SECANT VARIETIES OF SEGRE-VERONESE VARIETIES $\mathbb{P}^m \times \mathbb{P}^n$ EMBEDDED BY $\mathcal{O}(1,2)$

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Abstract. Let $X_{m,n}$ be the Segre-Veronese variety $\mathbb{P}^m \times \mathbb{P}^n$ embedded by the morphism given by $\mathcal{O}(1,2)$. In this paper, we provide two functions $\underline{s}(m,n) \leq \overline{s}(m,n)$ such that the s^{th} secant variety of $X_{m,n}$ has the expected dimension if $s \leq \underline{s}(m,n)$ or $\overline{s}(m,n) \leq s$. We also present a conjecturally complete list of defective secant varieties of such Segre-Veronese varieties.

1. Introduction

Let $X \subset \mathbb{P}^N$ be an irreducible non-singular variety of dimension d. Then the s^{th} *secant variety* of X, denoted $\sigma_s(X)$, is defined to be the Zariski closure of the union of the linear spans of all s-tuples of points of X. The study of secant varieties has a long history. The interest in this subject goes back to the Italian school at the turn of the 20^{th} century. This topic has received renewed interest over the past several decades, mainly due to its increasing importance in an ever widening collection of disciplines including algebraic complexity theory [Bürgisser et al. 1997, Landsberg 2006, Landsberg 2008], algebraic statistics [Garcia et al. 2005, Eriksson et al. 2005, Aoki et al. 2007], and combinatorics [Sturmfels and Sullivant 2006, Sullivant 2008].

The major questions surrounding secant varieties center around finding invariants of those objects such as dimension. A simple dimension count suggests that the expected dimension of $\sigma_s(X)$ is $\min\{N, s(d+1)-1\}$. We say that X has a *defective* s^{th} *secant variety* if $\sigma_s(X)$ does not have the expected dimension. In particular, X is said to be *defective* if X has a defective s^{th} secant variety for some s. For instance, the Veronese surface X in $\mathbb{P}^5$ is defective, because the dimension of $\sigma_2(X)$ is four while its expected dimension is five. A well known classification of the defective Veronese varieties was completed in a series of papers by Alexander and Hirschowitz [Alexander and Hirschowitz 1995, Brambilla and Ottaviani 2008]. There are corresponding conjecturally complete lists of defective Segre varieties [Abo et al. 2006] and defective Grassmann varieties [Baur et al. 2007]. Secant varieties of Segre-Veronese varieties are however less well-understood. In recent years, a lot of efforts have been made to develop techniques to study secant varieties of such varieties (see for example [Catalisano et al. 2005, Carlini and Chipalkatti 2003, Carlini and Catalisano 2007, Ottaviani 2006, Catalisano et al. 2008, Ballico 2006, Abrescia 2008]). But even the classification of defective two-factor Segre-Veronese varieties is still far from complete.

In order to classify defective Segre-Veronese varieties, a crucial step is to prove the existence of a large family of non-defective such varieties. A powerful tool to establish non-defectiveness of large classes of Segre-Veronese varieties is the inductive approach based on specialization techniques, which consist in placing a certain number of points on a chosen divisor. For a given $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, we write $\mathbb{P}^{\mathbf{n}}$ for $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$. Let $X^{\mathbf{a}}_{\mathbf{n}}$ be the Segre-Veronese variety $\mathbb{P}^{\mathbf{n}}$ embedded by the morphism given by $\mathcal{O}(\mathbf{a})$ with $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$. As we shall see in Section 2, the problem of determining the dimension of $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ is equivalent to the problem of determining the value of the Hilbert function $h_{\mathbb{P}^{\mathbf{n}}}(Z, \cdot)$ of a collection Z of s general double points in $\mathbb{P}^{\mathbf{n}}$ at $\mathbf{a}$,

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i.e.,

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) = \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}(\mathbf{a})) - \dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{I}_Z(\mathbf{a})).$$

Suppose that $a_k \geq 2$. Denote by $\mathbf{n}'$ and $\mathbf{a}'$ the k-tuples $(n_1, n_2, \ldots, n_k - 1)$ and $(a_1, a_2, \ldots, a_k - 1)$ respectively. Given a $\mathbb{P}^{\mathbf{n}'} \subset \mathbb{P}^{\mathbf{n}}$, we have an exact sequence

$$0 \to \mathcal{I}_{\widetilde{Z}}(\mathbf{a}') \to \mathcal{I}_Z(\mathbf{a}) \to \mathcal{I}_{Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbb{P}^{\mathbf{n}'}}(\mathbf{a}) \to 0,$$

where $\widetilde{Z}$ is the residual scheme of Z with respect to $\mathbb{P}^{\mathbf{n}'}$. This exact sequence gives rise to the so-called *Castelnuovo inequality*

$$h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a}) \geq h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}') + h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbf{a}).$$

Thus, we can conclude that

- if $h_{\mathbb{P}^{\mathbf{n}}}(\widetilde{Z}, \mathbf{a}')$ and $h_{\mathbb{P}^{\mathbf{n}'}}(Z \cap \mathbb{P}^{\mathbf{n}'}, \mathbf{a}')$ are the expected values and
- if the degrees of $\widetilde{Z}$ and $Z \cap \mathbb{P}^{\mathbf{n}'}$ are both less than or both greater than $\dim H^0(\mathbb{P}^{\mathbf{n}}, \mathcal{O}(\mathbf{a}'))$ and $\dim H^0(\mathbb{P}^{\mathbf{n}'}, \mathcal{O}(\mathbf{a}))$ respectively,

then $h_{\mathbb{P}^{\mathbf{n}}}(Z, \mathbf{a})$ is also the expected value. By semicontinuity, the Hilbert function of a general collection of s double points in $\mathbb{P}^{\mathbf{n}}$ has the expected value at $\mathbf{a}$. This enables one to check whether or not $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension by induction on $\mathbf{n}$ and $\mathbf{a}$.

To apply this inductive approach, we need some initial cases regarding either dimensions or degrees. The class of secant varieties of two-factor Segre-Veronese varieties embedded by the morphism given by $\mathcal{O}(1, 2)$ can be viewed as one of such initial cases. In fact, the above-mentioned specialization technique involves secant varieties of two-factor Segre varieties, most of which are known to be defective, and thus we cannot apply this technique to find $\dim \sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ for $\mathbf{n} = (m, n)$ and $\mathbf{a} = (1, 2)$. To sidestep this problem, we therefore need an *ad hoc* approach.

This paper will be devoted to studying secant varieties of Segre-Veronese varieties $\mathbb{P}^m \times \mathbb{P}^n$ embedded by the morphism given by $\mathcal{O}(1, 2)$. Let

$$q(m, n) = \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor.$$

Our main goal is to prove the following theorem:

Theorem 1.1. *Let $\mathbf{n} = (m, n)$ and let $\mathbf{a} = (1, 2)$. If n is sufficiently large, then $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension for any $s \leq q(m, n)$.*

In order to prove Theorem 1.1, it suffices to show that $\sigma_{q(m,n)}(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension. To prove this, we show that if $m \leq n + 2$, then $\sigma_{\underline{s}(m,n)}(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension, where

$$\underline{s}(m, n) = \begin{cases} (m+1)\left\lfloor \frac{n}{2} \right\rfloor - \frac{(m-2)(m+1)}{2} & \text{if } n \text{ is even;} \\ (m+1)\left\lfloor \frac{n}{2} \right\rfloor - \frac{(m-3)(m+1)}{2} & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)\left\lfloor \frac{n}{2} \right\rfloor - \frac{(m-3)(m+1)+1}{2} & \text{if } m \text{ is even and if } n \text{ is odd.} \end{cases}$$

Theorem 1.1 then follows immediately, because $\underline{s}(m, n) = q(m, n)$ for a sufficiently large n (an explicit bound for n can be found just before Corollary 3.14).

To prove hat $\sigma_{\underline{s}(m,n)}(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension, we will use double induction on m and n. More precisely, we will show the following two claims:

(i) Let $\mathbf{n} = (n+1, n)$. Then the secant variety $\sigma_{\underline{s}(n+1,n)}(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension. Note that the case $\mathbf{n} = (n + 2, n)$ is trivial since $\underline{s}(n + 2, n) = 0$.

(ii) Let $\mathbf{n}' = (m, n - 2)$ and $\mathbf{n} = (m, n)$. If $\sigma_{\underline{s}(m,n-2)}(X^{\mathbf{a}}_{\mathbf{n}'})$ has the expected dimension, then $\sigma_{\underline{s}(m,n)}(X^{\mathbf{a}}_{\mathbf{n}})$ has also the expected dimension.

Claim (i) can be proved by an inductive approach that specializes a certain number of points on a subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ of the form $\mathbb{P}^{m'} \times \mathbb{P}^n$ (see Section 2 for more details). Note that a similar approach was successfully applied to study secant varieties of Segre varieties (see for example [Abo et al. 2006]).

The proof of (ii) relies on a different specialization technique which allows to place a certain number of points on a two-codimensional subvariety of $\mathbb{P}^m \times \mathbb{P}^n$ of the form $\mathbb{P}^m \times \mathbb{P}^{n-2}$ (see Section 3 for more details). This approach can be regarded as a modification of the approach introduced in [Brambilla and Ottaviani 2008] that simplifies the proof of the Alexander-Hirschowitz theorem for cubic Veronese varieties. We also would like to mention that the same approach was extended to secant varieties of Grassmannians of planes in [Abo et al. 2008].

In Section 4, we will modify the above techniques to prove the following theorem:

Theorem 1.2. *Let $\mathbf{n} = (m,n)$ and let $\mathbf{a} = (1,2)$ and let*

$$\bar{s}(m,n) = \begin{cases} (m+1)\left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } n \text{ is even;} \\ (m+1)\left\lfloor \frac{n}{2} \right\rfloor + 3 & \text{otherwise.} \end{cases}$$

Then $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension for any $s \geq \bar{s}(m,n)$.

Theorems 1.1 and 1.2 complete the classification of defective Segre-Veronese varieties $X^{1,2}_{m,n}$ for $m = 1, 2$. To be more precise, the following is an immediate consequence of these theorems:

Corollary 1.3. *Let $\mathbf{n} = (m,n)$ and let $\mathbf{a} = (1,2)$.*

(i) *If $m = 1$, then $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension for any s.*
(ii) *If $m = 2$, then $\sigma_s(X^{\mathbf{a}}_{\mathbf{n}})$ has the expected dimension unless $(n,s) = (2k+1, 3k+2)$ with $k \geq 1$.*

Note that (i) is well known (see for example [Carlini and Chipalkatti 2003]). But, to our best knowledge, (ii) was previously unknown. The defectiveness of the $(3k+2)^{\text{nd}}$ secant variety of $X^{1,2}_{2,2k+1}$ has already been established (see [Carlini and Chipalkatti 2003, Ottaviani 2006] for the proofs). Thus Corollary 1.3 (ii) completes the classification of defective secant varieties of $X^{1,2}_{2,n}$.

In Section 5, we will give a conjecturally complete list of defective secant varieties of $X^{1,2}_{m,n}$. Evidence for the conjecture was provided by results in [Catalisano et al. 2005, Carlini and Chipalkatti 2003, Ottaviani 2006]. Further evidence in support of the conjecture was obtained via the computational experiments we carried out. Thus the first part of this section will be devoted to explaining these experiments, which were done with the computer algebra system `Macaulay2` developed by Dan Grayson and Mike Stillman [Grayson and Stillman]. The proofs of Lemmas 3.10 and 4.5 are also based on computations in `Macaulay2`. All the `Macaulay2` scripts needed to make thees computations are available at `http://www.webpages.uidaho.edu/~abo/programs.html`.

2. Splitting Theorem

Let V be an $(m+1)$-dimensional vector space over $\mathbb{C}$ and let W be an $(n+1)$-dimensional vector space over $\mathbb{C}$. For simplicity, we write $\mathbb{P}^{m,n}$ for $\mathbb{P}^m \times \mathbb{P}^n = \mathbb{P}(V) \times \mathbb{P}(W)$. In this section, we indicate by $X_{m,n}$ for the Segre-Veronese variety $\mathbb{P}^{m,n}$ embedded by the morphism $\nu_{1,d}$ given by $\mathcal{O}(1,2)$ for simplicity. Let $T_p(X_{m,n})$ be the affine cone over the tangent space $\mathbb{T}_p(X_{m,n})$ to $X_{m,n}$ at a point $p \in X_{m,n}$.

For each $p \in X_{m,n}$, there are $u \in V \setminus \{0\}$ and $v \in W \setminus \{0\}$ such that $p = [u \otimes v^d] \in \mathbb{P}(V \otimes S_d(W))$. In this case, p can be identified with $([u],[v]) \in \mathbb{P}^{m,n}$ through $\nu_{1,d}$. Thus p also denotes $([u],[v])$. Let $p = [u \otimes v^d] \in X_{m,n}$. Then $T_p(X_{m,n}) = V \otimes v^d + u \otimes v^{d-1}W$. We denote by $Y_p(X_{m,n})$ (or just by Y_p) the $(m+1)$-dimensional subspace $V \otimes v^d$ of $V \otimes S_d(W)$.

Definition 2.1. Let $p_1, \ldots, p_s, q_1, \ldots, q_t$ be general points of $X_{m,n}$ and let $U_{m,n}$ be the subspace of $V \otimes S_d(W)$ spanned by $\sum_{i=1}^s T_{p_i}(X_{m,n})$ and $\sum_{i=1}^t Y_{q_i}(X_{m,n})$. Then $U_{m,n}$ is expected to have dimension

$$\min\left\{ s(m+n+1) + t(m+1), (m+1)\binom{n+d}{d} \right\}.$$

We say that $S(m,n;1,d;s;t)$ *is true* if $U_{m,n}$ has the expected dimension. For simplicity, we denote $S(m,n;1,d;s;0)$ by $T(m,n;1,d;s)$.

Remark 2.2. Let $q_1,\dots,q_t$ be general points of $X_{m,n}$ and let $\sigma_s(X_{m,n})$ be the s^{th} secant variety of $X_{m,n}$. By Terracini's lemma [Terracini 1911], the span of the tangent spaces to $X_{m,n}$ at s generic points is equal to the tangent space to $\sigma_s(X_{m,n})$ at the generic z point on the linear subspace spanned by the s points. Thus the vector space $U_{m,n}$ can be thought of as the affine cone over the tangent space to the join $J(\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t}),\sigma_s(X_{m,n}))$ of $\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t})$ and $\sigma_s(X_{m,n})$ at a general point on the linear subspace spanned by $q_1,\dots,q_t$ and z. Therefore, $S(m,n;1,d;s;t)$ is true if and only if $J(\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t}),\sigma_s(X_{m,n}))$ has the expected dimension. In particular, $\sigma_s(X_{m,n})$ has the expected dimension if and only if $S(m,n;1,d;s;0)$ is true.

Remark 2.3. Let $N=(m+1)\binom{n+d}{d}$. Then $H^0(\mathbb{P}^{m,n},\mathcal{O}(1,d))$ can be identified with the set of hyperplanes in $\mathbb{P}^{N-1}$. Since the condition that a hyperplane $H\subset\mathbb{P}^N$ contains $\mathbb{T}_p(X_{m,n})$ is equivalent to the condition that H intersects $X_{m,n}$ in the first infinitesimal neighborhood of p, the elements of $H^0(\mathbb{P}^{m,n},\mathcal{I}_{p^2}(1,d))$ can be viewed as hyperplanes containing $\mathbb{T}_p(X_{m,n})$. Let $q\in X_{m,n}$. A similar argument shows that the elements of $H^0(\mathbb{P}^{m,n},\mathcal{I}_{q^2|_{\mathbb{P}(Y_q)}}(1,d))$ can be identified with hyperplanes containing Y_q, where $q^2|_{\mathbb{P}(Y_q)}$ is a zero-dimensional subscheme of $X_{m,n}$ of length $m+1$.

Let $p_1,\dots,p_s,q_1,\dots,q_t\in X_{m,n}$ and let $Z=\{p_1^2,\dots,p_s^2,q_1^2|_{\mathbb{P}(Y_{q_1})},\dots,q_t^2|_{\mathbb{P}(Y_{q_t})}\}$. Recall that Terracini's lemma says that the linear subspace spanned by $\mathbb{T}_{p_1}(X_{m,n}),\dots,\mathbb{T}_{p_s}(X_{m,n})$ is the tangent space to $\sigma_s(X_{m,n})$ at a general point on the linear subspace spanned by $p_1,\dots,p_s$. Thus implies that $\dim J(\mathbb{P}(Y_{q_1}),\dots,\mathbb{P}(Y_{q_t}),\sigma_s(X_{m,n}))$ equals the value of the Hilbert function $h_{\mathbb{P}^{m,n}}(Z,\cdot)$ of Z at $(1,d)$, i.e.,

$$h_{\mathbb{P}^{m,n}}(Z,(1,d))=\dim H^0(\mathbb{P}^{m,n},\mathcal{O}(1,d))-\dim H^0(\mathbb{P}^{m,n},\mathcal{I}_Z(1,d)).$$

In particular,

$$h_{\mathbb{P}^{m,n}}(Z,(1,d))=\min\{s(m+n+1)+t(m+1),\ N\}.$$

if and only if $S(m,n;1,d;s;t)$ is true.

Definition 2.4. A six-tuple $(m,n;1,d;s;t)$ is called *subabundant* (resp. *superabundant*)

$$s(m+n+1)+t(m+1)\le(m+1)\binom{n+d}{d}\ \text{(resp. }\ge).$$

We say that $(m,n;1,d;s;t)$ is *equiabundant* if it is both subabundant and superabundant. We write statements such as $(m,n;1,d;s)$ is subabundant (resp. superabundant, resp. equiabundant) when we really mean that $(m,n;1,d;s;0)$ is subabundant (resp. superabundant, resp. equiabundant).

Remark 2.5. Given two vectors (s,t) and (s',t'), we say that $(s,t)\ge(s',t')$ if $s\ge s'$ and $t\ge t$. Suppose that $S(m,n;1,2;s;t)$ is true and that $(m,n;1,2;s;t)$ is subabundant (resp. superabundant). Then $S(m,n;1,2;s';t')$ is true for any choice of s' and t' with $(s,t)\ge(s',t')$ (resp. with $(s,t)\le(s',t')$).

Remark 2.6. Suppose that $m=0$. We make the following simple remarks:

(i) Let $q\in X_{0,m}$. Then $\mathbb{P}(Y_q(X_{0,n}))$ is just q itself. Thus if $q_1,\dots,q_t$ are general points of $X_{0,n}$ and if $(0,n;1,d;s;t)$ is subabundant, then $S(0,n;1,d;s;t)$ is true if and only if $T(0,n;1,d;s)$ is true.

(ii) Since $S(0,n;1,d;n+1;0)$ is true, if $(0,n;1,d;s)$ is superabundant and if $s\ge n+1$, then $S(0,n;1,d;s)$ is true.

Theorem 2.7. *Let $m=m'+m''+1$ and let $s=s'+s''$. If $(m',n;1,d;s';s''+t)$ and $(m'',n;1,d;s'';s'+t)$ are subabundant (resp. superabundant, resp. equiabundant) and if $S(m',n;1,2;s':s''+t)$ and $S(m,n;1,d;s'';s'+t)$ are true, then $(m,n;1,d;s;t)$ is subabundant (resp. superabundant, resp. equiabundant) and $S(m,n;1,d;s;t)$ is true.*

Proof. Here we only prove the theorem for the case where $(m', n; 1, d; s'; s'' + t)$ and $(m'', n; 1, d; s''; s' + t)$ are both subabundant, because the remaining cases can be proved in a similar manner. Let V' and V'' be subspaces of V of dimensions $m' + 1$ and $m'' + 1$ respectively. Suppose that V is the direct sum of V' and V''. Let $p = [u \otimes v^d] \in X_{m,n}$. If $u \in V''$, then we have

$$\begin{aligned} T_p(X_{m,n}) &= V \otimes v^d + u \otimes v^{d-1} W \\ &= (V' \otimes v^d + u \otimes v^{d-1} W) \oplus (V'' \otimes v^d) \\ &= T_p(X_{m',n}) \oplus Y''_p(X_{m'',n}) \end{aligned}$$

for some $p'' \in X_{m'',n}$ (p'' must be of the form $[u'' \otimes v^d]$ with $u'' \in V''$). Similarly, one can prove that if $u \in V''$, then $T_p(X_{m,n}) = Y'_p(X_{m',n}) \oplus T_p(X_{m'',n})$ for some $p' \in X_{m',n}$.

Let $q = [u' \otimes v'^2] \in X_{m,n}$. Then there exist $q' \in X_{m',n}$ and $q'' \in X_{m'',n''}$ such that

$$\begin{aligned} Y_q(X_{m,n}) &= V \otimes v'^d \\ &= (V' \otimes v'^d) \oplus (V'' \otimes v'^d) \\ &= Y_{q'}(X_{m',n}) \oplus Y_{q''}(X_{m'',n}). \end{aligned}$$

Thus one can conclude that $U_{m,n} \simeq U_{m',n} \oplus U_{m'',n}$. By assumption, $\dim U_{m',n} = s'(m' + n + 1) + (s'' + t)(m' + 1)$ and $\dim U_{m'',n} = s''(m'' + n + 1) + (s' + t)(m'' + 1)$. Thus $\dim U_{m,n} = \dim U_{m',n} + \dim U_{m'',n} = s(m+n+1) + t(m+1) \leq (m'+1)\binom{n+d}{d} + (m''+1)\binom{n+d}{d} = (m+1)\binom{n+d}{d}$, and hence $(m, n; 1, d; s, t)$ is subabundant and $S(m, n; 1, d; s; t)$ is true □

We will discuss three examples to illustrate how to use Theorem 2.7 below. These examples will be used in later sections.

Example 2.8. In this example, we apply Theorem 2.7 to prove that $T(2, 2; 1, 2; s)$ is true for every $s \leq 3$. Note that $(2, 2; 1, 2; s)$ is subabundant. Thus it suffices to show that $T(2, 2; 1, 2; 3)$ is true. Since $(1, 2; 1, 2; 2; 1)$ and $(0, 2; 1, 2; 1; 2)$ are both subabundant, one can reduce $T(2, 2; 1, 2; 3)$ to $S(1, 2; 1, 2; 2; 1)$ and $S(0, 2; 1, 2; 1; 2)$. The statement $S(1, 2; 1, 2; 2; 1)$ can be reduced to twice $S(0, 2; 1, 2; 1; 2)$. This means that $T(2, 2; 1, 2; 3)$ is reduced to triple $S(0, 2; 1, 2; 1; 2)$. Clearly $S(0, 2; 1, 2; 1; 0)$ is true, and so is $S(0, 2; 1, 2; 1; 2)$. Hence we completed the proof.

Example 2.9. We prove that $T(m, 1; 1, 2; 3)$ is true for any m. The proof is by induction. It has been already proved that $T(1, 1; 2, 3; 3)$ is true (see [Catalisano et al. 2005]). Now suppose that $T(m-1, 1; 1, 2; 3)$ is true for some m. Note that $(m, 1; 1, 2; 3)$ is superabundant. Since $(m-1, 1; 1, 2; 3; 0)$ and $(0, 1; 1, 2; 0; 3)$ are also superabundant, we can reduce $T(m, 1; 1, 2; 3)$ to $S(m-1, 1; 1, 2; 3; 0)$ and $S(0, 1; 1, 2; 0; 3)$. Clearly, $S(0, 1; 1, 2; 0; 3)$ is true. Since $T(m-1, 1; 1, 2; 3)$, and hence $S(m-1, 1; 1, 2; 3; 0)$, is true by induction hypothesis, $T(m, 1; 1, 2; 3)$ is also true.

Example 2.10. Here we prove that $T(n+1, n; 1, 2; s)$ is true for any $s \leq \left\lfloor \frac{n+1}{2} \right\rfloor + 1$. Note that $(n+1, n; 1, 2; s)$ is subabundant for such an s. Thus it is sufficient to prove that $T(n+1, n; 1, 2; s)$ is true if $s = \left\lfloor \frac{n+1}{2} \right\rfloor + 1$.

First suppose that n is even, i.e., $n = 2k$ for some integer k. Then $s = k + 1$. Since $(2k, 2k; 1, 2; k, 1)$ and $(0, 2k; 1, 2; 1; k)$ are both subabundant, $T(2k+1, 2k; 1, 2; k+1) = S(2k+1, 2k : 1, 2; k+1; 0)$ can be reduced to $S(2k, 2k; 1, 2; k; 1)$ and $S(0, 2k; 1, 2; 1; k)$. For the same reason, $S(2k, 2k; 1, 2; k; 1)$ can be reduced to $S(2k-1, 2k; 1, 2; k-1; 2)$ and $S(0, 2k; 1, 2; 1; k)$. That means $T(2k+1, 2k; 1, 2; k+1)$ is now reduced to $S(2k-1, 2k; 1, 2; k-1; 2)$ and twice $S(0, 2k; 1, 2; 1; k)$ (we will denote it by $2 * S(0, 2k; 1, 2; 1; k)$). We can repeat the same process $k-2$ times to reduce to reduce $T(2k+1, 2k; 1, 2; k+1)$ to $S(k, 2k; 1, 2; 0; k+1)$ and $(k+1) * S(0, 2k; 1, 2; 1; k)$. The statement $S(k, 2k; 1, 2; 0; k+1)$ can be reduced to $S(k-1, 2k; 1, 2; 0; k+1)$ and $S(0, 2k; 1, 2; 0; k+1)$. $S(k-1, 2k; 1, 2; 0; k+1)$ can be reduced to $S(k-2, 2k; 1, 2; 0; k+1)$ and $S(0, 2k; 1, 2; 0; k+1)$. Repeating the same process $k-2$ times, we can reduce $S(k, 2k; 1, 2; 0; k+1)$ to $(k+1) * S(0, 2k; 1, 2; 0; k+1)$. Recall that $(0, 2k; 1, 2; 1; k)$ and $(0, 2k; 1, 2; 0; k+1)$ are subabundant. Thus $S(0, 2k; 1, 2; 1; k)$ and

$S(0,2k;1,2;0;k+1)$ are true because $S(0,2k;1,2;1;0)$ and $S(0,2k;1,2;0;0)$ are true. This implies that $T(2k+1,2k;1,2;k+1)$ is true.

In the same way, we can also prove that $T(n+1,n;1,2;s)$ is true if n is odd. Indeed, $T(2k+2,2k+1;1,2;k+2)$ can be reduced to $(k+2)*S(0,2k+1;1,2;1:k+1)$ and $(k+1)*S(0,2k+1;1,2;0;k+2)$. Since $S(0,2k+1;1,2;1:k+1)$ and $S(0,2k+1;1,2;0;k+2)$ are true, so is $T(2k+2,2k+1;1,2;k+2)$.

As immediate consequences of Theorem 2.7, we can prove the following two propositions:

Proposition 2.11. *$T(m,n;1,2;s)$ is true if $s \le m+1$ and $m \le \binom{n+1}{2}$ or if $s \ge (m+1)(n+1)$.*

Proof. We first prove that $T(m,n;1,2;s)$ is true for any $s \le m+1$. Note that $(m,n;1,2;s)$ is subabundant for such an s, it is enough to prove that $T(m,n;1,2;m+1)$ is true. This statement can be reduced to $(m+1)*S(0,n;1,2;1;m)$. Since $m \le \binom{n+1}{2}$,

$$(n+1)+m \ge \binom{n+2}{2}.$$

Thus $(0,n;1,2;1;m)$ is subabundant. Since $S(n;1,2;1;0)$ is true, so is $S(0,n;1,2;1;m)$. This implies that $T(m,n;1,2;m+1)$ is true.

To show that $T(m,n;1,2;s)$ is true for any $s \ge (m+1)(n+1)$, it is enough to prove that $T(m,n;1,2;(m+1)(n+1))$ is true. Note that $(m,n;1,2;(m+1)(n+1))$ is superabundant. The statement can be reduced to $(m+1)*S(0,n;1,2;(m+1);(m+1)n)$. Since $(0,n;1,2;(m+1);0;0)$ is superabundant and $T(0,n;1,2;(m+1))$ is true, $(0,n;1,2;(m+1);(m+1)n)$ is superabundant and $S(0,n;1,2;(m+1);(m+1)n)$ is true. Thus $T(m,n;1,2;(m+1)(n+1))$ is true. □

Remark 2.12. In Sections 3 and 4, we will use different techniques to improve the bounds given in Proposition 2.11.

Proposition 2.13. *Suppose that $m \ge 1$ and $d \ge 3$. Let $\ell = \left\lfloor \frac{\binom{n+d}{d}}{m+n+1} \right\rfloor$ and let $h = \left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$.*

(i) *$T(m,n;1,d;s)$ is true for any $s \le \ell(m+1)$.*
(ii) *If $(n,d) \ne (2,4)$, $(3,4)$, $(4,3)$, $(4,4)$ and if $s \ge h(m+1)$, then $T(m,n;1,d;s)$ is true.*
(iii) *If $(n,d) = (2,4)$, $(3,4)$, $(4,3)$ or $(4,4)$, then $T(m,n;1,d;s)$ is true for any $s \ge (h+1)(m+1)$.*

Proof. (i) Suppose that $s = \ell(m+1)$. Then $T(m,n;1,d;s)$ can be reduced to $(m+1)*S(0,n;1,d;\ell;\ell m)$. Since

$$\ell(n+1)+\ell m = \ell(m+n+1) \le \frac{\binom{n+d}{d}}{m+n+1}(m+n+1) = \binom{n+d}{d},$$

$(0,n;1,d;\ell;\ell m)$ is subabundant (this implies that $(m,n;1,d;s)$ is subabundant too). Furthermore, since $\ell < \left\lfloor \frac{\binom{n+d}{d}}{n+1} \right\rfloor$, $S(0,n;1,d;\ell;0)$ is true by the Alexander-Hirschowitz theorem [Alexander and Hirschowitz 1995]. Thus $S(0,n;1,d;\ell;\ell m)$ is true, which implies that $T(m,n;1,d;s)$ is true.

(ii) Let $s = h(m+1)$. Then $(m,n;1,d;s)$ is clearly superabundant. The statement $T(m,n;1,d;s)$ can be reduced to $(m+1)*S(0,n;1,d;h;hm)$. Suppose that $n \ne 3,4$. Then the Alexander-Hirschowitz theorem says that $S(0,n;1,d;h;0)$ is true, and so is $S(0,n;1,d;h;hm)$.

(iii) Suppose that $(n,d) = (2,4)$, $(3,4)$, $(4,3)$ or $(4,4)$. Then $S(0,n;1,d;h+1;0)$ is true by the Alexander-Hirschowitz theorem, and thus $S(0,n;1,d;h;(h+1)m)$ is also true. Therefore the same argument as above proves that $T(m,n;1,d;s)$ is true if $s = (h+1)(m+1)$. □

3. Segre-Veronese varieties $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$: Subabundant Case

Let V be an $(m+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{e_0, \dots, e_m\}$ and let W be an $(n+1)$-dimensional vector space over $\mathbb{C}$ with basis $\{f_0, \dots, f_m\}$. As in the previous section, $X_{m,n}$ denotes $X^{1,2}_{m,n}$.

Definition 3.1. Let $k = \lfloor \frac{n}{2} \rfloor$ and let

$$\underline{s} = \begin{cases} (m+1)k - \frac{(m-2)(m+1)}{2} & \text{if } n \text{ is even;} \\ (m+1)k - \frac{(m-3)(m+1)}{2} & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)k - \frac{(m-3)(m+1)+1}{2} & \text{if } m \text{ is even and if } n \text{ is odd.} \end{cases}$$

Note that $\underline{s} = 0$ if $m = n + 2$.

The goal of this section is to prove that $T(m, n; 1, 2; s)$ is true for any $s \leq \underline{s}$ if $m \leq n + 2$. Since $(m, n; 1, 2; s)$ is subabundant, it is sufficient to prove that $T(m, n; 1, 2; \underline{s})$ is true. The proof is by double induction on m and n.

It is obvious that $T(m, m-2; 1, 2; 0)$ is true. It also follows from Example 2.10 that $T(m, m-1; 1, 2; \underline{s})$ is true. Thus it remains only to show that if $T(m, n-2; 1, 2; \underline{s} - (m+1))$ is true, then so is $T(m, n; 1, 2; \underline{s})$. To do this, we need to introduce some auxiliary statements $\underline{R}(m, n)$ and $Q(m, n)$ (see Definitions 3.2 and 3.6 for further details).

Let U_L be a two-codimensional subspace of W and let $L = \mathbb{P}(V) \times \mathbb{P}(U_L)$. Note that if p is a point of $\nu_{1,2}(L)$, then the affine cone $T_p(X_{m,n})$ over the tangent space to $X_{m,n}$ at p modulo $V \otimes S_2(U_L)$ has dimension $(m+n+1) - (m+n-2+1) = 2$.

Definition 3.2. Let k and $\underline{s}$ be the integers given in Definition 3.1, let $p_1, \dots, p_{\underline{s}-(m+1)}$ be general points of L, let $q_1, \dots, q_{m+1}$ be general points of $\mathbb{P}^{m,n} \setminus L$ and let $V_{m,n}$ be the vector space $\langle V \otimes S_2(U_L), \sum_{i=1}^{\underline{s}-(m+1)} T_{p_i}(X_{m,n}), \sum_{i=1}^{m+1} T_{q_i}(X_{m,n})\rangle$. Then the following inequality holds:

$$\begin{aligned} \dim V_{m,n} &\leq (m+1)\binom{n}{2} + 2[\underline{s} - (m+1)] + (m+1)(m+n+1) \\ &= \begin{cases} (m+1)\binom{n+2}{2} & \text{if } n \text{ is even, or if } m \text{ and } n \text{ are odd;} \\ (m+1)\binom{n+2}{2} - 1 & \text{if } m \text{ is even and if } n \text{ is odd.} \end{cases} \end{aligned}$$

We say $\underline{R}(m, n)$ *is true* if the equality holds. Remark 2.3 implies that $\underline{R}(m, n)$ is true if and only if

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L}(1,2)) = \begin{cases} 0 & \text{if } n \text{ is even or if } m \text{ and } n \text{ are odd;} \\ 1 & \text{if } m \text{ is even and if } n \text{ is odd,} \end{cases}$$

where $Z = \{p_1^2, \dots, p^2_{\underline{s}-(m+1)}, q_1^2, \dots, q^2_{m+1}\}$.

Proposition 3.3. *If $\underline{R}(m, n)$ is true and if $T(m, n-2; 1, 2; \underline{s} - (m+1))$ is true, then $T(m, n; 1, 2; \underline{s})$ is true.*

Proof. Let $k = \lfloor \frac{n}{2} \rfloor$, let $p_1, \dots, p_{\underline{s}} \in \mathbb{P}^{m,n}$ and let $Z = \{p_1^2, \dots, p_{\underline{s}}^2\}$. Then

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \geq \begin{cases} (m+1) + \frac{(m-2)(m+1)^2}{2} & \text{if } n \text{ is even;} \\ 3(m+1) + \frac{(m-3)(m+1)(m+2)}{2} & \text{if } m \text{ and } n \text{ are odd;} \\ k + 3(m+1) + \frac{(m-3)(m+1)(m+2)+(m+2)}{2} & \text{otherwise.} \end{cases}$$

Suppose that $p_1, \dots, p_{\underline{s}-(m+1)} \in L$ and that $p_{\underline{s}-m}, \dots, p_{\underline{s}} \in \mathbb{P}^{m,n} \setminus L$. Let $Z = \{p_1^2, \dots, p_{\underline{s}}^2\}$. Let $Z' = Z \cap L = \{p_1^2, \dots, p^2_{\underline{s}-(m+1)}\}$. Then we have the following short exact sequence:

$$0 \to \mathcal{I}_{Z \cup L}(1,2) \to \mathcal{I}_Z(1,2) \to \mathcal{I}_{Z',L}(1,2) \to 0.$$

Taking cohomology, we have

$$0 \to H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L}(1,2)) \to H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \to H^0(L, \mathcal{I}_{Z'}(1,2)).$$

Thus we must have

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \leq \dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z\cup L}(1,2)) + \dim H^0(L, \mathcal{I}_{Z'}(1,2)).$$

Since $\underline{R}(m,n)$ and $T(m,n;1,2;\underline{s}-(m+1))$ are true, we have

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_Z(1,2)) \leq \begin{cases} \dim H^0(L, \mathcal{I}_{Z'}(1,2)) + 1 & \text{if } m \text{ is even and if } n \text{ is odd;} \\ \dim H^0(L, \mathcal{I}_{Z'}(1,2)) & \text{otherwise,} \end{cases}$$

from which the proposition follows. □

To prove that $T(m,n;1,2;\underline{s})$ is true, it is therefore enough to show that $\underline{R}(m,n)$ is true if $m \leq n$. The proof is by double induction on m and n. To be more precise, we first prove $\underline{R}(m,m)$ and $\underline{R}(m,m+1)$ are true. Then we show that if $\underline{R}(m,n-2)$ is true, then $\underline{R}(m,n)$ is also true.

Proposition 3.4. *$\underline{R}(m,m)$ is true for any $m \geq 1$.*

Proof. Without loss of generality, we may assume that $U_L = \langle f_2, \dots, f_{m+1}\rangle$. Let $p_0 \dots, p_m \in \mathbb{P}^{m,m} \setminus L$. For each $i \in \{0,\dots,m\}$, we can consider p_i as a pair (u_i, v_i) of a vector u_i in V and v_i in $W \setminus U_L$. Recall that

$$T_{p_i}(X_{m,m}) = V \otimes v_i^2 + u_i \otimes v_i W.$$

To prove the proposition, we will find explicit vectors u_i's and v_i's such that

$$V \otimes S_2(W) \equiv \sum_{i=0}^{m} T_{p_i}(X_{m,m}) \pmod{V \otimes S_2(U_L)}.$$

Let $u_i = e_i$ for each $i \in \{0,\dots,m\}$ and let

$$v_i = \begin{cases} f_i & \text{for } i = 0,1; \\ if_0 + f_1 + f_i & \text{for } 2 \leq i \leq m. \end{cases}$$

Then we have

$$T_{p_i}(X_{m,m}) = \begin{cases} \left\langle e_0 \otimes f_0^2, \dots, e_m \otimes f_0^2, e_0 \otimes f_0 f_1, \dots, e_0 \otimes f_0 f_m \right\rangle & \text{if } i = 0; \\ \left\langle e_0 \otimes f_1^2, \dots, e_m \otimes f_1^2, e_1 \otimes f_0 f_1, \dots, e_1 \otimes f_1 f_m \right\rangle & \text{if } i = 1; \\ \left\langle e_0 \otimes (if_0 + f_1 + f_i)^2, \dots, e_m \otimes (if_0 + f_1 + f_i)^2, \right. & \\ \left. e_i \otimes (if_0 + f_1 + f_i) f_0, \dots, e_i \otimes (if_0 + f_1 + f_i) f_m \right\rangle & \text{if } i \geq 2. \end{cases}$$

Now we prove that every monomial in $\{\ e_i \otimes f_j f_k \mid 0 \leq i,j \leq 1, j \leq k \leq m\ \}$ lies in $\langle V \otimes S_2(U_L),\ \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$.

For each $i \in \{2,\dots,m\}$, we have

$$\begin{aligned} e_0 \otimes (if_0 + f_1 + f_i)^2 &\equiv e_0 \otimes (i^2 f_0^2 + f_1^2 + f_i^2 + 2if_0 f_1 + 2if_0 f_i + 2f_1 f_i) \\ &\equiv e_0 \otimes 2f_1 f_i \pmod{\langle V \otimes S_2(U_L),\ T_{p_1}(X_{m,m}),\ T_{p_2}(X_{m,m})\rangle}. \end{aligned}$$

Indeed, $e_0 \otimes f_0^2, e_0 \otimes f_0 f_1$ and $e_0 \otimes f_0 f_i$ are in $T_{p_1}(X_{m,m})$; $e_0 \otimes f_1^2$ is in $T_{p_2}(X_{m,m})$; $e_0 \otimes f_i^2$ is in $V \otimes S_2(U_L)$. Similarly, one can prove that

$$e_1 \otimes (if_0 + f_1 + f_i)^2 \equiv e_1 \otimes 2if_0 f_i \pmod{\langle V \otimes S_2(U_L),\ T_{p_1}(X_{m,m}),\ T_{p_2}(X_{m,m})\rangle}$$

for each $i \in \{2,\dots,m\}$. So we have proved that $e_i \otimes f_j f_k \in \sum_{i=0}^m T_{p_i}(X_{m,m})$ if $i,j \in \{0,1\}$ and $k \in \{0,\dots,m\}$.

Note that, for each $i \in \{2,\dots,m\}$,

$$\begin{aligned} e_i \otimes (if_0 + f_1 + f_i) f_0 &\equiv ie_i \otimes f_0 f_1 + e_i \otimes f_0 f_i; \\ e_i \otimes (if_0 + f_1 + f_i)^2 &\equiv 2ie_i \otimes f_0 f_1 + 2ie_i \otimes f_0 f_i + 2e_i \otimes f_1 f_i; \\ e_i \otimes (if_0 + f_1 + f_i) f_1 &\equiv ie_i \otimes f_0 f_1 + e_i \otimes f_1 f_i \end{aligned}$$

modulo $\langle V \otimes S_2(U_L),\ \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$. Thus

$$e_i \otimes (if_0 + f_1 + f_i)^2 - 2e_i \otimes (if_0 + f_1 + f_i) f_0 - \frac{2i-2}{i} e_i \otimes (if_0 + f_1 + f_i) f_1$$

is congruent $\frac{2}{i}e_i \otimes f_1 f_i$ modulo $\langle V \otimes S_2(U_L),\ \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$. Thus $e_i \otimes f_1 f_i$, and hence $e_i \otimes f_0 f_1$ and $e_i \otimes f_0 f_i$, is in $\langle V \otimes S_2(U_L),\ \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$.

For every integer j such that $i \neq j$ and $j \geq 2$, we have

$$\begin{array}{rcl} e_i \otimes (if_0 + f_1 + f_i)f_j & \equiv & ie_i \otimes f_0 f_j + e_i \otimes f_1 f_j; \\ e_i \otimes (jf_0 + f_1 + f_j)^2 & \equiv & 2je_i \otimes f_0 f_j + 2e_i \otimes f_1 f_j \end{array}$$

modulo $\langle V \otimes S_2(U_L),\ \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$. Hence

$$e_i \otimes (jf_0 + f_1 + f_j)^2 - \frac{2j}{i} e_i \otimes (if_0 + f_1 + f_i)f_j \equiv \frac{2(i-j)}{i} e_i \otimes f_1 f_j.$$

This implies that $e_i \otimes f_1 f_j$, and hence $e_i \otimes f_0 f_j$, is contained in $\langle V \otimes S_2(U_L),\ \sum_{i=0}^m T_{p_i}(X_{m,m})\rangle$, which completes the proof. □

Proposition 3.5. *$\underline{R}(m, m+1)$ is true for any $m \geq 1$.*

Proof. We prove that the statement is true if m is even, because the other case can be proved in the same way.

Since m is even, $\underline{s} = \frac{3}{2}m + 1$. Let $p_1, \dots, p_{\frac{1}{2}m} \in L$ and let $q_1, \dots, q_{m+1} \in \mathbb{P}^{m,m+1} \setminus L$. Choose a subvariety $\mathbb{P}^{m,m} = \mathbb{P}(V) \times \mathbb{P}(W') \subset \mathbb{P}^{m,m+1}$ in such a way that the intersection of $\mathbb{P}^{m,m}$ with L is $\mathbb{P}^{m,m-2}$. We denote it by H. Specialize $q_1, \dots, q_{m+1}$ in $H \setminus L$. Suppose that $p_1, \dots, p_{\frac{1}{2}m} \notin H$. Let $Z = \{p_1^2, \dots, p_{\frac{1}{2}m}^2, q_1^2, \dots, q_{m+1}^2\}$. Then we have an exact sequence

$$0 \to \mathcal{I}_{Z \cup L \cup H}(1,2) \to \mathcal{I}_{Z \cup L}(1,2) \to \mathcal{I}_{(Z \cup L) \cap H, H}(1,2) \to 0.$$

By Proposition 3.4, $\underline{R}(m,m)$ is true. Thus $\dim H^0(\mathcal{I}_{(Z\cup L)\cap H,H}(1,2)) = 0$. So we have

$$\dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z \cup L \cup H}(1,2)) = \dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z \cup L}(1,2)).$$

Thus we need to prove that $\dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z\cup L\cup H}(1,2)) = 1$.

Let $\widetilde{Z}$ be the residual of $Z \cup L$ by H. Then $H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{Z\cup L\cup H}(1,2)) \simeq H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{\widetilde{Z}}(1,1))$. Note that $\widetilde{Z}$ consists of $\frac{m}{2}$ double points $p_1^2, \dots, p_{\frac{m}{2}}^2$, $m+1$ single points $q_1, \dots, q_{m+1}$ and L.

We denote by $X'_{m,m+1}$ the Segre variety $X^{1,1}_{m,n}$ obtained from $\mathbb{P}^{m,m+1}$ by embedding by the morphism given by $\mathcal{O}(1,1)$. The condition that $\dim H^0(\mathbb{P}^{m,m+1}, \mathcal{I}_{\widetilde{Z}}(1,1)) = 1$, i.e., $h_{\mathbb{P}^{m,m+1}}(\widetilde{Z},(1,1)) = (m+1)(m+2)-1$, is equivalent to the condition that the following subspace of $V \otimes W$ has dimension $(m+1)(m+2)-1$:

$$\left\langle V \otimes U_L,\ \sum_{i=1}^{m/2} T_{p_i}(X'_{m,m+1}),\ \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i\ \rangle \right\rangle,$$

where $T_{p_i}(X'_{m,m+1}) = V \otimes v_i + u_i \otimes W$ if $p_i = [u_i \otimes v_i] \in X'_{m,m+1}$ and where $q_i = [u'_i \otimes v'_i]$. We may assume that $U_L = \langle f_0, \dots, f_{m-1}\rangle$ and that $W = \langle f_1, \dots, f_{m+1}\rangle$. Then

$$T_{p_i}(X'_{m,m+1}) \equiv u_i \otimes \langle f_m, f_{m+1}\rangle \pmod{V \otimes U_L},$$

which implies that

$$\langle V \otimes U_L, T_{p_i}(X'_{m,m+1})\rangle = (V \otimes f_0) \oplus \left\langle V \otimes (U_L \cap W),\ \sum_{i=1}^{m/2} u_i \otimes \langle f_m, f_{m+1}\rangle \right\rangle.$$

Thus

$$\begin{aligned} &\left\langle V \otimes U_L,\ \sum_{i=1}^{m/2} T_{p_i}(X'_{m,m+1}),\ \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i\ \rangle \right\rangle \\ &= (V \otimes f_0) \oplus \left\langle V \otimes (U_L \cap W'),\ \sum_{i=1}^{m/2} u_i \otimes \langle f_m, f_{m+1}\rangle,\ \sum_{i=1}^{m+1} \langle u'_i \otimes v'_i\ \rangle \right\rangle. \end{aligned}$$

Note that $T_1 = \{ \ e_i \otimes f_0 \mid 0 \leq i \leq m+1 \ \}$ and $T_2 = \{ \ e_i \otimes f_j \mid 0 \leq i \leq m, 1 \leq j \leq m-1 \ \}$ are bases for $V \otimes f_0$ and $V \otimes (U_L \cap W')$ respectively. Let $u_i = e_{i-1}$ for every $i \in \left\{1, \dots, \frac{m}{2}\right\}$. Then $T_3 = \left\{ \ e_i \otimes f_j \mid 0 \leq i \leq \frac{m}{2} - 1, m \leq j \leq m+1 \right\}$. Let T_4 be the standard basis for $V \otimes W$ setminus $T_1 \cup T_2 \cup T_3$. Then T_4 consists of $m+2$ distinct non-zero vectors. Choose $m+1$ distinct elements of T_4 as $u_i' \otimes v_i'$'s. Then $\bigcup_{i=1}^4 T_i$ spans an $(m+1)(m+2)-1$-dimensional vector space, which completes the proof. □

Let U_M be another two-codimensional subspaces of W and let M be the subvariety of $\mathbb{P}^{m,n}$ of the forms $\mathbb{P}(V) \times \mathbb{P}(U_M)$. If L and M are general, then we have

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{L \cup M}(1,2)) = (m+1)\left[\binom{n+2}{2} - 2\binom{n}{2} + \binom{n-2}{2}\right] = 4(m+1).$$

This is equivalent to the condition that the subspace of $V \otimes W$ spanned by $V \otimes U_L$ and $V \otimes U_M$ has codimension $4(m+1)$.

Definition 3.6. Let $p_1, \dots, p_{m+1}$ be general points of L and let $q_1, \dots, q_{m+1}$ be general points of M. We denote by $W_{m,n}$ the subspace of $V \otimes S_2(W)$ spanned by $V \otimes S_2(U_L)$, $V \otimes S_2(U_M)$, $\sum_{i=1}^3 T_{p_i}(X_{m,n})$ and $\sum_{i=1}^3 T_{q_i}(X_{m,n})$. Then $\dim W_{m,n}$ is expected to be $(m+1)\binom{n+2}{2}$. We say that $Q(m,n)$ *is true* if $W_{m,n}$ has the expected dimension.

Remark 3.7. Keeping the same notation as in the previous definition, we denote by Z the zero-dimensional subscheme $\{p_1^2, \dots, p_{m+1}^2, q_1^2, \dots, q_{m+1}^2\}$. Then $Q(m,n)$ is true if and only if $\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L \cup M}(1,2)) = 0$.

Proposition 3.8. *If $Q(m,n)$ and $\underline{R}(m,n-2)$ are true, then $\underline{R}(m,n)$ is also true.*

Proof. Let $p_1, \dots, p_{s-(m+1)} \in L$ and let $q_1, \dots, q_{m+1} \in \mathbb{P}^{m,n} \setminus L$. Suppose that $p_1, \dots, p_{s-2(m+1)} \in L \cap M$, $p_{s-2m-1} \dots, p_{s-(m+1)} \in L \setminus L \cap M$ and $q_1, \dots, q_{m+1} \in M$. Let $Z' = Z \cap M$. Then we have an exact sequence

$$0 \to \mathcal{I}_{Z \cup L \cup M}(1,2) \to \mathcal{I}_{Z \cup L}(1,2) \to \mathcal{I}_{Z' \cup (L \cap M), M}(1,2) \to 0.$$

Taking cohomology gives rise to the following exact sequence:

$$0 \to H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L \cup M}(1,2)) \to H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L}(1,2)) \to H^0(M, \mathcal{I}_{Z' \cup (L \cap M), M}(1,2)).$$

By the assumption that $Q(m,n)$ is true, $\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L \cup M}(1,2)) = 0$. Thus we have

$$\dim H^0(\mathbb{P}^{m,n}, \mathcal{I}_{Z \cup L}(1,2)) \leq \dim H^0(M, \mathcal{I}_{Z' \cup (L \cap M), M}(1,2)).$$

Hence if $\underline{R}(m,n-2)$ is true, then so is $\underline{R}(m,n)$. □

Lemma 3.9. *If $Q(m-2,n)$ and $Q(1,n)$ are true, then $Q(m,n)$ is also true.*

Proof. Let V' be a $(m-1)$-dimensional subspace of V and let V'' be a two-dimensional subspace of V. Suppose that V can be written as the direct sum of V' and V''. Let $U = \langle V \otimes S_2(U_L), V \otimes S_2(U_M), T_p(X_{m,n}) \rangle$. Suppose that if $p = ([u],[v^2]) \in \mathbb{P}^{m-2,n} = \mathbb{P}(V') \times \mathbb{P}(U_L)$. Then $V \otimes v^2 \subset V \otimes S_2(W)$. Thus

$$T_p(X_{m,n}) \equiv T_p(X_{m-2,n}) \ (\mathrm{mod}\ V \otimes S_2(U_L)).$$

Similarly, it can be proved that

$$T_q(X_{m,n}) \equiv T_q(X_{1,n}) \ (\mathrm{mod}\ V \otimes S_2(U_L))$$

if $q = ([u],[v]) \in \mathbb{P}(V'') \times \mathbb{P}(W)$.

This means that if $p_1, \dots, p_{m+1} \in \mathbb{P}(V') \times \mathbb{P}(W)$ and if $q_1, \dots, q_{m+1} \in \mathbb{P}(V'') \times \mathbb{P}(W)$, then

$$\left\langle V \otimes S_2(U_L), \sum_{i=1}^{m+1} T_{p_i}(X_{m,n}), \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}) \right\rangle$$
$$= \left\langle V' \otimes S_2(U_L), \sum_{i=1}^{m+1} T_{p_i}(X_{m-2,n}) \right\rangle \oplus \left\langle V'' \otimes S_2(U_L), \sum_{i=1}^{m+1} T_{q_i}(X_{1,n}) \right\rangle.$$

In other words, $W_{m,n} \simeq W_{m-2,n} \oplus W_{1,n}$. Thus if $Q(m-2,n)$ and $Q(1,n)$ are true, so is $Q(m,n)$. □

Lemma 3.10. *Let $n \geq 3$. Then $Q(1,n)$ and $Q(2,n)$ are true.*

Proof. Here we only prove that $Q(1,n)$ is true for any $n \geq 3$, because the proof of the remaining case follows the same path.

To prove that $Q(1,n)$ is true, it is enough to prove that $Q(1,3)$ is true, because $\dim H^0(\mathbb{P}^{1,3}, \mathcal{I}_{Z\cup L\cup M}(1,2)) \geq \dim H^0(\mathbb{P}^{1,n}, \mathcal{I}_{Z\cup L\cup M}(1,2))$. Let p_1 and p_2 be general points of L and let q_1 and q_2 be general points of M. To do so, we directly prove that

$$W_{1,3} = \langle V \otimes S_2(U_L), V \otimes S_2(U_M), T_{p_1}(X_{1,3}), T_{p_2}(X_{1,3}), T_{q_1}(X_{1,3}), T_{q_2}(X_{1,3})\rangle . \tag{3.1}$$

Recall that $T_p(X_{1,3})$ for $p = [u \otimes v^2]$ is isomorphic to $V \otimes v^2 + u \otimes vW$. Thus we can check equality (3.1) as follows. Let $S = \mathbb{C}[e_0, e_1, f_0, \dots, f_3]$. Choose randomly $u_1, \dots, u_4 \in V$, $v_1, v_2 \in U_L$ and $v_3, v_4 \in U_M$. For each $i \in \{1, \dots, 4\}$, let T_i be the ideal of S generated by $u_i \otimes v_i^2, e_1 \otimes v_i^2, u_i \otimes v_i f_0, \dots, u_i \otimes v_i f_3$. Let I_L and I_M be the ideals of S generated by $V \otimes S_2(U_L)$ and $V \otimes S_2(U_M)$ respectively and let $I = \sum_{i=1}^4 T_i + I_L + I_M$. The minimal set of generators for I can be computed in `Macaulay2` and we checked that the members of the minimal generating set form a basis for $V \otimes S_2(W)$. □

Theorem 3.11. *Let $n \geq 3$. Then $Q(m,n)$ is true for any m.*

Proof. The proof is by two-step-induction on m. Since we have already proved this proposition for $m = 1$ and 2, we may assume that $m \geq 3$. The statement $Q(m,n)$ can be reduced to $Q(m-2,n)$ and $Q(1,n)$. By induction hypothesis, $Q(m-2,n)$ is true. Since $Q(1,n)$ is true by Lemma 3.10, it immediately follows from Proposition 3.9 that $Q(m,n)$ is true. □

As mentioned earlier, the following is an immediate consequence of Theorem 3.11:

Corollary 3.12. *Let $m \leq n$. Then $\underline{R}(m,n)$ is true.*

Proof. The proof is by induction on n. By Proposition 3.4, $\underline{R}(m,m)$ is true. The statement $\underline{R}(m,m+1)$ is also true by Proposition 3.5. Assume that $\underline{R}(m,n)$ is true for some $n \geq m$. We may also assume that $n \geq 3$. From Proposition 3.8 and Theorem 3.11 it follows, therefore, that $\underline{R}(m,n)$ is true. Thus we have completed the proof. □

Theorem 3.13. *Suppose that $m \leq n+2$. Then $T(m,n;1,2;s)$ is true for any $s \leq \underline{s}$.*

Proof. Since $(m,n;1,2;s)$ is subabundant, it is enough to prove that $T(m,n;1,2;\underline{s})$ is true. As claimed, the proof is by two-step-induction on n. If $n = m-2$, then $\underline{s}(m,m-2) = 0$. Thus $T(m,m-2;1,2;0)$ is clearly true. If $n = m-1$, then $\underline{s}(m,m-1) = \left\lfloor \frac{m-1}{2} \right\rfloor + 1$. By Example 2.10, $T(m,m-1;1,2;s)$ is true for any $s \leq \left\lfloor \frac{m}{2} \right\rfloor + 1$.

Now suppose that the statement is true for some $m \leq n$. By Proposition 3.3, $T(m,n;1,2;\underline{s})$ reduces to $T(m,n-2;1,2;\underline{s}-(m+1))$ and $\underline{R}(m,n)$. By Corollary 3.12, $\underline{R}(m,n)$ is true for any $m \leq n$. It follows therefore that $T(m,n;1,2;\underline{s})$ is true, which completes the proof. □

Define a function $r(m,n)$ as follows:

$$r(m,n) = \begin{cases} m^3 - 2m & \text{if } m \text{ is even and if } n \text{ is odd;} \\ \frac{(m-2)(m+1)^2}{2} & \text{otherwise.} \end{cases}$$

Corollary 3.14. *Suppose that $n > r(m,n)$. Then $T(m,n;1,2;s)$ is true for any $s \leq \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor$.*

Proof. Since $(m,n;1,2;s)$ is subabundant, it suffices to show that $T(m,n;1,2;s)$ is true for $s = \left\lfloor \frac{(m+1)\binom{n+2}{2}}{m+n+1} \right\rfloor$. Note that

$$s = \begin{cases} (m+1)k - \frac{(m-2)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor & \text{if } n \text{ is even;} \\ (m+1)k - \frac{(m-3)(m+1)}{2} + \left\lfloor \frac{m^3-m}{2(m+n+1)} \right\rfloor & \text{if } m \text{ and } n \text{ are odd;} \\ (m+1)k - \frac{(m-3)(m+1)+1}{2} + \left\lfloor \frac{n+m^3+2}{2(m+n+1)} \right\rfloor & \text{otherwise.} \end{cases}$$

It is straightforward to show that if $n > r(m,n)$, then $s = \underline{s}$. Thus it follows immediately from Theorem 3.13 that $T(m,n;1,2;s)$ is true. □

Remark 3.15. If $m = 1$, then $r(1) < 0$. Since $\underline{s} = n+1$, $T(1,n;1,2;n+1)$ is true. Since $(1,n;1,2;n+1)$ is equiabundant, $T(1,n;1,2;s)$ is therefore true for any s.

4. Segre-Veronese varieties $\mathbb{P}^m \times \mathbb{P}^n$ embedded by $\mathcal{O}(1,2)$: Superabundant Case

In this section, we keep the same notation as in Section 3. Let $k = \lfloor \frac{n}{2} \rfloor$ and let

$$\bar{s} = \begin{cases} (m+1)k+1 & \text{if } n \text{ is even;} \\ (m+1)k+3 & \text{otherwise.} \end{cases}$$

It is straightforward to show that $(m,n;1,2;\bar{s})$ is superabundant. The main goal of this section is to prove that $T(m,n;1,2;\bar{s})$ is true, which implies that $T(m,n;1,2;s)$ is true for any $s \geq \bar{s}$.

Definition 4.1. Let $p_1, \dots, p_{\bar{s}-(m+1)}$ be general points of L, let $q_1, \dots, q_{m+1}$ be general points of $\mathbb{P}^{m,n} \setminus L$ and let $\overline{V}_{m,n}$ be the vector space $\left\langle V \otimes S_2(U_L), \sum_{i=1}^{\bar{s}-(m+1)} T_{p_i}(X_{m,n}), \sum_{i=1}^{m+1} T_{q_i}(X_{m,n}) \right\rangle$. Then the following inequality holds:

$$\dim \overline{V}_{m,n} \leq (m+1)\binom{n+2}{2}.$$

We say that $\overline{R}(m,n)$ *is true* if the equality holds.

Remark 4.2. In the same way as in the proofs of Propositions 3.3 and 3.8, one can prove the following:

(i) If $\overline{R}(m,n)$ and $T(m,n-2;1,2;\bar{s}-(m+1))$ are true, then $T(m,n;1,2;\bar{s})$ is true.
(ii) If $Q(m,n)$ and $\overline{R}(m,n-2)$ are true, then $\overline{R}(m,n)$ is true. In particular, if $\overline{R}(m,n-2)$ is true, then $\overline{R}(m,n)$ is true, because $Q(m,n)$ is true for $n \geq 3$.

Definition 4.3. Suppose that $(m,n) \neq (1,1)$. A 4-tuple $(m,n;1,d)$ is said to be *balanced* if

$$m \leq \binom{n+d}{d} - d.$$

Otherwise, we say that $(m,n;1,d)$ is *unbalanced.*

Remark 4.4. The notion of "unbalanced" was first introduced for Segre varieties (see for example [Catalisano et al. 2002, Abo et al. 2006]). Then it was extended to Segre-Veronese varieties in [Catalisano et al. 2008]. In the same paper it is also proved that if $(m,n;1,d)$ is unbalanced, then $T(m,n;1,d;s)$ fails if and only if

$$\binom{n+d}{d} - n < s < \min\left\{m+1, \binom{n+d}{d}\right\}. \tag{4.1}$$

In particular, $T(m,2;1,2;m+1)$ is true if $m \geq 5$ and $T(m,3;1,2;m+1)$ is true if $m \geq 8$.

Here we would like to briefly explain why if s satisfies the above inequalities, then $\sigma_s(X_{m,n})$ is defective. Let $p_1, \ldots, p_s$ be generic points on $X_{m,n}$. By assumption, we have $s < n+1$. Thus there is a proper subvariety of $\mathbb{P}^{m,n}$ of type $\mathbb{P}^{s-1,n}$ that contains $p_1, \ldots, p_s$. Thus we have

$$\begin{aligned} \dim \sigma_s(X_{m,n}) &\leq s(\dim \mathbb{P}^{m,n} - \dim \mathbb{P}^{s-1,n}) + s\binom{n+d}{d} \\ &= s\left[\binom{n+d}{d} + m + 1 - s\right]. \end{aligned}$$

It is straightforward to show that if s fulfills inequalities (4.1), then

$$s\left[\binom{n+d}{d} + m + 1 - s\right] < \min\left\{s(m+n+1),\ (m+1)\binom{n+d}{d}\right\}.$$

Thus $\sigma_s(X_{m,n})$ is defective. This also says that, for such an s, the expected dimension of $\sigma_s(X_{m,n})$ is $s\left[\binom{n+d}{d} + m + 1 - s\right]$.

Lemma 4.5. (i) *If $m \geq 3$, then $\overline{R}(m,n)$ is true for any $n \geq 2$;*
(ii) *$\overline{R}(2,n)$ is true for any $n \geq 3$.*

Proof. Theorem 3.11 and Remark 4.2 (ii) imply that, to prove Claim (i), it suffices to show that $\overline{R}(m,2)$ and $\overline{R}(m,3)$ are true for any $m \geq 3$. We first prove (i) for $m \geq 8$. Suppose that $n \in \{2,3\}$. If $m \geq 8$, then $(m,n;1,2)$ is unbalanced. Furthermore, $(m,n;1,2;m+1)$ is superabundant. Thus $\overline{R}(m,n)$ can be reduced to $T(m,n;1,2;m+1)$. By Remark 4.4, $T(m,n;1,2;m+1)$ is true for $n \in \{2,3\}$. Thus $\overline{R}(m,n)$ is also true for $m \geq 8$.

The remaining cases of (i) can be checked directly as follows: Let $S = \mathbb{C}[e_0, \ldots, e_m, f_0, \ldots, f_n]$. Choose randomly $u_1, \ldots, u_{\bar{s}} \in V$, $v_1, \ldots, v_{\bar{s}-(m+1)} \in U_L$ and $v_{\bar{s}-m}, \ldots, v_{\bar{s}} \in W$. For each $i \in \{1, \ldots, \bar{s}\}$, let T_i be the ideal of S generated by $e_0 \otimes v_i^2, \ldots, e_m \otimes v_i^2$, $u_i \otimes v_i f_0, \ldots, u_i \otimes v_i f_n$ and let I_L be the ideal generated by $V \otimes S_2(U_L)$. Let $I = \sum_{i=1}^{\bar{s}} T_i + I_L$. Computing the minimal generating set of I, we can check in `Macaulay2` that the vector space spanned by homogeneous elements of I of the multi-degree $(1,2)$ coincides with $V \otimes S_2(W)$. Claim (ii) can be also checked in the same way. □

Theorem 4.6. *$T(m,n;1,2;s)$ is true for any $s \geq \bar{s}$.*

Proof. In Example 2.9, we showed that $T(m,1;1,2;3)$ is true for any m. One can directly check that $T(2,2;1,2;4)$ is true. So, since $\overline{R}(2,n)$ is true for any $n \geq 3$ by Proposition 4.5, it follows from Remark 4.2 (i) that $T(2,n;1,2;\bar{s})$ is true for any $n \geq 1$.

Suppose now that $m \geq 3$. If $n = 0$, then $\bar{s} = 1$, and obviously $T(m,0;1,2;1)$ is true. If $n = 1$, $T(m,1;1,2;3)$ is true. Moreover by Proposition 4.5, we know that $\overline{R}(m,n)$ is true for any $n \geq 2$. Hence, from Remark 4.2 (i) it follows that $T(m,n;1,2;\bar{s})$ is true for any n and any $m \geq 3$. This concludes the proof. □

5. Conjecture

Let $X_{m,n}$ be the Segre-Veronese variety $\mathbb{P}^{m,n}$ embedded by the morphism given by $\mathcal{O}(1,2)$. The main purpose of this section is to give a conjecturally complete list of defective secant varieties of $X_{m,n}$.

Let V be an m-dimensional vector space over $\mathbb{C}$ with basis $\{e_0, \ldots, e_m\}$ and let W be an n-dimensional vector space over $\mathbb{C}$ with basis $\{f_0, \ldots, f_n\}$. As mentioned at the beginning of Section 2, for a given point $p = [u \otimes v^2] \in X_{m,n}$, the affine cone $T_p(X_{m,n})$ over the tangent space to $X_{m,n}$ at p is isomorphic to $V \otimes v^2 + u \otimes vW$. Let $A(p)$ be the $(m+1) \times (m+1)\binom{n+2}{2}$ matrix whose i^{th} row corresponds to $e_i \otimes v^2$ and let $B(p)$ be the $(n+1) \times (m+1)\binom{n+2}{2}$ matrix whose i^{th} row corresponds to $u \otimes vf_i$. Then $T_p(X_{m,n})$ is represented by the $(m+n+2) \times (m+1)\binom{n+2}{2}$ matrix $C(p)$ obtained by stacking $A(p)$ and $B(p)$:

$$C(p) = (\ A(p) \ || \ B(p)\ ).$$

For randomly chosen $p_1, \dots, p_s \in X_{m,n}$, let $T_s(X_{m,n}) = \sum_{i=1}^{s} T_{p_i}(X_{m,n})$. Then $T_s(X_{m,n})$ is represented by the $s(m+n+2) \times (m+1)\binom{n+2}{2}$ matrix $C(p_1, \dots, p_s)$ defined by

$$C(p_1, \dots, p_s) = (\ C(p_1) \ || \ C(p_2) \ || \ \cdots \ || \ C(p_s)\ ).$$

Thus Remark 2.2 and semicontinuity imply that if

$$\text{rank } C(p_1, \dots, p_s) = \min \left\{ s(m+n+1),\ (m+1)\binom{n+2}{2} \right\},$$

then $\sigma_s(X_{m,n})$ has the expected dimension.

We programed this in `Macaulay2` and computed the dimension of $\sigma_s(X_{m,n})$ for $m, n \leq 10$ to detect "potential" defective secant varieties of $X_{m,n}$. This experiment shows that $X_{m,n}$ is non-defective except for

- $(m, n; 1, 2)$ unbalanced;
- $(m, n) = (2, n)$, where n is odd and $n \leq 10$;
- $(m, n) = (4, 3)$.

Remark 5.1. The defective cases we found in the experiments are all well known. In Remark 4.4, we gave an explanation of why if $(m, n; 1, 2)$ is unbalanced, then $X_{m,n}$ is defective. Here we will discuss the remaining known defective cases.

(i) It is classically known that $\sigma_5(X_{2,3})$ is defective (see [Carlini and Chipalkatti 2003] and [Carlini and Catalisano 2007] for modern proofs). Carlini an Chipalkatti proved in their work on Waring's problem for several algebraic forms [Carlini and Chipalkatti 2003] that $T(2, 5; 1, 2; 8)$ is false. In [Ottaviani 2006], Ottaviani then proved, as a generalization of the Strassen theorem [Strassen 1983] on three-factor Segre varieties, that $T(2, n; 1, 2; s)$ fails if $(n, s) = (2k+1, 3k+2)$ for any $k \geq 1$. Here we sketch his proof for the defectiveness of $X_{2,2k+1}$. Recall that $X_{2,2k+1}$ is the image of the Segre-Veronese embedding

$$\nu_{1,2} : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes S^2W),$$

where V and W have dimensions 3 and $2k+2$ respectively. For every tensor $\phi \in V \otimes S^2W$, let $S_\phi : V \otimes W^\vee \to \wedge^2 V \otimes W \cong V^\vee \otimes W$ be the contraction operator induced by ϕ. If P, Q and R are the three symmetric slices of ϕ, then S_ϕ can be written as a skew-symmetric matrix of order $3(2k+2)$ of the form

$$S_\phi = \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix}.$$

The rank of S_ϕ is $3(2k+2)$ for a general tensor $\phi \in V \otimes S^2W$. On the other hand, since the contraction operator corresponding to a decomposable tensor has rank 2, we have rank $S_\phi \leq 2s$, if ϕ is the sum of s decomposable tensors. Since the decomposable tensors correspond to the points of the Segre-Veronese variety, we can deduce that if $s = 3k+2$, then $\sigma_s(X_{2,2k+1})$ does not fill $\mathbb{P}^{3\binom{2k+3}{2}-1}$.

(ii) The defectiveness of $\sigma_6(X_{4,3})$ can be proved by the existence of a certain rational normal curve in $X_{4,3}$ passing through generic six points of $X_{4,3}$. Let $\pi_1 : \mathbb{P}^{4,3} \to \mathbb{P}^4$ and $\pi_2 : \mathbb{P}^{4,3} \to \mathbb{P}^3$ be the canonical projections. Given generic points $p_1, \dots, p_6 \in \mathbb{P}^{4,3}$, there is a unique twisted cubic $\nu_3 : \mathbb{P}^1 \to C_3 \subset \mathbb{P}^3$ that passes through $\pi_2(p_1), \dots, \pi_2(p_6)$. Let $q_i = \nu_3^{-1}(\pi_2(p_i))$ for each $i \in \{1, \dots, 6\}$. Since any ordered subset of six points in general position in $\mathbb{P}^4$ is projectively equivalent to the ordered set $\{\pi_1(p_1), \dots, \pi_1(p_6)\}$, there is a rational quartic curve $\nu_4 : \mathbb{P}^1 \to C_4 \subset \mathbb{P}^4$ such that $\nu_4(q_i) = \pi_1(p_i)$ for all $i \in \{1, \dots, 6\}$. Let $\nu = (\nu_4, \nu_3)$ and let $C = \nu(\mathbb{P}^1)$. Then C passes through $p_1, \dots, p_6$. The image of C

under the morphism given by $\mathcal{O}(1,2)$ is a rational normal curve of degree $10(= 4\cdot 1 + 2\cdot 3)$ in $\mathbb{P}^{10}$. Thus we have

$$\dim \sigma_6(X_{4,3}) \le 10 + 6(7-1) = 46 < 6(4+3+1) - 1 = 47,$$

and so $\sigma_6(X_{4,3})$ is defective. See [Carlini and Chipalkatti 2003] for an alternative proof.

The experiments with our program and Remark 5.1 suggest the following conjecture:

Conjecture 5.2. *Let $X_{m,n}$ be the Segre-Veronese variety $\mathbb{P}^{m,n}$ embedded by the morphism given by $\mathcal{O}(1,2)$. Then $\sigma_s(X_{m,n})$ is defective if and only if (m,n,s) falls into one of the following cases:*

(a) *$(m,n;1,2)$ is unbalanced and $\binom{n+2}{2} - n < s < \min\left\{m+1,\ \binom{n+2}{2}\right\}$;*
(b) *$(m,n,s) = (2, 2k+1, 3k+2)$ with $k \ge 1$;*
(c) *$(m,n,s) = (4,3,6)$.*

It is known that the conjecture is true for $m = 1$ (see [Carlini and Chipalkatti 2003]). Here we prove that the conjecture is true for $m = 2$ as a consequence of Theorems 3.13 and 4.6.

Theorem 5.3. *$T(2,n;1,2;s)$ is true for any s except $(n,s) = (2k+1, 3k+2)$ with $k \ge 1$.*

Proof. Assume first that $n = 2k$ is even. Then we have $\overline{s} = \underline{s} = 3k+1$. Hence, from Theorems 3.13 and 4.6, it follows that $T(2,2k;1,2;s)$ is true for any s.

Suppose now that $n = 2k+1$ is odd. Then we have $\underline{s} = 3k+1$ and $\overline{s} = 3k+3$. Thus $T(2,n;1,2;s)$ is true for any $s \le 3k+1$, by Theorem 3.13, and for any $s \ge 3k+3$, by Theorem 4.6.

If $n = 1$, then $\underline{s} = 1$ and $\overline{s} = 3$. So it remains only to prove that also $T(2,1;1,2;2)$ is true. But this has been already proved in Example 2.10. So we completed the proof. □

In [Ottaviani 2006] it is also claimed that $\sigma_{3k+2}(X_{2,2k+1})$ is a hypersurface if $k \ge 1$ and that this can be proved by modifying Strassen's argument in [Strassen 1983]. Then it follows that the equation of $\sigma_{3k+2}(X_{2,2k+1})$ is given by the pfaffian of S_ϕ, where S_ϕ is the skew-symmetric matrix introduced in Remark 5.1 (i). We conclude this paper by giving an alternative proof of the fact that $\sigma_{3k+2}(X_{2,2k+1})$ is a hypersurface for $k \ge 1$.

Definition 5.4. Suppose that n is odd. Let $s = 3\left\lfloor \frac{n}{2} \right\rfloor + 2$, let $p_1, \ldots, p_{s-3}$ be general points of L, let q_1, q_2, q_3 be general points of $\mathbb{P}^{2,n} \setminus L$ and let $V_{2,n}$ be the vector space $\left\langle V \otimes S_2(U_L), \sum_{i=1}^{s-3} T_{p_i}(X_{2,n}), \sum_{i=1}^{3} T_{q_i}(X_{2,n}) \right\rangle$. Then the following inequality holds:

$$\dim V_{2,n} \le 3\binom{n+2}{2}.$$

We say that $R(2,n)$ *is true* if the equality holds.

Lemma 5.5. *Let n be a positive odd integer greater than or equal to 3. Then $R(2,n)$ is true.*

Proof. The proof is very similar to that of Proposition 3.5. One can easily prove that if $Q(2,n)$ is true and if $R(2,n-2)$ is true, then $R(2,n)$ is true. Since we have already proved that $Q(2,n)$ is true, it suffices to show that $R(2,3)$ is true.

Let $p_1, p_2 \in L$ and let $q_1, q_2, q_3 \in \mathbb{P}^{2,3}$. Choose a subvariety H of $\mathbb{P}^{2,3}$ of the form $\mathbb{P}^{2,2} = \mathbb{P}(V) \times \mathbb{P}(W')$ such that $\mathbb{P}^{2,2}$ intersects L in $\mathbb{P}^{2,0}$. Suppose that $p_1, p_2 \notin H$. Specializing q_1, q_2 and q_3 in $H \setminus L$, we obtain an exact sequence:

$$0 \to \mathcal{I}_{Z\cup L\cup H}(1,2) \to \mathcal{I}_{Z\cup L}(1,2) \to \mathcal{I}_{(Z\cup L)\cap H, H}(1,2) \to 0,$$

where $Z = \{p_1^2, p_2^2, q_1^2, q_2^2, q_3^2\}$. Since we have already proved that $\underline{R}(2,2)$ is true, we can conclude that $\dim H^0(\mathcal{I}_{(Z\cup L)\cap H,H}(1,2)) = 0$. Thus it is enough to prove that $H^0(\mathcal{I}_{Z\cup L\cup H}(1,2)) = 0$ or $H^0(\mathcal{I}_{\widetilde{Z}}(1,1)) = 0$, where $\widetilde{Z}$ is the residual of $Z \cup L$ by H. Note that $\widetilde{Z}$ consists of two double points p_1^2, p_2^2, three single points q_1, q_2, q_3 on H and L. Let $X'_{2,3}$ be the Segre-Veronese variety $\mathbb{P}^{2,3}$ embedded by $\mathcal{O}(1,1)$. We want to prove that L, $\sum_{i=1}^{2} T_{p_i}(X'_{2,3})$ and $\sum_{i=1}^{3} T_{q_i}(X'_{2,3})$ span $V \otimes W$. Note that if $p = u \otimes v$, then $T_p(X'_{2,3}) = V \otimes v + u \otimes W$. Now assume the following:

- $U_L = \langle f_0, f_1 \rangle$ and $W' = \langle f_1, f_2, f_3 \rangle$;
- $p_1 = e_0 \otimes f_0, p_2 = e_1 \otimes f_1 \in V \otimes U_L$;
- $q_1 = e_2 \otimes f_2, q_2 = e_2 \otimes f_3 \in V \otimes W'$.

For any non-zero $q_3 \in V \otimes W'$, one can show that

$$V \otimes W = \left\langle L, \sum_{i=1}^{2} T_{p_i}(X'_{2,3}), \sum_{i=1}^{3} T_{q_i}(X'_{2,3}) \right\rangle.$$

Thus we complete the proof. □

Proposition 5.6. *If $(n, s) = (2k+1, 3k+2)$ for $k \geq 1$, then $\dim \sigma_s(X_{2,n}) = 3\binom{n+2}{2} - 2$.*

Proof. The proof is by induction on k. It is well known that $\sigma_5(X_{2,3})$ is a hypersurface. Thus we may assume that $k \geq 2$. Let $p_1, \ldots, p_s \in \mathbb{P}^{2,n}$. Then there is a subvariety L of $\mathbb{P}^{2,n}$ of the form $\mathbb{P}^{2,n-2}$ such that $p_1, p_2, p_3 \in L$. Let us suppose that $p_4, \ldots, p_s \in \mathbb{P}^{2.n} \setminus L$. Then we have an exact sequence

$$0 \to \mathcal{I}_{Z \cup L}(1,2) \to \mathcal{I}_Z(1,2) \to \mathcal{I}_{Z \cap L, L}(1,2) \to 0.$$

Taking cohomology, we get

$$\dim H^0(\mathcal{I}_Z(1,2)) \leq \dim H^0(\mathcal{I}_{Z \cup L}(1,2)) + \dim H^0(\mathcal{I}_{Z \cap L, L}(1,2)).$$

By Lemma 5.5, $\dim H^0(\mathcal{I}_{Z \cup L}(1,2)) = 0$. Thus, by induction hypothesis,

$$\dim H^0(\mathcal{I}_Z(1,2)) \leq \dim H^0(\mathcal{I}_{Z \cap L, L}(1,2)) \leq 1.$$

As already claimed, it is known that $T(2, n; 1, 2; s)$ does not hold, i.e. $\dim H^0(\mathcal{I}_Z(1,2)) \geq 1$. It follows that $\sigma_s(X_{2,n})$ is a hypersurface in the ambient space $\mathbb{P}^{3\binom{n+2}{2}-1}$. □

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