradiction concludes the proof. \( \square \)

Proof of Theorem 1.1 when \( K \) is not strictly convex. Let \( H \) be a convex body with \( g_H = g_K. \) Let \( F_K(u) \) be a segment. After, possibly, a translation or a reflection, we may suppose (see Lemma 3.2) that
\[ F_K(u) = F_H(u) \quad \text{and} \quad F_K(-u) = F_H(-u). \]

Let \( U \subset S^1 \) be the maximal interval containing \( u \) where (3.4) holds; \( U \) is not empty.

Let \( A \) and \( B \) be the arcs defined in Lemma 3.3. Lemma 4.1 proves that if \( H \neq K, \) then \( H \) is a reflection of \( K. \)

\( \square \)

5. Proof of Theorem 1.1: bodies not \( C^1 \)

In this section we may assume that \( H \) and \( K \) are strictly convex, due to the result proved in the previous section. We recall (see Section 2) that \( a_H(\cdot, v) \) is concave on its support.

**Lemma 5.1.** Let \( H \) be a plane strictly convex body, and, given \( v \in S^1 \) and \( r > 0, \) let \( a_H(r, v) \) be the distance between the two chords of \( H \) which are parallel to \( v \) and have length \( r. \)

(1) Let \( m, \varepsilon > 0 \) and let \( r \) be such that \( r \geq m, \) \( a_H(r, v) \geq m. \) There exists \( \gamma, \delta > 0, \) which depends only on \( H, m \) and \( \varepsilon \) such that, if
\[ \left( \frac{\partial^+}{\partial r} - \frac{\partial^-}{\partial r} \right) a_H(r,v) < -\varepsilon, \]
then in at least one endpoint \( p \) of the chords of \( H \) which are parallel to \( v \) and have length \( r, \) the opening of the normal cone \( N_H(p) \) is larger than \( \delta. \) The opposite applies if the opening of \( N_H(p) \) is larger than \( \delta \) in one endpoint \( p. \) Then
\[ \left( \frac{\partial^+}{\partial r} - \frac{\partial^-}{\partial r} \right) a_H(r,v) < -\gamma. \]

(2) Let \( H \in C^1. \) If \( \partial H \) admits curvature at all the four endpoints of the two corresponding chords, then \( \frac{\partial^2}{\partial r^2} a_H(r,v) \) exists. If, on the other hand, \( \partial H \) does not admit curvature at exactly one of the endpoints, then \( \frac{\partial^2}{\partial r^2} a_H(r,v) \) does not exist.
By the opening of a cone $C \subset \mathbb{R}^2$, we mean the length of $C \cap S^1$. The symbols $\partial^+ / \partial r$ and $\partial^- / \partial r$ denote right and left derivatives respectively.

Lemma 5.1(2) is proved in [4, Lemma 5.3].

Proof of Lemma 5.1(1). Let us choose a reference system with $\nu = (0,1)$ and let $H = \{(x,y): a \leq x \leq b, f(x) \leq y \leq g(x)\}$ for suitable $a, b \in \mathbb{R}$ and functions $f$ and $g$ defined in $[a,b]$. Let $x_1, x_2$, with $x_1 \leq x_2$, be the two solutions of
\[ g(x) - f(x) = r, \quad (5.1) \]
and, for $i = 1, 2$, let the angles $\alpha^+_i(f), \alpha^-_i(f), \alpha^+_i(g), \alpha^-_i(g) \in (-\pi/2, \pi/2)$ be defined by
\[
\frac{d^+ g}{dx}(x_i) = \tan \alpha^+_i(g), \quad \frac{d^- g}{dx}(x_i) = \tan \alpha^-_i(g), \\
\frac{d^+ f}{dx}(x_i) = \tan \alpha^+_i(f), \quad \frac{d^- f}{dx}(x_i) = \tan \alpha^-_i(f).
\]
Then, by (5.1),
\[
\left(\frac{\partial^+}{\partial r} - \frac{\partial^-}{\partial r}\right) a_H = \left(\frac{d^+}{dr} - \frac{d^-}{dr}\right)(x_2 - x_1)
\]
\[= \sum_{i=1,2} \left( \frac{1}{\frac{dx}{dr}(g-f)(x_i)} - \frac{1}{\frac{dx}{dr}(g-f)(x_i)} \right) \]
\[= \sum_{i=1,2} \sin(\alpha^+_i(g) - \alpha^-_i(g)) \cos(\alpha^-_i(f) \cos(\alpha^+_i(f)) \sin(\alpha^+_i(g) - \alpha^-_i(g)) \sin(\alpha^-_i(g) \cos(\alpha^+_i(g)) \sin(\alpha^-_i(g) - \alpha^+_i(g))) \sin(\alpha^-_i(g) \cos(\alpha^+_i(g)) \sin(\alpha^-_i(g) - \alpha^+_i(g))). \quad (5.2) \]

The convexity and compactness of $H$ imply, via standard arguments, that there exists $\beta > 0$ such that
\[
\frac{\pi}{2} - \beta \geq \alpha^-_i(g) \geq \alpha^+_i(g) \geq -\frac{\pi}{2} + \beta \\
\frac{\pi}{2} - \beta \geq \alpha^+_i(f) \geq \alpha^-_i(f) \geq -\frac{\pi}{2} + \beta \\
\pi - \beta \geq \alpha^-_i(g) - \alpha^-_i(f) \geq \beta \\
\pi - \beta \geq \alpha^+_i(g) - \alpha^+_i(f) \geq \beta
\]
uniformly with respect to $\nu \in S^1$ and $r$ which satisfies $r \geq m$, $a_H(r, \nu) \geq m$.

Each summand in the right-hand side of (5.2) is thus non-positive. If the term $(\partial^+ / \partial r - \partial^- / \partial r) a_H$ is less than $-\varepsilon$, then at least one of the summands is less than $-\varepsilon/4$. Since $\cos(\alpha^-_i(f))$, $\cos(\alpha^+_i(f))$, $\cos(\alpha^-_i(g))$, $\cos(\alpha^+_i(g))$, $\sin(\alpha^-_i(g) - \alpha^-_i(f))$ and $\sin(\alpha^+_i(g) - \alpha^+_i(f))$ are bounded uniformly from above and below by positive constants, at least one of the differences $\alpha^+_i(g) - \alpha^-_i(g)$ or $\alpha^-_i(f) - \alpha^+_i(f)$, $i = 1, 2$, is less than $-\delta$ for a suitable $\delta > 0$. This is the required condition on the openings of the normal cones to $H$ in the endpoints of the chords which correspond to $r$ and $\nu$. The converse holds too.

\[ \square \]

Lemma 5.2. Let $H, K, U, A$ and $B$ be as in Lemma 3.3. If $A$ and $B$ are strictly convex arcs, then $H$ is a reflection of $K$. 

Proof. After, possibly, a translation, we may suppose that $A = -B$. First we prove that if $p \in \partial K \setminus \partial H$ is close to $A$, then $-p \in \partial H$.

Let $p_2 \in \text{relint} A$ and let $P$ be the parallelogram inscribed in $K$ with an edge equal to $[p, p_2]$. Let $p_3, p_4$ be the other vertices of $P$, with $p_3 - p_4 = p - p_2$; see Figure 3.

We claim that if $p$ is close enough to $A$, then $p_3 \in \text{relint} B$. This is a consequence of the continuous dependence of the chord $[p_3, p_4]$ on the choice of $p$. If $p$ were an endpoint of $A$, then $[p_3, p_4] = -[p_2, p_3]$, and thus $p_3 \in \text{relint} B$.

Moreover, $[p_2, p_3], -[p_2, p_3]$ is the only chord of $H$ which is a translation of $[p_2, p_3]$, by the strict convexity of $A$ and $B$ and the identity $A = -B$ (see Figure 3).

Let $Q$ be a translate of $P$ which is inscribed in $H$. The diagonal of $Q$ which corresponds to the diagonal $[p_2, p_3]$ of $P$ is equal to either $[p_2, p_3]$ or $-[p_2, p_3]$. In the first case, $P = Q$; this cannot happen since $p \notin \partial H$. Thus $Q = -P$. This proves that $-p \in \partial H$.

To prove that $H = -K$, one argues exactly as in the corresponding part of the proof of Lemma 4.1. \qed

Proof of Proposition 1.4. If $\partial H \cap \partial K$ contains a line segment, then the statement is an immediate consequence of Lemmas 3.2, 3.3 and 4.1. If, on the contrary, $\partial H \cap \partial K$ is strictly convex, then Lemmas 3.1, 3.3 and 5.2 imply the thesis. \qed

Proof of Theorem 1.1 when $K$ is not $C^1$. Let $p_1 \in \partial K$ be a point where the tangent line does not exist, and let $I \subset [0, 2\pi]$ be a closed interval such that for $\theta \in I$, there exists a parallelogram $P(\theta)$ which is inscribed in $K$, and has a vertex in $p_1$ and an edge parallel to $(\cos \theta, \sin \theta)$. Let $p_2(\theta)$ be the vertex of $P(\theta)$, with the
property that the chord \([p_2,p_1]\) is parallel to \((\cos \theta, \sin \theta)\), and let \(p_3(\theta)\), and \(p_4(\theta)\) be the other vertices of \(P(\theta)\) with \(p_4 - p_3 = p_2 - p_1\).

Let \(Q(\theta)\) denote a translate of \(P(\theta)\) which is inscribed in \(H\), and, for \(i = 1, \ldots, 4\), let \(q_i(\theta)\) denote the vertex of \(Q(\theta)\) which corresponds to \(p_i(\theta)\).

We claim that both \(P\) and \(Q\) depend continuously on \(\theta\). This is a consequence of standard compactness arguments and of the fact that there exist only two chords with given length and direction which are inscribed in a strictly convex body. Also \(q_i\), for \(i = 1, \ldots, 4\), depends continuously on \(\theta\).

We claim that there exists \(i\) such that \(q_i(\theta)\) is constant for \(\theta\) in a suitable subinterval \(J \subset I\). Let \(a_K\) and \(a_H\) be defined as in Lemma 5.1; \(a_H = a_K\) by (2.2). Let \(\delta\) denote the opening of the normal cone to \(\partial K\) in \(p_1\), and let

\[E = \{q \in \partial H : \text{the opening of } N_H(q) \text{ is at least } \beta\}.
\]

Lemma 5.1(1), states that if \(\beta > 0\) is chosen suitably, then, for each \(\theta \in I\), at least one of the \(q_i(\theta) \in E\). Since \(E\) is finite and each \(q_i\) depends continuously on \(\theta\), the claim follows.

If \(q_1(\theta)\) is constant, then the arc \(\{q_2(\theta) : \theta \in J\}\) is a translate of \(\{p_2(\theta) : \theta \in J\}\). If \(q_4(\theta)\) is constant, then \(\{q_3(\theta) : \theta \in J\}\) is a reflection of \(\{p_2(\theta) : \theta \in J\}\). A similar behavior holds if \(q_2\) or \(q_3\) are constant. In each case, after, possibly, a translation or a reflection, \(\partial H \cap \partial K\) contains an open arc. Proposition 1.4 concludes the proof.

\[\Box\]

6. Proof of Theorem 1.1: opposite \(C^2\) arcs

In this section we may assume that \(H\) and \(K\) are strictly convex and \(C^1\), due to the results proved in the previous sections.

We recall that an arc of the boundary of a convex body is \(C^2\) if it is a \(C^2\) curve and if its curvature is strictly positive at each point.


**Lemma 6.1.** Let \(K\) be a plane \(C^1\) strictly convex body. Let \(V \subset S^1\) be an open interval, and let us assume that the arcs

\[A = \bigcup_{v \in V} p_K(v) \text{ and } B = \bigcup_{v \in V} p_K(-v)\]

are \(C^2_+\). For each \(v \in V\), the covariogram \(g_K\) determines the nonordered pair \(\{\tau_K(v), \tau_K(-v)\}\) of curvatures at the points with outer normal \(v\) and \(-v\).

The next lemma is a ‘local’ version of [4, Proposition 5.1].

**Lemma 6.2.** Let \(H, K\) be plane \(C^1\) strictly convex bodies with equal covariograms. Let \(V \subset S^1\) be an open interval and let

\[A = \bigcup_{v \in V} p_K(v), \quad B = \bigcup_{v \in V} p_K(-v),\]

\[A' = \bigcup_{v \in V} p_H(v), \quad B' = \bigcup_{v \in V} p_H(-v).\]

If \(A\) and \(B\) are \(C^2_+\), then \(A'\) and \(B'\) are also \(C^2_+\).
Proof. Since $V$ is open and $H$ and $K$ are $C^1$, the arcs $A$, $B$, $A'$ and $B'$ are open. Let us first prove that $H$ admits curvature at each point of $A'$ and $B'$.

Let us assume by contradiction that $H$ does not admit curvature at a point $p \in A'$; let $v_0$ be the outer normal to $H$ in $p$, and let $q = p_H(−v_0) \in B'$. In a polar coordinate system centered at $p$, let $θ_0$ denote the angular coordinate of the point $q$. Let $ε > 0$ and, for any $θ \in (θ_0 − ε, θ_0 + ε) \setminus \{θ_0\}$, let $p_2(θ)$ be the point of $∂H$ with the property that the chord $[p_2, p]$ is parallel to $(cos θ, sin θ)$. Let $p_3(θ)$ and $p_4(θ)$ be points of $∂H$ with $p_4 − p_3 = p_2 − p$, $p_4 \neq p_2$, $p_3 \neq p$.

By Lemma 5.1, there exists a translate $Q$ of the parallellogram $P := conv(p, p_2, p_3, p_4)$ which is inscribed in $K$.

We claim that if $ε$ is small enough, all the vertices of $P$ (of $Q$) belong to $A' \cup B'$ (to $A \cup B$, respectively). To prove this claim, it suffices to observe that as $θ \rightarrow θ_0$, $P$ converges to the chord $[p, q]$, whose endpoints belong to $A' \cup B'$. $Q$ converges, by (3.1), to $[p_K(v_0), p_K(−v_0)]$, which is the only chord of $H$ which is a translate of $[p, q]$. Its endpoints belong to $A \cup B$ too.

Let $v = (cos θ, sin θ)$, $r = V([p, p_2])$ and $a_H(r, v)$, $a_K(r, v)$ be defined as in Lemma 5.1. Since $H$ has the same covariogram function as $K$, it follows from Lemma 5.1(2) that $\frac{∂^2}{∂v^2} a_H(r, v) = 0$.

Lemma 5.1 implies also that for any $θ \in (θ_0 − ε, θ_0 + ε)$, the curvature of $∂H$ does not exist in at least one of the points $p_i$, $i = 2, 3, 4$. We shall show that this contradicts the existence almost everywhere of the curvature of $∂H$.

Let us prove that there exists an interval $J \subset (θ_0 − ε, θ_0 + ε)$ such that each mapping $θ \rightarrow p_i$, $i = 2, 3, 4$, restricted to $J \setminus \{θ_0\}$ maps sets of positive measure in subsets of $∂H$ with positive measure.

This property is clear for the mapping $p_2(θ)$, for any choice of the interval $J$.

Also the mapping $p_3$ is not constant in any interval $J$. If it were constant, then the arcs described by $p_2(θ)$ and $p_4(θ)$ for $θ \in J$ would be translates of each other, and this would violate the strict convexity of $H$.

We claim that the mapping $p_4$ is not constant in $(θ_0 − δ, θ_0 + δ)$, for any choice of $δ > 0$. Otherwise,

$$\{p_3(θ) : θ \in (θ_0 − δ, θ_0 + δ)\} = −\{p_2(θ) : θ \in (θ_0 − δ, θ_0 + δ)\} + τ,$$

for some $τ \in \mathbb{R}^2$. Consequently, $\{p_3(θ) : θ \in (θ_0 − δ, θ_0 + δ)\}$ is a neighborhood of $p$ on $∂H$, and it is homothetic to an arc of $∂DH$. Such an arc of $∂DH$ is $C^2$, since $DH = DK$ and $A, B$ are $C^2$. This contradicts the choice of $p$.

Let $s_i(θ)$, $i = 3, 4$, be the length of the arc consisting of the points of $∂H$ which, in counterclockwise order, follow $p$ and precede $p_i$. [4, Proposition 5.1] proves that the functions $s_i(θ)$ are continuously differentiable. This regularity, together with the property of not being constant, implies that there is an interval $J$ where the derivatives of $s_3(θ)$ and $s_4(θ)$ with respect to $θ$ never vanish. With this choice of $J$, $p_2(θ)$, $p_3(θ)$ and $p_4(θ)$ map sets of positive measure in sets of positive measure.

In conclusion, let us denote by $I_i$, for $i = 2, 3, 4$, the subsets of $J$ where $∂H$ does not admit curvature at $p_i$, respectively. We have proved above that $I_2 \cup I_3 \cup I_4 = J$, and therefore at least one of the $I_i$ has positive measure. However, $p_i(I_i)$ would then be a subset of $∂H$ having positive measure, where $∂H$ had no curvature, which is impossible.

To prove that the curvature varies continuously, one can argue exactly as in the conclusive part of [4, Proposition 5.1].
Proof of Theorem 1.1 when \( \partial \mathcal{K} \) contains two opposite \( C^2 \) arcs. Let \( \bigcup_{v \in \mathcal{V}} \mathcal{P}_K(v) \) and \( \bigcup_{v \in \mathcal{V}} \mathcal{P}_K(-v) \) be the two \( C^2 \) opposite arcs on \( \partial \mathcal{K} \), for a suitable interval \( \mathcal{V} \subset \mathbb{S}^1 \). Since we may assume that \( \mathcal{K} \) is strictly convex, we may also suppose, after possibly substituting \( \mathcal{V} \) by a smaller subinterval, that these arcs are \( C^2_+ \).

By Lemma 6.1, \( \{ \tau_K(v), \tau_K(-v) \} \) is determined for each \( v \in \mathcal{V} \).

Suppose now that
\[
\tau_K(v_0) \neq \tau_K(-v_0)
\]
for some \( v_0 \in \mathcal{V} \). The invariance under translation allows us to fix for \( \mathcal{P}_K(v_0) \in \partial \mathcal{K} \) an arbitrary point in \( \mathbb{R}^2 \). The covariogram function determines \( \mathcal{P}_K(-v_0) \) (see Lemma 3.1). Up to reflection, we may assume that \( \partial \mathcal{K} \) has curvature \( \tau_K(v_0) \) at \( \mathcal{P}_K(v_0) \). Then, by continuity of the curvature, there is an open subinterval \( I \) of \( \mathcal{V} \) containing \( v_0 \) such that the curvatures at \( \mathcal{P}_K(v) \) and \( \mathcal{P}_K(-v) \) are different for all \( v \in I \). By continuity, these curvatures are determined by the choice of the curvature at \( \mathcal{P}_K(v_0) \), and this knowledge uniquely determines the arcs \( \bigcup_{v \in I} \mathcal{P}_K(v) \) and \( \bigcup_{v \in I} \mathcal{P}_K(-v) \) via the parametric representation (see for instance [5, p. 79])
\[
x(v) = x(v_0) + \int_{\theta(v_0)}^{\theta(v)} \frac{-\sin t}{\tau_K(\cos t, \sin t)} dt,
\]
\[
y(v) = y(v_0) + \int_{\theta(v_0)}^{\theta(v)} \frac{\cos t}{\tau_K(\cos t, \sin t)} dt,
\]
where \( \theta(v) \) denotes the angular coordinate of \( v \in \mathbb{S}^1 \).

If
\[
\tau_K(v) = \tau_K(-v)
\]
for each \( v \in \mathcal{V} \), then one arrives similarly at the same conclusion.

If there exists an \( \mathcal{H} \neq \mathcal{K} \) plane convex body with \( g_{\mathcal{K}} = g_{\mathcal{H}} \), then the arcs \( \bigcup_{v \in I} \mathcal{P}_H(v) \) and \( \bigcup_{v \in I} \mathcal{P}_H(-v) \) are \( C^2_+ \), by Lemma 6.2. Arguments similar to those above imply that the arcs \( \bigcup_{v \in I} \mathcal{P}_H(v) \) and \( \bigcup_{v \in I} \mathcal{P}_H(-v) \) are determined by \( g_{\mathcal{H}} \) and that, after, possibly, a translation or a reflection, they coincide with the corresponding arcs of \( \partial \mathcal{K} \). Proposition 1.4 concludes the proof. \( \square \)

7. A counterexample in \( \mathbb{R}^n, n \geq 4 \)

As mentioned in the introduction, given any finite sets \( \mathcal{E} \) and \( \mathcal{F} \) in \( \mathbb{R}^n \), the multisets \( \mathcal{E} + \mathcal{F} \) and \( \mathcal{E} + (-\mathcal{F}) \) are homometric. The proof of this property is immediate and purely algebraic. When one tries to construct counterexamples to Matheron’s conjecture via a similar procedure, one encounters immediately two problems: the requirement that \( \mathcal{E} + \mathcal{F} \) and \( \mathcal{E} + (-\mathcal{F}) \) be sets, not multisets, and the requirement that they be convex. The choice of \( \mathcal{E} \) and \( \mathcal{F} \) as arbitrary convex bodies in orthogonal subspaces solves these first problems. To prove that the sets that we obtain have the same covariogram, we use the formulation of the covariogram problem in terms of Fourier analysis.

Proof of Theorem 1.2. The property of having equal covariograms is invariant under linear maps, and the same is also true for the property of being equal after
Matheron’s conjecture for the covariogram problem

a translation or a reflection, since

\[ g_{LT}(x) = (\det L) g_T(L^{-1}x) \]
\[ LS = L(\pm T) + \tau \iff S = \pm T + L^{-1} \tau \]

for any linear map \( L \), vector \( x \) and \( \tau \) and convex body \( T \) and \( S \). We may thus prove the theorem assuming that \( L = I \).

By (1.2), to prove the first claim, it suffices to show that

\[ |\hat{1}_{K \times H}| = |\hat{1}_{K \times (-H)}|. \tag{7.1} \]

Let \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^n \times \mathbb{R}^k \) and \( x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^k \). Then

\[
\hat{1}_{K \times H}(\xi_1, \xi_2) = \int_{K \times H} e^{i\langle (\xi_1, \xi_2), (x_1, x_2) \rangle} \, dx_1 \, dx_2 \\
= \int_{K \times H} e^{i\langle \xi_1, x_1 \rangle} e^{i\langle \xi_2, x_2 \rangle} \, dx_1 \, dx_2 \\
= \int_K e^{i\langle \xi_1, x_1 \rangle} \, dx_1 \int_H e^{i\langle \xi_2, x_2 \rangle} \, dx_2 \\
= \hat{1}_K(\xi_1) \hat{1}_H(\xi_2), \tag{7.2}
\]

and the same formula also holds for \( K \times (-H) \):

\[
\hat{1}_{K \times (-H)}(\xi_1, \xi_2) = \hat{1}_K(\xi_1) \hat{1}_{-H}(\xi_2). \tag{7.3}
\]

Since \( \hat{1}_{-H}(\xi_2) = \hat{1}_H(\xi_2) \), formula (7.3) becomes

\[
\hat{1}_{K \times (-H)}(\xi_1, \xi_2) = \hat{1}_K(\xi_1) \hat{1}_H(\xi_2). \tag{7.4}
\]

Formulas (7.2) and (7.3) imply equality (7.1).

To prove the second claim, we may assume that both \( K \) and \( H \) have the center of mass at the origin. Since a translation of \( K \) and a translation of \( H \) give as a result a translation of \( K \times H \) and of \( K \times (-H) \), this assumption is not restrictive.

Let us assume by contradiction that

\[ K \times H = K \times (-H) + \tau \tag{7.5} \]

or

\[ K \times H = -(K \times (-H)) + \tau \tag{7.6} \]

for some vector \( \tau \in \mathbb{R}^{n+k} \). Since the origin is the center of mass of \( K \times H \) and of \( K \times (-H) \), \( \tau = 0 \). It is clear that if (7.5) holds, then \( H = -H \), that is, \( H \) is centrally symmetric. Similarly, if (7.6) holds, then \( K \) is centrally symmetric.

\[ \square \]

Acknowledgements. Some of the results in this paper were established while the author was visiting Richard Gardner at Western Washington University. The author is indebted to him for several stimulating discussions and for the warm hospitality.

References


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