Cokernel bundles and Fibonacci bundles

Maria Chiara Brambilla[1]

[1] Dipartimento di Matematica e Applicazioni per l'Architettura, Università di Firenze, piazza Ghiberti, 27-50122, Florence, Italy

We are interested in those bundles C on $\mathbb{P}^N$ which admit a resolution of the form

$$0 \longrightarrow \mathbb{C}^s \otimes E \xrightarrow{\ \mu\ } \mathbb{C}^t \otimes F \longrightarrow C \longrightarrow 0.$$

In this paper we prove that, under suitable conditions on (E, F), a generic bundle with this form is either simple or canonically decomposable. As applications we provide an easy criterion for the stability of such bundles on $\mathbb{P}^2$ and we prove the stability when $E = \mathcal{O}, F = \mathcal{O}(1)$ and C is an exceptional bundle on $\mathbb{P}^N$ for $N \geq 2$.

WILEY

REPRINT

Cokernel bundles and Fibonacci bundles

Maria Chiara Brambilla[*][1]

[1] Dipartimento di Matematica e Applicazioni per l'Architettura, Università di Firenze, piazza Ghiberti, 27-50122, Florence, Italy

We are interested in those bundles C on $\mathbb{P}^N$ which admit a resolution of the form

$$0 \longrightarrow \mathbb{C}^s \otimes E \xrightarrow{\mu} \mathbb{C}^t \otimes F \longrightarrow C \longrightarrow 0.$$

In this paper we prove that, under suitable conditions on (E, F), a generic bundle with this form is either simple or canonically decomposable. As applications we provide an easy criterion for the stability of such bundles on $\mathbb{P}^2$ and we prove the stability when $E = \mathcal{O}$, $F = \mathcal{O}(1)$ and C is an exceptional bundle on $\mathbb{P}^N$ for $N \geq 2$.

1 Introduction

In this paper we consider the family of those vector bundles C on $\mathbb{P}^N$, with $N \geq 2$, which admit a resolution of the form

$$0 \longrightarrow E^s \xrightarrow{\mu} F^t \longrightarrow C \longrightarrow 0, \tag{1.1}$$

for some bundles E, F and for $s, t \in \mathbb{N}$. Here and in the sequel we write E^s (resp. F^t) instead of $\mathbb{C}^s \otimes E$ (resp. $\mathbb{C}^t \otimes F$), and we assume $t \operatorname{rk}(F) - s \operatorname{rk}(E) \geq N$. Any bundle C in (1.1) is the cokernel of a morphism μ of bundles. Our purpose is to describe the properties of the bundles corresponding to generic morphisms in $\operatorname{Hom}(E^s, F^t)$. In particular we want to find criteria of simplicity, rigidity and decomposability.

Throughout the paper we will assume that E and F are two different vector bundles on $\mathbb{P}^N = \mathbb{P}(V)$, with $N \geq 2$, which verify the following *basic hypotheses*:

$$E \text{ and } F \text{ are simple, and } \operatorname{Hom}(F, E) = \operatorname{Ext}^1(F, E) = 0, \tag{1.2}$$

$$\text{the sheaf } E^* \otimes F \text{ is globally generated, and } W = \operatorname{Hom}(E, F) \text{ has dimension } w \geq 3. \tag{1.3}$$

The first instance one can consider is $E = \mathcal{O}$ and $F = \mathcal{O}(1)$: in this case we obtain the family of *Steiner bundles*, where the morphism μ is a $(t \times s)$-matrix whose entries are homogeneous linear polynomials. In [3] we studied this case and we obtained a criterion for the simplicity of Steiner bundles. In [4] we extended this result, describing the canonical decomposition of generic non-simple Steiner bundles. In particular we proved that the indecomposable elements which appear in such a decomposition are exceptional bundles.

Exceptional bundles were introduced by Drézet and Le Potier in [7] as a class of bundles on $\mathbb{P}^2$ without deformation. Later the school of Rudakov generalized the concept of exceptional bundles and introduced mutations in order to construct them, in the setting of derived categories (see for example [12]).

The first result we obtain here (Theorem 4.3 below) states that if C is the cokernel of a generic map $\mu \in \operatorname{Hom}(E^s, F^t)$, then

$$C \text{ is simple} \quad \Longleftrightarrow \quad s^2 - wst + t^2 \leq 1.$$

* e-mail: brambilla@math.unifi.it

This result allows us to obtain a criterion for the stability of the cokernel bundles C on $\mathbb{P}^2$. In fact Drézet and Le Potier obtained an important criterion for the stability of all bundles on $\mathbb{P}^2$ (see [7]), but their result is very difficult to apply. In this paper, using another result of Drézet (see [6]), we get a new criterion for the stability of the bundles C with resolution (1.1) on $\mathbb{P}^2$, which is much easier to apply.

Our second result is a canonical decomposition for non-simple cokernel bundles. In this context, the main tool is a new family of bundles, here referred to as *Fibonacci bundles*, which play the role of the exceptional bundles, but which are much more general. In Theorem 5.1 we define Fibonacci bundles by means of mutations, and in Theorem 5.2 we prove that they admit a resolution (1.1) in which the coefficients are related to the numbers of Fibonacci (this motivates our choice of their name).

Under additional conditions on the pair (E, F) we prove that all the Fibonacci bundles are simple and rigid. These two crucial properties allow us to find a canonical decomposition of a generic non-simple cokernel bundle C in (1.1) in terms of Fibonacci bundles. More precisely, we add the following conditions on (E, F):

$$E \text{ and } F \text{ are rigid, and } \mathrm{Ext}^2(F, E) = 0, \tag{1.4}$$

and we call, for brevity, *hypotheses* (R) the union of conditions (1.2), (1.3) and (1.4). Theorem 6.3 states that if (E, F) satisfies (R), then for a generic C in (1.1) we have

$$s^2 - wst + t^2 \geq 1 \quad \Longrightarrow \quad C \cong C_k^n \oplus C_{k+1}^m,$$

where C_k, C_{k+1} are Fibonacci bundles and $n, m \in \mathbb{N}$. We stress that, in this case, any generic non-simple bundle is rigid and homogeneous.

Finally, as an application of our results, we prove the following

Theorem 1.1 *Any exceptional Steiner bundle on $\mathbb{P}^N$ is stable for all $N \geq 2$.*

We recall that exceptional bundles are known to be stable on $\mathbb{P}^2$ ([7]) and on $\mathbb{P}^3$ ([13]), but the stability of exceptional bundles on $\mathbb{P}^N$ with $N > 3$ is an open problem.

The plan of the article is as follows: in Section 2 we present some basic examples, and in Section 3 we recall the case of Steiner bundles and their interpretation in terms of matrices. Section 4 is devoted to the criterion for simplicity and Section 5 to Fibonacci bundles. In Section 6 we give the decomposition theorem for non-simple bundles and in Section 7 we describe some applications of our results. Finally Section 8 is devoted to our results on stability.

2 Preliminaries

For a fixed $N \geq 2$, we are interested in the vector bundles C on $\mathbb{P}^N = \mathbb{P}(V)$ with resolution (1.1), where E, F satisfy the basic hypotheses (1.2) and (1.3) and $s, t \in \mathbb{N}$ verify $t \, \mathrm{rk}(F) - s \, \mathrm{rk}(E) \geq N$.

We say that C is *generic* when the morphism μ is generic in the space $H = \mathrm{Hom}(E^s, F^t) \cong \mathbb{C}^s \otimes \mathbb{C}^t \otimes \mathrm{Hom}(E, F)$. The morphism μ can be represented by a $(t \times s)$-matrix, whose entries are morphisms from E to F.

Let us see some examples. As in (1.3) we denote by W the vector space $\mathrm{Hom}(E, F)$ and by w its dimension.

Example 2.1 If $E = \mathcal{O}$ and $F = \mathcal{O}(d)$, it is easy to check that conditions (1.2) and (1.3) are satisfied for any $d \geq 1$. Hence we deal with bundles with resolution

$$0 \longrightarrow \mathcal{O}^s \xrightarrow{\ \mu\ } \mathcal{O}(d)^t \longrightarrow C \longrightarrow 0, \tag{2.1}$$

where μ is a matrix whose entries are homogeneous polynomials of degree d. In this case $W = \mathrm{H}^0(\mathcal{O}(d)) = S^d V$ and $w = \binom{N+d}{d}$. In particular when $d = 1$ we obtain the case of Steiner bundles, studied in [3].

Example 2.2 For any $p \geq 0$, let us denote $\Omega^p(p) = \wedge^p \Omega^1(1)$. Given $0 \leq p < N$, we consider $E = \mathcal{O}(-1)$ and $F = \Omega^p(p)$ and we obtain bundles of the form

$$0 \longrightarrow \mathcal{O}(-1)^s \xrightarrow{\ \mu\ } \Omega^p(p)^t \longrightarrow C \longrightarrow 0. \tag{2.2}$$

In this case $W = \wedge^{N-p} V^*$, $w = \binom{N+1}{N-p}$ and the entries of the matrix μ are $(N - p)$-forms. Analogously we consider $E = \Omega^p(p)$ and $F = \mathcal{O}$, where $0 < p \leq N$, and we obtain bundles of the form

$$0 \longrightarrow \Omega^p(p)^s \xrightarrow{\ \mu\ } \mathcal{O}^t \longrightarrow C \longrightarrow 0, \tag{2.3}$$

where μ is a matrix of p-forms.

Example 2.3 On $\mathbb{P}^2 = \mathbb{P}(V)$ we denote $Q = T(-1) = \Omega^1(2)$, i.e.

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \otimes V^* \longrightarrow Q \longrightarrow 0,$$

and $S^p Q(d) = \mathrm{Sym}^p Q \otimes \mathcal{O}(d)$, where $p \geq 1$ and $d \in \mathbb{Z}$. Let $E = S^p Q$ and $F = S^r Q(d)$, for some fixed $p, r \geq 1$ and $d \in \mathbb{Z}$, and consider the bundles C of the form

$$0 \longrightarrow (S^p Q)^s \xrightarrow{\mu} (S^r Q(d))^t \longrightarrow C \longrightarrow 0. \tag{2.4}$$

In this case $E^* \otimes F = S^p(Q^*) \otimes S^r Q(d) \cong S^p Q \otimes S^r Q \otimes \mathcal{O}(d - p)$, and hypotheses (1.2) and (1.3) hold true if $d > p + 1$.

2.1 The "Fibonacci" sequences

Given any integer $w \geq 3$, we introduce the following sequence of numbers:

$$a_{w,k} = \frac{\left(\frac{w + \sqrt{w^2 - 4}}{2}\right)^k - \left(\frac{w - \sqrt{w^2 - 4}}{2}\right)^k}{\sqrt{w^2 - 4}}, \tag{2.5}$$

for $k \geq 0$. This sequence satisfies the recurrence

$$\begin{cases} a_{w,0} = 0, \\ a_{w,1} = 1, \\ a_{w,k+1} = w a_{w,k} - a_{w,k-1}. \end{cases}$$

In the following for brevity we will write a_k instead of $a_{w,k}$, when the value of w is clear from the context.

Remark 2.4 In the case $w = 3$, the sequence $\{a_{w,k}\}$ is exactly the odd part of the well-known Fibonacci sequence. Also if $w > 3$ the numbers $a_{w,k}$ satisfy some good relations, analogously to Fibonacci numbers. More precisely, for any fixed $w \geq 3$, we can easily prove by induction that the following equalities hold for all $k \geq 1$:

$$a_{k-1}^2 + a_k^2 - w a_{k-1} a_k = 1, \tag{2.6}$$

$$a_k^2 - a_{k+1} a_{k-1} = 1, \tag{2.7}$$

$$a_{k+1} a_k - a_{k-1} a_{k+2} = w. \tag{2.8}$$

From (2.7), it also follows that $(a_k, a_{k-1}) = 1$, for all $k \geq 1$.

Remark 2.5 It is possible to prove that the pairs $(s, t) = (a_k, a_{k+1})$ are the unique integer solutions of the diophantine equation $s^2 + t^2 - wst = 1$. For more details see Lemma 3.4 of [3].

2.2 Exceptional bundles

Exceptional bundles were defined by Drézet and Le Potier in [7] as a class of bundles on $\mathbb{P}^2$ without deformation. These bundles appeared as some sort of exceptional points in the study of the stability of bundles on $\mathbb{P}^2$. Drézet and Le Potier showed that these vector bundles are uniquely determined by their slopes, and they described the set of all the possible slopes. Later, the school of Rudakov (see for example [12]) generalized the definition of exceptional bundles on $\mathbb{P}^N$ and other varieties, with an axiomatic presentation in the setting of derived categories. Following Gorodentsev and Rudakov ([8]) we give the following definition:

Definition 2.6 A bundle E on $\mathbb{P}^N$ is *exceptional* if

$$\mathrm{Hom}(E, E) = \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^i(E, E) = 0 \ \text{for all} \ i > 1.$$

We recall that a bundle is called *semi-exceptional* when it is a direct sum of exceptional bundles.

3   Steiner bundles and matrices

In this section we recall some results concerning Steiner bundles on $\mathbb{P}^N = \mathbb{P}(V)$, with $N \geq 2$, i.e. the bundles S which admit a resolution of the form

$$0 \longrightarrow \mathcal{O}^s \overset{\mu}{\longrightarrow} \mathcal{O}(1)^t \longrightarrow S \longrightarrow 0, \tag{3.1}$$

with $t - s \geq N$. In this case μ belongs to the space $H = \mathrm{Hom}(\mathcal{O}^s, \mathcal{O}(1)^t) = \mathbb{C}^s \otimes \mathbb{C}^t \otimes V$, which can be seen as the space of $(t \times s)$-matrices whose entries are homogeneous linear forms in $N + 1$ variables or, alternatively, as the space of $(s \times t \times (N + 1))$-matrices of numbers.

We consider the following action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$ on H:

$$\mathrm{GL}(s) \times \mathrm{GL}(t) \times H \longrightarrow H, \quad (A, B, \mu) \longmapsto B^{-1} \mu A.$$

Given $\mu \in H$, we denote by $(\mathrm{GL}(s) \times \mathrm{GL}(t))\mu$ the orbit of μ and by $\mathrm{Stab}(\mu)$ the stabilizer of μ with respect to the action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$.

In order to describe the orbits of this action, we introduce the following definitions concerning multidimensional matrices. We say that two matrices $\mu, \mu' \in H$ are $\mathrm{GL}(s) \times \mathrm{GL}(t)$-equivalent if they are in the same orbit with respect to the action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$ on H. This corresponds to perform Gaussian elimination on a $(s \times t)$-matrix of linear polynomials.

Definition 3.1 If $a_k = a_{N+1,k}$ is the sequence defined in (2.5), we call *block of type B_k* a matrix in $\mathbb{C}^{a_{k-1}} \otimes \mathbb{C}^{a_k} \otimes \mathbb{C}^{N+1}$. Given $n, m \in \mathbb{N}$, let $s = na_{k-1} + ma_k$ and $t = na_k + ma_{k+1}$. We say that a matrix $\mu \in \mathbb{C}^s \otimes \mathbb{C}^t \otimes \mathbb{C}^{N+1}$ is a *canonical matrix* if there exist decompositions

$$\mathbb{C}^s = \mathbb{C}^{na_{k-1}} \oplus \mathbb{C}^{ma_k} \quad \text{and} \quad \mathbb{C}^t = \mathbb{C}^{na_k} \oplus \mathbb{C}^{ma_{k+1}},$$

such that the matrix μ is zero except for n blocks of type B_k and m blocks of type B_{k+1} on the diagonal. We denote such a matrix by $B_k^n \oplus B_{k+1}^m$.

The following theorem describes the elements of H with respect to the action above. For the proof we refer to [3], [4], and to Theorem 4 of [10].

Theorem 3.2 *Let $H = \mathbb{C}^s \otimes \mathbb{C}^t \otimes \mathbb{C}^{N+1}$ be endowed with the natural action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$.*

(i) *If $s^2 + t^2 - (N + 1)st \leq 1$, then the stabilizer of a generic element of H has dimension 1. In particular if $s^2 + t^2 - (N + 1)st = 1$, there is a dense orbit in H.*

(ii) *If $s^2 + t^2 - (N + 1)st \geq 1$, then a generic element of H is $\mathrm{GL}(s) \times \mathrm{GL}(t)$-equivalent to a canonical matrix $B_k^n \oplus B_{k+1}^m$ for unique $n, m, k \in \mathbb{N}$.*

Remark 3.3 After [3] and [4] have been written, we learned that our results on matrices turn out to be connected to a theorem of Kac, framed in the setting of quiver theory. More precisely, in [10] the quiver with two vertices and w arrows from the first vertex to the second one is considered, and a representation of this quiver is exactly a w-uple of linear maps from one vector space into another. In Theorem 4 of [10], Kac describes the isomorphism classes of representations of this quiver. Notice that the proofs given in [3] and [4] are independent from techniques of quiver theory.

The previous theorem implies the following classification of Steiner bundles, proved in [3] and [4]. Here we omit the proof, since we will prove the same result in a more general framework later (see Theorems 4.3 and 6.3). Given $h \in \mathbb{N}$ we denote by $a_h = a_{N+1,h}$ and by S_h the exceptional Steiner bundle with resolution

$$0 \longrightarrow \mathcal{O}^{a_{h-1}} \longrightarrow \mathcal{O}(1)^{a_h} \longrightarrow S_h \longrightarrow 0.$$

Theorem 3.4 *Let S be a generic Steiner bundle on $\mathbb{P}^N$ with resolution (3.1) and $t - s \geq N \geq 2$.*

(i) *If $s^2 + t^2 - (N + 1)st \leq 0$ i.e. if $t \leq \left(\frac{N+1+\sqrt{(N+1)^2 - 4}}{2} \right) s$, then the bundle S is simple,*

(ii) *if $s^2 + t^2 - (N + 1)st \geq 1$ i.e. if $t > \left(\frac{N+1+\sqrt{(N+1)^2 - 4}}{2} \right) s$, then the bundle S is isomorphic to $S_k^n \oplus S_{k+1}^m$,*

for some unique $n, m, k \in \mathbb{Z}$.

Notice that the bundles of the form $S_k^n \oplus S_{k+1}^m$, which appear in the previous theorem, correspond to canonical matrices.
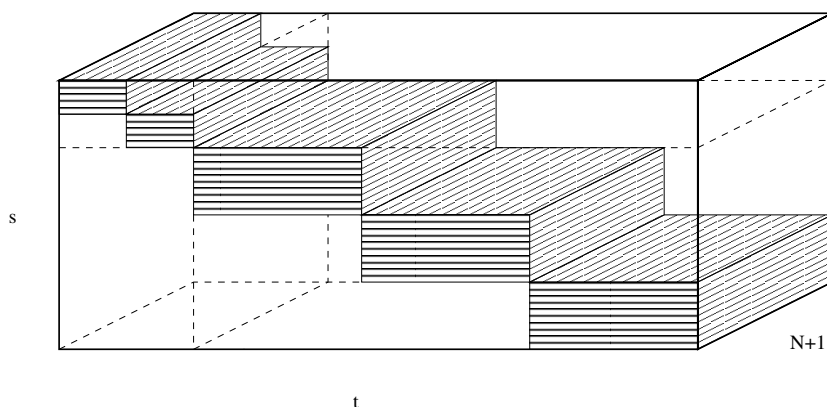
Fig. 1   Example of canonical matrix, with $n = 2, m = 3$

4   Simplicity

In this section we begin to study the generic cokernel bundles C with resolution (1.1) on $\mathbb{P}^N$, focusing on the simplicity property. As in Section 3, we consider the natural action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$ on the space $H = \mathrm{Hom}(E^s, F^t)$ and we denote by $\mathrm{Stab}(\mu)$ the stabilizer of μ.

Lemma 4.1 *If C is a bundle with resolution* (1.1) *and* $\dim \mathrm{Stab}(\mu) = 1$*, then C is simple.*

P r o o f.   Assume by contradiction that C is not simple. Then there exists $\phi : C \to C$ nontrivial. Applying the functor $\mathrm{Hom}(-, C)$ to the sequence (1.1), we get that ϕ induces $\widetilde{\phi}$ nontrivial in $\mathrm{Hom}(F^t, C)$.

Now applying the functor $\mathrm{Hom}(F^t, -)$ again to the same sequence and using hypothesis (1.2), we get $\mathrm{Hom}(F^t, F^t) \cong \mathrm{Hom}(F^t, C)$, hence $\widetilde{\phi}$ induces a nontrivial morphism in $\mathrm{Hom}(F^t, F^t)$. Since F is simple, this nontrivial morphism induces a complex $(t \times t)$-matrix $B \neq \lambda \, \mathrm{Id}$, such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & E^s & \overset{\mu}{\longrightarrow} & F^t & \longrightarrow & C & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle B} \quad {\scriptstyle \widetilde{\phi}} & & \downarrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & E^s & \overset{\mu}{\longrightarrow} & F^t & \longrightarrow & C & \longrightarrow & 0
\end{array}
$$

By restricting B to E^s and by the simplicity of E we obtain a complex $(s \times s)$-matrix A, such that $BM = MA$. Let $0 \neq \rho \in \mathbb{C}$ be different from any eigenvalue of B and A. If we define $\widetilde{A} = A - \rho \, \mathrm{Id}$ and $\widetilde{B} = B - \rho \, \mathrm{Id}$, we get that the pair $\big(\widetilde{A}, \widetilde{B}\big)$ belongs to $\mathrm{Stab}(M) \subset \mathrm{GL}(s) \times \mathrm{GL}(t)$. Since $\widetilde{A}$ is not a scalar matrix, it follows that $\big(\widetilde{A}, \widetilde{B}\big) \neq (\lambda \, \mathrm{Id}, \lambda \, \mathrm{Id})$ i.e. $\dim \mathrm{Stab}(M) > 1$, which is a contradiction. $\square$

Lemma 4.2 *If C is a bundle with resolution* (1.1) *and* $\mathrm{Hom}(C, F) = 0$*, then* $\dim \mathrm{Stab}(\mu) = \dim \mathrm{Hom}(C, C)$.

P r o o f.   Let us apply the functor $\mathrm{Hom}(-, E^s)$ to the sequence (1.1). By hypothesis (1.2), we obtain the following relation

$$
\mathrm{End}\,\mathbb{C}^s = \mathbb{C}^s \otimes \mathbb{C}^{s*} \otimes \mathrm{Hom}(E, E) = \mathrm{Hom}(E^s, E^s) = \mathrm{Ext}^1(C, E^s).
$$

By applying $\mathrm{Hom}(-, F^t)$ to (1.1) and using hypothesis $\mathrm{Hom}(C, F) = 0$ and the simplicity of F, we get

$$
0 \to \mathrm{End}\,\mathbb{C}^t \to \mathbb{C}^t \otimes \mathbb{C}^s \otimes \mathrm{Hom}(E, F) \to \mathbb{C}^t \otimes \mathrm{Ext}^1(C, F),
$$

and applying $\mathrm{Hom}(C, -)$ to (1.1) we get

$$
0 \to \mathrm{Hom}(C, C) \to \mathrm{Ext}^1(C, E^s) \to \mathrm{Ext}^1(C, F^t).
$$

The previous results together give the following commutative diagram

$$
\begin{array}{c}
0 \\
\downarrow \\
\operatorname{End} \mathbb{C}^t \\
\downarrow {\scriptstyle r_\mu} \\
\mathbb{C}^t \otimes \mathbb{C}^s \otimes \operatorname{Hom}(E,F) = H \\
\downarrow {\scriptstyle \pi} \\
0 \longrightarrow \operatorname{Hom}(C,C) \xrightarrow{\ i\ } \operatorname{End} \mathbb{C}^s \longrightarrow \mathbb{C}^t \otimes \operatorname{Ext}^1(C,F) \\
\downarrow \\
0
\end{array}
$$

where $l_\mu(A) = \mu A$ and $r_\mu(B) = B\mu$. Notice that the tangent space to the stabilizer of μ is

$$\mathrm{T}(\operatorname{Stab}(\mu)) = \{(A,B) \in \operatorname{End}\mathbb{C}^s \times \operatorname{End}\mathbb{C}^t \mid l_\mu(A) = r_\mu(B)\}.$$

We want to prove that $\dim \operatorname{Stab}(\mu) = \dim \mathrm{T}(\operatorname{Stab}(\mu)) = \dim \operatorname{Hom}(C,C)$. Let us suppose that $A \in \operatorname{End}\mathbb{C}^s$ satisfies $l_\mu(A) \in \operatorname{Im}(r_\mu)$. Since the map r_μ is injective, there exists a unique $B \in \operatorname{End}\mathbb{C}^t$ such that (A,B) is in the stabilizer. Moreover $\pi(l_\mu(A)) = 0$, and thus, since the diagram is commutative, there exists $\phi = i^{-1}(A) \in \operatorname{Hom}(C,C)$ which is unique, since i is injective. Conversely, we associate to every $\phi \in \operatorname{Hom}(C,C)$ a unique $A = i(\phi)$. Since the sequences are exact and the diagram commutes, we have $l_\mu(A) \in \operatorname{Ker}\pi = \operatorname{Im} r_\mu$, i.e. there exists B such that the pair (A,B) is in the stabilizer. Moreover, B is unique, since r_μ is injective by hypothesis $\operatorname{Hom}(C,F) = 0$.

Since this correspondence is one-to-one and linear, it follows that $\dim \operatorname{Stab}(\mu) = \dim \operatorname{Hom}(C,C)$. $\qquad\square$

In the next result we provide a criterion for the simplicity of cokernel bundles.

Theorem 4.3 *Let C be a generic bundle with resolution* (1.1)*. Then the following statements are equivalent:*
 (i) C *is simple,*
(ii) $s^2 - wst + t^2 \leq 1$.

P r o o f. From Theorem 3.2 and Lemma 4.1 it follows that (ii) implies (i).

To prove the other implication suppose that C is simple. Then, since F is simple and $\operatorname{Hom}(F,C) \neq 0$, it follows that $\operatorname{Hom}(C,F) = 0$. Hence applying Lemma 4.2, we get that $\dim \operatorname{Hom}(C,C) = \dim \operatorname{Stab}(\mu)$. Clearly

$$\dim \operatorname{Stab}(\mu) \geq \dim(\mathrm{GL}(s) \times \mathrm{GL}(t)) - \dim H = s^2 + t^2 - wst,$$

thus $1 = \dim \operatorname{Hom}(C,C) = \dim \operatorname{Stab}(\mu) \geq s^2 + t^2 - wst$. Hence (i) implies (ii). $\qquad\square$

The following result will be used in Section 8.

Proposition 4.4 *Assume that E and F are rigid, and $\operatorname{Ext}^2(F,E) = 0$. If $s^2 + t^2 - wst \leq 1$, then the property of admitting resolution of the form* (1.1) *is invariant under small deformations.*

P r o o f. Since C is simple, the dimension of the space of matrices in $\operatorname{Hom}(E^s, F^t)$ up to the action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$ is

$$\dim H - \dim(\mathrm{GL}(s) \times \mathrm{GL}(t)) + \dim \operatorname{Stab}(\mu) = wst - s^2 - t^2 + 1 \geq 0.$$

We know that $\dim \operatorname{Ext}^1(C,C) \geq wst - s^2 - t^2 + 1 \geq 0$. On the other hand we prove that $\dim \operatorname{Ext}^1(C,C) \leq wst - s^2 - t^2 + 1$. Indeed, by applying the functor $\operatorname{Hom}(C,-)$ to the resolution of C we obtain

$$0 \longrightarrow \mathrm{Hom}(C, E^s) \longrightarrow \mathrm{Hom}(C, F^t) \longrightarrow \mathrm{Hom}(C, C) \longrightarrow \mathrm{Ext}^1(C, E^s) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^1(C, F^t) \longrightarrow \mathrm{Ext}^1(C, C) \longrightarrow \mathrm{Ext}^2(C, E^s) \longrightarrow Q \longrightarrow 0.$$

Hence by the assumptions on E and F and the simplicity of C it follows

$$\dim \mathrm{Ext}^1(C, C) \leq s \left(\dim \mathrm{Hom}(C, E) - \dim \mathrm{Ext}^1(C, E) + \dim \mathrm{Ext}^2(C, E) \right)$$
$$+ t \left(\dim \mathrm{Ext}^1(C, E) - \dim \mathrm{Hom}(C, E) \right) + \dim \mathrm{Hom}(C, C)$$
$$= -s^2 + t(ws - t) + 1,$$

which completes the proof. $\qquad \square$

5 Fibonacci bundles on $\mathbb{P}^N$

In this section we introduce the family of *Fibonacci bundles*, which will replace exceptional bundles in the canonical decomposition (see Section 6). In fact these bundles satisfy some properties of exceptional bundles, but in general they are not exceptional.

Given a bundle G, we denote by G_z the fiber of G at the point $z \in \mathbb{P}^N$. Given a map of bundles $f : G \to L$, we denote by f_z the restriction of the map f to the fiber at z, i.e. $f_z : G_z \to L_z$.

Theorem 5.1 *For any pair* (E, F) *satisfying the basic hypotheses* (1.2) *and* (1.3)*, there exist the following sequences of bundles:*

- $0 \longrightarrow C_{k-1} \xrightarrow{i_k} C_k \otimes W^* \xrightarrow{p_k} C_{k+1} \longrightarrow 0$ *if k is odd,*

- $0 \longrightarrow C_{k-1} \xrightarrow{i_k} C_k \otimes W \xrightarrow{p_k} C_{k+1} \longrightarrow 0$ *if k is even,*

where $C_0 = E$, $C_1 = F$, and the map i_k is recursively defined as follows:

- $i_1 : E \longrightarrow F \otimes W^* = F \otimes \mathrm{Hom}(E, F)^*$ *is the canonical map and*

- $i_k = (p_{k-1} \otimes id) \circ (id \otimes d) : C_{k-1} \otimes \mathbb{C} \longrightarrow C_{k-1} \otimes W \otimes W^* \longrightarrow C_k \otimes W$ *(resp. $C_k \otimes W^*$), where $d : \mathbb{C} \to W \otimes W^*$ is the diagonal map.*

We call the bundles C_k "Fibonacci bundles corresponding to (E, F)".

P r o o f. In order to prove the theorem, we will go through the following recursive steps for any k:
I. We define the map $i_k : C_{k-1} \to C_k \otimes W^*$ if k is odd, $i_k : C_{k-1} \to C_k \otimes W$ if k is even;
II. We prove by induction the property

$$\begin{array}{l} \text{for any } z \in \mathbb{P}^N, \text{ for any } 0 \neq c \in C_{k-1,z} \text{ the rank of} \\ i_{k,z}(c) \in \mathrm{Hom}(C_{k,z}^*, W), \text{ resp. } \mathrm{Hom}(C_{k,z}^*, W^*), \text{ is bigger than 1;} \end{array} \qquad (P_k)$$

III. We prove that i_k is injective, i.e. that the rank of i_k is constant;
IV. We define $C_{k+1} := \mathrm{Coker}(i_k)$.

If $k = 1$ the map i_1 is canonical, hence the property (P_1) holds and the fact that $E^* \otimes F$ is globally generated implies the injectivity of i_1.

Now, let us assume the bundles C_h to be defined for all $h \leq k + 1$, the map i_h to be defined for all $h \leq k$, to satisfy (P_h) and to be injective.

Let k be odd. First, we define the map i_{k+1}. By induction we have

$$0 \longrightarrow C_{k-1} \xrightarrow{i_k} C_k \otimes W^* \xrightarrow{p_k} C_{k+1} \longrightarrow 0$$

where p_k is the projection induced by i_k.

By tensoring by W, we get the diagram

$$
\begin{array}{ccc}
& & 0 \\
& & \downarrow \\
C_k \otimes \mathbb{C} & == & C_k \otimes \mathbb{C} \\
\downarrow {\scriptstyle id \otimes d} & & \downarrow \\
0 \longrightarrow C_{k-1} \otimes W \xrightarrow{i_k \otimes id} C_k \otimes W^* \otimes W \xrightarrow{p_k \otimes id} C_{k+1} \otimes W \longrightarrow 0 \\
\| & & \downarrow \\
C_{k-1} \otimes W \longrightarrow C_k \otimes \mathrm{Ad}(W) \\
& & \downarrow {\scriptstyle i_k} \\
& & 0
\end{array}
$$

where $d : \mathbb{C} \to W \otimes W^*$ is the diagonal map; more explicitly if $\{e_1, \dots, e_w\}$ is a basis of W and $\{e_1^*, \dots, e_w^*\}$ the dual basis, then

$$
d(1) = \sum_{i=1}^{w} e_i \otimes e_i^*.
$$

We define the map i_{k+1} as the following composition

$$
i_{k+1} = (p_k \otimes id) \circ (id \otimes d) : C_k \otimes \mathbb{C} \longrightarrow C_{k+1} \otimes W.
$$

Now we prove the property (P_{k+1}). For any $z \in \mathbb{P}^N$ we have $C_{k+1,z} = \frac{C_{k,z} \otimes W^*}{i_{k,z}(C_{k-1,z})}$. Hence for any $c \in C_{k,z}$ we get

$$
c \xrightarrow{(id \otimes d)_z} \sum_{i=1}^{w} c \otimes e_i \otimes e_i^* \xrightarrow{(p_k \otimes id)_z} \sum_{i=1}^{w} \frac{c \otimes e_i^*}{i_{k,z}(C_{k-1,z})} \otimes e_i,
$$

hence

$$
i_{k+1,z}(c) = \sum_{i=1}^{w} \frac{c \otimes e_i^*}{i_{k,z}(C_{k-1,z})} \otimes e_i.
$$

If there exists $0 \neq c \in C_{k,z}$ such that the rank of $i_{k+1,z}(c) \in \mathrm{Hom}(C_{k+1,z}^*, W)$ is 1, then for any $i \neq j$ there exist $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$ such that

$$
\alpha_{ij} \frac{c \otimes e_i^*}{i_{k,z}(C_{k-1,z})} = \beta_{ij} \frac{c \otimes e_j^*}{i_{k,z}(C_{k-1,z})},
$$

that is $\alpha_{ij} c \otimes e_i^* - \beta_{ij} c \otimes e_j^* = c \otimes (\alpha_{ij} e_i^* - \beta_{ij} e_j^*) \in i_{k,z}(C_{k-1})$, which contradicts (P_k). Therefore (P_{k+1}) is true.

Now in order to prove the injectivity of i_{k+1}, we show that

$$
\mathrm{Im}(i_k \otimes id)_z \cap \mathrm{Im}(id \otimes d)_z = \{0\}, \quad \text{for all} \quad z \in \mathbb{P}^N. \tag{5.1}
$$

Indeed, for any $z \in \mathbb{P}^N$, an element of $\mathrm{Im}(id \otimes d)_z$ is of the form $\sum_{i=1}^{w} c \otimes e_i \otimes e_i^*$ for some $c \in C_{k,z}$ and if $\sum_{i=1}^{w} c \otimes e_i \otimes e_i^* \in \mathrm{Im}(i_k \otimes id)_z$ then there exists an element

$$
b = \sum_{i,j} \gamma_{ij} b_i \otimes e_j \in C_{k-1,z} \otimes W,
$$

where $\{b_i\}$ is a basis of $C_{k-1,z}$ and $\gamma_{ij} \in \mathbb{C}$, such that

$$\sum_{i=1}^{w} c \otimes e_i^* \otimes e_i = (i_k \otimes id)_z(b).$$

It follows

$$\sum_{i=1}^{w} c \otimes e_i^* \otimes e_i = \sum_{i,j} \gamma_{ij}\, i_{k,z}(b_i) \otimes e_j$$

and, projecting this equation on e_j, we get

$$c \otimes e_j^* = \sum_i \gamma_{ij}\, i_{k,z}(b_i) = i_{k,z}\left(\sum_i \gamma_{ij} b_i\right)$$

which contradicts (P_k). Hence (5.1) is proved, and this implies the injectivity of the map i_{k+1} as a bundle map.

Finally we can define the bundle $C_{k+2} := \mathrm{Coker}(i_{k+1})$, and we get the exact sequence

$$0 \longrightarrow C_k \xrightarrow{i_{k+1}} C_{k+1} \otimes W \longrightarrow C_{k+2} \longrightarrow 0.$$

If k is even, we repeat the same argument interchanging W and W^* and this yields the following exact sequence

$$0 \longrightarrow C_k \xrightarrow{i_{k+1}} C_{k+1} \otimes W^* \longrightarrow C_{k+2} \longrightarrow 0.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Theorem 5.2 *For every $k \geq 1$, a Fibonacci bundle C_k on $\mathbb{P}^N$ corresponding to (E, F) admits the following resolution*

$$0 \longrightarrow E^{a_{k-1}} \longrightarrow F^{a_k} \longrightarrow C_k \longrightarrow 0, \tag{5.2}$$

with $\{a_k\} = \{a_{w,k}\}$ as in (2.5).

P r o o f. We prove the statement by induction on k. If $k = 1$ the sequence (5.2) is $0 \to F \to C_1 \to 0$, and the claim is true.

Now, we suppose that every C_h admits a resolution of the form (5.2) for any $h \leq k$ and we prove the same assertion for C_{k+1}. First assume k odd. By the sequence

$$0 \longrightarrow C_{k-1} \longrightarrow C_k \otimes W^* \longrightarrow C_{k+1} \longrightarrow 0,$$

and by induction hypothesis, we have:

where we define the map α as the composition of the known maps.

Since $\mathrm{Ext}^1(F, E) = 0$, the map α induces a map $\widetilde{\alpha} : F^{a_{k-1}} \to F^{a_k} \otimes W^*$ such that the diagram commutes. Moreover if $\widetilde{\beta}$ is the restriction of $\widetilde{\alpha}$ to $E^{a_{k-2}}$ the following diagram commutes:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & C_{k-1} & \longrightarrow & C_k \otimes W^* & \longrightarrow & C_{k+1} & \longrightarrow & 0 \\
 & & \uparrow & \nearrow{\alpha} & \uparrow & & & & \\
 & & F^{a_{k-1}} & \xrightarrow{\widetilde{\alpha}} & F^{a_k} \otimes W^* & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & E^{a_{k-2}} & \xrightarrow{\widetilde{\beta}} & E^{a_{k-1}} \otimes W^* & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

This diagram implies that $\mathrm{Ker}(\widetilde{\alpha}) \cong \mathrm{Ker}(\widetilde{\beta})$, but since E and F are simple, $\mathrm{Ker}(\widetilde{\alpha}) \cong F^a$ and $\mathrm{Ker}(\widetilde{\beta}) \cong E^b$ for some $a, b \in \mathbb{N}$. Hence since E and F are indecomposable and $E \not\cong F$, by the Krull–Schmidt theorem for vector bundles (see [2]), we get $\mathrm{Ker}(\widetilde{\alpha}) \cong \mathrm{Ker}(\widetilde{\beta}) = 0$. Thus $\widetilde{\alpha}$ is injective and we can complete the diagram as follows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_{k-1} & \longrightarrow & C_k \otimes W^* & \longrightarrow & C_{k+1} & \longrightarrow & 0 \\
 & & \uparrow & \nearrow{\alpha} & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F^{a_{k-1}} & \xrightarrow{\widetilde{\alpha}} & F^{a_k} \otimes W^* & \longrightarrow & F^{a_{k+1}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E^{a_{k-2}} & \xrightarrow{\widetilde{\beta}} & E^{a_{k-1}} \otimes W^* & \longrightarrow & E^{a_k} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

It follows that C_{k+1} has the resolution

$$
0 \longrightarrow E^{a_k} \longrightarrow F^{a_{k+1}} \longrightarrow C_{k+1} \longrightarrow 0.
$$

If k is even, we replace W with W^* and we obtain the same result. $\square$

We remark that it is possible to describe more explicitly the resolutions of Fibonacci bundles. Indeed for every $k \geq 0$, a Fibonacci bundle C_k corresponding to (E, F) on $\mathbb{P}^N$ has the following resolution

$$
0 \longrightarrow E \otimes A_k \longrightarrow F \otimes B_k \longrightarrow C_k \longrightarrow 0,
$$

where

$$
A_1 = 0, \quad B_1 = \mathbb{C}, \quad A_2 = \mathbb{C}, \quad B_2 = W^*,
$$

and

$$
A_{k+1} = \frac{A_k \otimes W}{j_k(A_{k-1})} \ \text{if } k \ \text{even}; \qquad A_{k+1} = \frac{A_k \otimes W^*}{j_k(A_{k-1})} \ \text{if } k \ \text{odd};
$$

$$B_{k+1} = \frac{B_k \otimes W}{u_k(B_{k-1})} \text{ if } k \text{ even}; \qquad B_{k+1} = \frac{B_k \otimes W^*}{u_k(B_{k-1})} \text{ if } k \text{ odd},$$

where j_k and u_k are recursively defined, in a similar way to the definition of i_k in the statement of Theorem 5.1.

More explicitly, we define $j_1 : 0 \to W$ as the zero map, $u_1 = d : \mathbb{C} \to W^* \otimes W$ as the diagonal map. For any $k \geq 1$, we define q_k and r_k such that

$$0 \longrightarrow A_{k-1} \xrightarrow{j_k} A_k \otimes U_k \xrightarrow{q_k} A_{k+1} \longrightarrow 0$$

and

$$0 \longrightarrow B_{k-1} \xrightarrow{u_k} B_k \otimes U_k \xrightarrow{r_k} B_{k+1} \longrightarrow 0,$$

where for brevity we denote $U_k = W$ if k is even, $U_k = W^*$ if k is odd. Now we define, for any $k \geq 2$,

$$j_k = (q_{k-1} \otimes id) \circ (id \otimes d) : A_{k-1} \otimes \mathbb{C} \longrightarrow A_{k-1} \otimes U_k \otimes U_k^* \longrightarrow A_k \otimes U_k$$

and

$$u_k = (r_{k-1} \otimes id) \circ (id \otimes d) : B_{k-1} \otimes \mathbb{C} \longrightarrow B_{k-1} \otimes U_k \otimes U_k^* \longrightarrow B_k \otimes U_k.$$

Remark 5.3 It is easy to check that $A_k \cong B_{k-1}^*$ as $\mathrm{SL}(V)$-representations, since all sequences of $\mathrm{SL}(V)$-modules split. However it is possible that this isomorphism is not canonical, because when A_k and B_k are decomposed as sums of irreducible representations, some summand can appear with multiplicity bigger than one.

In order to clarify the situation look at an example. Let $W = S^2V$ and $N = 4$. We denote by $\Gamma(a_1 a_2 a_3 a_4)$ the irreducible representation of $\mathrm{SL}(V)$ with highest weight $\sum_i a_i \omega_i$, where ω_i are the fundamental weights. With this notation we have for example $A_3 = S^2V = \Gamma(2000)$ and $B_2 = S^2V^* = \Gamma(0002)$. Going on, we can compute

$$A_5 = 2\Gamma(3001) + 2\Gamma(1101) + \Gamma(2000) + \Gamma(0100) + \Gamma(4002) + \Gamma(2102) + \Gamma(0202)$$

and

$$B_4 = 2\Gamma(1003) + 2\Gamma(1011) + \Gamma(0002) + \Gamma(0010) + \Gamma(2004) + \Gamma(2012) + \Gamma(2020).$$

In this example it is evident that $A_5 \cong B_4^*$, nevertheless the isomorphism need not be canonic, since two terms in the sum have multiplicity two.

Remark 5.4 Since $a_{k-1}^2 + a_k^2 - w a_{k-1} a_k = 1$, from Theorem 4.3 it follows that any generic bundle with resolution (5.2) is simple. In general this does not imply that any Fibonacci bundle is simple. However with additional assumptions on (E, F), we will prove the simplicity of any Fibonacci bundle.

Lemma 5.5 *Assume that E is rigid. If C_k is a Fibonacci bundle corresponding to (E, F), then*

$$\dim \mathrm{Hom}(F, C_k) = a_k \quad \text{and} \quad \dim \mathrm{Hom}(E, C_k) = a_{k+1}.$$

P r o o f. By applying respectively the functors $\mathrm{Hom}(F, -)$ and $\mathrm{Hom}(E, -)$ to the sequence (5.2), we easily obtain that $\dim \mathrm{Hom}(F, C_k) = a_k$ and $\dim \mathrm{Hom}(E, C_k) = w a_k - a_{k-1} = a_{k+1}$, as claimed. $\qquad \square$

Recall that by hypotheses (R) we mean the union of conditions (1.2), (1.3) and (1.4).

Corollary 5.6 *Assume that the pair (E, F) satisfies the conditions (R). Then if a corresponding Fibonacci bundle C_k is simple, it is also rigid.*

P r o o f. Assume that C_k is simple, i.e. $\dim \mathrm{Hom}(C_k, C_k) = 1$. Since $\mathrm{Ext}^1(F, F) = 0$ and $\mathrm{Ext}^2(F, E) = 0$, it follows $\mathrm{Ext}^1(F, C_k) = 0$. Then by applying $\mathrm{Hom}(-, C_k)$ to the resolution (5.2) and by Lemma 5.5 we get

$$\dim \mathrm{Ext}^1(C_k, C_k) = \dim \mathrm{Hom}(C_k, C_k) - a_k \dim \mathrm{Hom}(F, C_k) + a_{k-1} \dim \mathrm{Hom}(E, C_k)$$
$$= 1 - a_k^2 + a_{k-1} a_{k+1} = 0,$$

hence C is rigid. $\qquad \square$

Lemma 5.7 *Assume that (E, F) satisfies (R) and, for any $k \geq 0$, let C_k be the corresponding Fibonacci bundle. Then the following properties (I_k), (II_k) and (III_k) are satisfied for any $k \geq 1$:*

$$(I_k) \qquad \operatorname{Hom}(C_k, C_k) \cong \mathbb{C},$$

$$(II_k) \qquad \operatorname{Hom}(C_k, C_{k-1}) = 0, \quad \operatorname{Ext}^1(C_k, C_{k-1}) = 0,$$

$$(III_k) \qquad \operatorname{Hom}(C_{k-1}, C_k) \cong W, \quad \textit{if } k \textit{ odd,}$$

$$\operatorname{Hom}(C_{k-1}, C_k) \cong W^*, \quad \textit{if } k \textit{ even.}$$

P r o o f. We prove the lemma by induction on k.

Recall that $C_0 = E$ and $C_1 = F$. Hence by hypotheses (1.2) and (1.3) we know that C_1 is simple, $\operatorname{Hom}(C_0, C_1) = W$ and $\operatorname{Hom}(C_1, C_0) = \operatorname{Ext}^1(C_1, C_0) = 0$.

Now suppose that (I_h), (II_h) and (III_h) hold for all $h \leq k$. Let us prove (I_{k+1}), (II_{k+1}) and (III_{k+1}). First, suppose k even. By the definition of Fibonacci bundles we get the sequence

$$0 \longrightarrow C_{k-1} \longrightarrow C_k \otimes W \longrightarrow C_{k+1} \longrightarrow 0. \tag{5.3}$$

By applying the functor $\operatorname{Hom}(C_k, -)$ we get

$$0 \longrightarrow \operatorname{Hom}(C_k, C_{k-1}) \longrightarrow \operatorname{Hom}(C_k, C_k) \otimes W \longrightarrow \operatorname{Hom}(C_k, C_{k+1}) \longrightarrow \operatorname{Ext}^1(C_k, C_{k-1})$$

and from (I_k) and (II_k) we get $\operatorname{Hom}(C_k, C_{k+1}) \cong W$. On the other hand if k is odd, we consider the sequence

$$0 \longrightarrow C_{k-1} \longrightarrow C_k \otimes W^* \longrightarrow C_{k+1} \longrightarrow 0,$$

and we obtain with the same argument $\operatorname{Hom}(C_k, C_{k+1}) \cong W^*$, hence (III_{k+1}) follows.

Now, suppose k even and let us apply $\operatorname{Hom}(-, C_k)$ to the sequence (5.3):

$$0 \longrightarrow \operatorname{Hom}(C_{k+1}, C_k) \longrightarrow \operatorname{Hom}(C_k, C_k) \otimes W^* \xrightarrow{\ \alpha\ } \operatorname{Hom}(C_{k-1}, C_k) \longrightarrow$$
$$\longrightarrow \operatorname{Ext}^1(C_{k+1}, C_k) \longrightarrow \operatorname{Ext}^1(C_k, C_k) \otimes W.$$

Since $\operatorname{Hom}(C_k, C_k) \cong \mathbb{C}$ and α is the canonical isomorphism $\operatorname{Hom}(C_{k-1}, C_k) \cong W^*$, we get

$$\operatorname{Hom}(C_{k+1}, C_k) = 0.$$

Moreover from (I_k) we know that C_k is simple and by Corollary 5.6 it follows that C_k is rigid, i.e.

$$\operatorname{Ext}^1(C_k, C_k) = 0.$$

It implies that $\operatorname{Ext}^1(C_{k+1}, C_k) = 0$, and the property (II_{k+1}) follows.

Now let us prove (I_{k+1}). First we apply $\operatorname{Hom}(-, C_{k-1})$ to (5.3), and we have

$$\operatorname{Hom}(C_k, C_{k-1}) \otimes W^* \longrightarrow \operatorname{Hom}(C_{k-1}, C_{k-1})$$
$$\longrightarrow \operatorname{Ext}^1(C_{k+1}, C_{k-1}) \longrightarrow \operatorname{Ext}^1(C_k, C_{k-1}) \otimes W^*,$$

then (II_k) and (I_{k-1}) imply that $\operatorname{Ext}^1(C_{k+1}, C_{k-1}) \cong \mathbb{C}$. Finally by applying $\operatorname{Hom}(C_{k+1}, -)$ to (5.3), we get

$$\operatorname{Hom}(C_{k+1}, C_k) \otimes W \longrightarrow \operatorname{Hom}(C_{k+1}, C_{k+1})$$
$$\longrightarrow \operatorname{Ext}^1(C_{k+1}, C_{k-1}) \longrightarrow \operatorname{Ext}^1(C_{k+1}, C_k) \otimes W$$

and, using (II_{k+1}), we obtain

$$\operatorname{Hom}(C_{k+1}, C_{k+1}) \cong \operatorname{Ext}^1(C_{k+1}, C_{k-1}) \cong \mathbb{C},$$

hence (I_{k+1}) holds.   □

As a consequence of Corollary 5.6 and Lemma 5.7, we obtain the following theorem.

Theorem 5.8 *For any bundle C on $\mathbb{P}^N$, with $N \geq 2$, the following are equivalent:*
(i) *C is a Fibonacci bundle corresponding to some pair (E, F) satisfying hypotheses (R),*
(ii) *C is simple and rigid.*

P r o o f. From Lemma 5.7 and from Corollary 5.6 it follows that property (i) implies (ii). The other implication is easy to prove, because it suffices to choose the bundles $E = C(-d)$ and $F = C$, with $d \gg 0$ such that the pair (E, F) satisfies conditions (R). □

Remark 5.9 Notice that, in particular, all the exceptional bundles are Fibonacci bundles with respect to some pair (E, F).

Lemma 5.10 *Assume that (E, F) satisfies (R) and, for any $k \geq 0$, let C_k be the corresponding Fibonacci bundle. Then $\operatorname{Ext}^1(C_{k-1}, C_k) = 0$ for any $k \geq 1$.*

P r o o f. Let us apply $\operatorname{Hom}(-, C_k)$ to the resolution of C_{k-1}. Since $\operatorname{Ext}^1(F, C_k) = 0$, by Lemma 5.5, we get

$$\dim \operatorname{Ext}^1(C_{k-1}, C_k) = \dim \operatorname{Hom}(C_{k-1}, C_k) - a_{k-1} \dim \operatorname{Hom}(F, C_k) + a_{k-2} \dim \operatorname{Hom}(E, C_k)$$
$$= w - a_{k-1} a_k + a_{k-2} a_{k+1}$$
$$= 0.$$
□

6 Non-simple bundles

In this section we investigate a generic bundle C on $\mathbb{P}^N$ with resolution (1.1) in the case $s^2 + t^2 - wst \geq 1$. By Theorem 4.3 we know that such a bundle C is simple only if $s^2 + t^2 - wst = 1$, that is only if C is a deformation of a Fibonacci bundle. Here we prove that when $s^2 + t^2 - wst \geq 1$ and the pair (E, F) satisfies hypotheses (R), then any generic bundle C is decomposable as a sum of Fibonacci bundles. In particular C is simple if and only if it is a Fibonacci bundle (if and only if $s^2 + t^2 - wst = 1$).

Remark 6.1 Since $E^* \otimes F$ is globally generated, we have

$$\operatorname{rk} E \operatorname{rk} F \leq w = \dim \operatorname{H}^0(E^* \otimes F).$$

The following lemma is a consequence of the second part of Theorem 3.2. Here we give another elementary proof.

Lemma 6.2 *For any $s, t \in \mathbb{N}$ satisfying $t \operatorname{rk}(F) - s \operatorname{rk}(E) \geq N$, and*

$$s^2 + t^2 - wst > 0,$$

there exist unique $k, n, m \in \mathbb{N}$ such that the bundle $C_k^n \oplus C_{k+1}^m$ admits a resolution of the form

$$0 \longrightarrow E^s \longrightarrow F^t \longrightarrow C_k^n \oplus C_{k+1}^m \longrightarrow 0, \tag{6.1}$$

where C_k and C_{k+1} are Fibonacci bundles.

P r o o f. By Remark 6.1 and conditions $t \operatorname{rk}(F) - s \operatorname{rk}(E) \geq N$ and $s^2 + t^2 - wst > 0$, it follows that $t > \left(\frac{w + \sqrt{w^2 - 4}}{2} \right) s$.

Fix s, t such that $t > \left(\frac{w + \sqrt{w^2 - 4}}{2} \right) s$. Let $\{a_k\} = \{a_{w,k}\}$ be the sequence defined in (2.5). It is easy to check that the sequence $\left\{ \frac{a_{k+1}}{a_k} \right\}$ is decreasing to $\frac{w + \sqrt{w^2 - 4}}{2}$. It follows that there exists $k \geq 1$ such that

$$\text{either} \quad \frac{a_k}{a_{k-1}} = \frac{t}{s} \quad \text{or} \quad \frac{a_{k+1}}{a_k} < \frac{t}{s} < \frac{a_k}{a_{k-1}}.$$

In the first case, since $(a_k, a_{k-1}) = 1$ by Remark 2.4, there exists $n > 1$ such that $t = na_k, s = na_{k-1}$, i.e. the bundle C_k^n admits resolution (6.1), with $m = 0$. In the second case, we solve the following system

$$
\begin{cases}
t = na_k + ma_{k+1}, \\
s = na_{k-1} + ma_k.
\end{cases}
$$

This system has discriminant $\Delta = a_k^2 - a_{k+1}a_{k-1} = 1$, thus it admits a pair of integer solutions (n, m). In particular, $n > 0$ because $\frac{t}{s} > \frac{a_{k+1}}{a_k}$, and $m > 0$ because $\frac{t}{s} < \frac{a_k}{a_{k-1}}$. It follows that the bundle $C_k^n \oplus C_{k+1}^m$ has resolution (6.1). $\square$

Theorem 6.3 *Let (E, F) satisfy hypotheses (R), and let $s, t \in \mathbb{N}$ satisfy $t \operatorname{rk}(F) - s \operatorname{rk}(E) \geq N$. Let C be a generic bundle on $\mathbb{P}^N$ with resolution (1.1). Then*

$$
s^2 + t^2 - wst > 0 \quad \Longrightarrow \quad C \cong C_k^n \oplus C_{k+1}^m
$$

where C_k and C_{k+1} are Fibonacci bundles and $n, m \in \mathbb{N}$ are unique.

P r o o f. It suffices to prove that the space of matrices $\mu \in H$ such that $\operatorname{Coker}(\mu) \cong C_k^n \oplus C_{k+1}^m$ is a dense subset of the vector space $\operatorname{Hom}(E^s, F^t)$. Let us compute $\dim \operatorname{Ext}^1\left(C_k^n \oplus C_{k+1}^m, C_k^n \oplus C_{k+1}^m\right)$. By the property (II) of Lemma 5.7, by Theorem 5.8 and Lemma 5.10, we obtain that $\dim \operatorname{Ext}^1\left(C_k^n \oplus C_{k+1}^m, C_k^n \oplus C_{k+1}^m\right) = 0$ for all $k \geq 1$ and for all $n, m \in \mathbb{N}$. Hence the bundles $C_k^n \oplus C_{k+1}^m$ are rigid. It follows that the set of matrices μ such that $\operatorname{Coker}(\mu)$ is isomorphic to $C_k^n \oplus C_{k+1}^m$ is open, hence dense in H. This completes the proof. $\square$

Remark 6.4 Notice that a generic bundle which satisfies the hypotheses of Theorem 6.3 is rigid, hence homogeneous.

7   Some applications

7.1   First example

If $E = \mathcal{O}$ and $F = \mathcal{O}(d)$ on $\mathbb{P}^N$ (with $N \geq 2$) the bundles with resolution (2.1) are cokernels of matrices of homogeneous polynomials of degree d.

Notice that hypotheses (R) are satisfied if either $N \geq 3$, or $N = 2$ and $1 \leq d \leq 2$. Then all the Fibonacci bundles on $\mathbb{P}^N$ with either $N \geq 3$ or $N = 2$ and $1 \leq d \leq 2$ are simple and rigid. Any generic deformation of a Fibonacci bundle on $\mathbb{P}^2$ with $d > 2$ is simple. More precisely we get the following classification:

Corollary 7.1 *Let C be a generic bundle on $\mathbb{P}^N$ with resolution (2.1). If either $N \geq 3$ or $N = 2$ and $1 \leq d \leq 2$, then*

$$
s^2 + t^2 - \binom{N+d}{d} st = 1 \quad \Longleftrightarrow \quad C \text{ is a Fibonacci bundle} \quad \Longleftrightarrow \quad C \text{ is simple and rigid.}
$$

If $N = 2$ and $d > 2$,

$$
s^2 + t^2 - \binom{N+d}{d} st = 1 \quad \Longleftrightarrow \quad C \text{ is a deformation of a Fibonacci} \quad \Longrightarrow \quad C \text{ is simple.}
$$

P r o o f. It is easy to check that if either $N \geq 3$ or $N = 2$ and $1 \leq d \leq 2$, then $\dim \operatorname{Hom}(C, C) - \dim \operatorname{Ext}^1(C, C) = s^2 + t^2 - \binom{N+d}{d} st$. Hence the corollary follows from Theorem 4.3 and from the results of Section 5. $\square$

Let C_k be a Fibonacci bundle on $\mathbb{P}^N$ corresponding to $\mathcal{O}, \mathcal{O}(d)$ with $d \geq 1$. Then by applying the functor $\operatorname{Hom}(-, C_k)$ to the resolution of C_k, one can easily check the following properties:

$$
\operatorname{Hom}(C_k, C_k) = \mathbb{C},
$$
$$
\operatorname{Ext}^i(C_k, C_k) = 0, \quad \text{for all} \quad 1 \leq i \leq N - 2,
$$

$$\operatorname{Ext}^{N-1}(C_k, C_k) \cong \operatorname{H}^N(\mathcal{O}(-d))^{a_k a_{k-1}}.$$

Thus all the Fibonacci bundles with $1 \leq d \leq N$ are exceptional. In particular if $d = 1$, they are exactly the exceptional Steiner bundles studied in [3]. Notice that the Fibonacci bundles with $d > N$ are not exceptional. This argument yields the following classification:

Corollary 7.2 *All the exceptional bundles on $\mathbb{P}^N$ with resolution* (2.1) *are exactly the Fibonacci bundles corresponding to $\mathcal{O}, \mathcal{O}(d)$ for $1 \leq d \leq N$. All the semi-exceptional bundles on $\mathbb{P}^N$ with resolution* (2.1) *are of the form C_k^n, where C_k is a Fibonacci bundle as above.*

7.2 Bundles on $\mathbb{P}^1$

In this paper we have always supposed $N \geq 2$, because the case $N = 1$, corresponding to bundles on $\mathbb{P}^1$, is nowadays trivial as it was solved by Kronecker in [11].

In this case Theorem 4.3 does not hold, since the fact that $\dim \operatorname{Stab}(\mu) = 1$ does not imply the simplicity of $\operatorname{Coker}(\mu)$. In fact, since any bundle C on $\mathbb{P}^1$ is decomposable as a sum of line bundles, C is simple if and only if C has rank 1 if and only if C is exceptional.

On the other hand, there exists a canonical decomposition for all bundles on $\mathbb{P}^1$ with resolution (2.1) for any $d \geq 1$. Let us prove that a generic bundle with resolution (2.1) is isomorphic to $\mathcal{O}(a)^n \oplus \mathcal{O}(a + 1)^m$. If $\frac{dt}{t-s}$ is integer, then we choose $a = \frac{dt}{t-s}$, $n = t - s$ and $m = 0$. If $\frac{dt}{t-s}$ is not integer then we choose the unique integer $\frac{dt}{t-s} - 1 < a < \frac{dt}{t-s}$. Then, as in the proof of Theorem 6.2, we see that the system

$$\begin{cases} n + m = t - s, \\ na + m(a + 1) = dt \end{cases}$$

admits a pair of integer positive solutions (n, m). It follows that the bundle $C = \mathcal{O}(a)^n \oplus \mathcal{O}(a + 1)^m$ has resolution (2.1). Since $\dim \operatorname{Ext}^1(C, C) = 0$, a generic bundle with resolution (2.1) is isomorphic to C and this implies that there is a canonical reduction for matrices of polynomials in two variables.

Remark 7.3 It follows that when we restrict any generic bundle on $\mathbb{P}^N$ with resolution (2.1) to a generic $\mathbb{P}^1 \subset \mathbb{P}^N$, the splitting type is of the form $\mathcal{O}(a)^n \oplus \mathcal{O}(a + 1)^m$, hence it is as balanced as possible.

7.3 Second example

Given $0 < p < N$, let us consider one of the following pairs of bundles on $\mathbb{P}^N = \mathbb{P}(V)$, with $N \geq 2$,

- $E = \mathcal{O}(-1)$ and $F = \Omega^p(p)$,

- $E = \Omega^p(p)$ and $F = \mathcal{O}$.

It is easily seen that in these two cases hypotheses (R) hold. Then we can apply Theorems 4.3 and 6.3 to the corresponding cokernel bundles and we get the following consequences.

Corollary 7.4 *Given $0 < p < N$, let C be a generic bundle with resolution either* (2.2) *or* (2.3). *Then*
  (i) *C is simple $\iff s^2 - wst + t^2 \leq 1$,*
  (ii) *C is simple and rigid $\iff C$ is a Fibonacci bundle C_k,*
  (iii) *$s^2 - wst + t^2 \geq 1 \implies C \cong C_k^n \oplus C_{k+1}^m$ for unique $n, m, k \in \mathbb{N}$,*
where $w = \binom{N+1}{N-p}$ in case (2.2)*, and $w = \binom{N+1}{p}$ in case* (2.3)*.*

Notice that also exceptional Steiner bundles belong to this class. More precisely we have:

Proposition 7.5 *Any exceptional Steiner bundle S_{k+1} on $\mathbb{P}^N$ of the form*

$$0 \longrightarrow \mathcal{O}(-2)^{a_k} \longrightarrow \mathcal{O}(-1)^{a_{k+1}} \longrightarrow S_{k+1} \longrightarrow 0$$

is isomorphic to a Fibonacci bundle C_k associated to the pair $\mathcal{O}(-1), \Omega^{N-1}(N - 1)$, i.e.

$$0 \longrightarrow \mathcal{O}(-1)^{a_{k-1}} \longrightarrow \Omega^{N-1}(N - 1)^{a_k} \longrightarrow C_k \longrightarrow 0.$$

P r o o f. It suffices to apply the theorem of Beĭlinson (see for example [1]) to the bundle S_{k+1}.

Let us compute the dimension $h^i(F(-j))$ for $0 \leq i, j \leq N$. From the resolution of S_{k+1}, it is easily seen that $h^i(S_{k+1}(-j)) = 0$ for any $i \geq 0$ and $0 \leq j \leq N - 2$. Moreover $h^i(S_{k+1}(-N + 1)) = 0$ for any $i \neq N - 1$, and $h^{N-1}(S_{k+1}(-N + 1)) = a_k$. Finally $h^i(S_{k+1}(-N)) = 0$ for any $i \leq N - 2$, and $h^{N-1}(S_{k+1}(-N)) - h^N(S_{k+1}(-N)) = a_{k-1}$. By Serre duality we know that $h^N(S_{k+1}(-N)) = h^0(S_{k+1}^*(N - N - 1))$. Since S_{k+1} is simple and $H^0(S_{k+1}(1)) \neq 0$, then $H^0(S_{k+1}^*(-1)) = 0$. Hence $h^{N-1}(S_{k+1}(-N)) = a_{k-1}$.

Then by applying the theorem of Beĭlinson, we get that S_{k+1} admits the resolution

$$0 \longrightarrow \mathcal{O}(-1)^{a_{k-1}} \longrightarrow \Omega^{N-1}(N - 1)^{a_k} \longrightarrow S_{k+1} \longrightarrow 0.$$

Hence by the rigidity of the Fibonacci bundles we conclude that $S_{k+1} \cong C_k$. $\square$

7.4 Third example

On $\mathbb{P}^2$, we consider $E = S^p Q$ and $F = S^r Q(d)$ and the bundles with resolution (2.4) for $d > p+1$. By applying Theorem 4.3 to the corresponding cokernel bundles we deduce the following result.

Corollary 7.6 *Given $r, p \geq 1$ and $d > p + 1$, a generic bundle C with resolution (2.4) is simple if and only if $s^2 - wst + t^2 \leq 1$, where $w = \dim H^0(S^p Q \otimes S^r Q \otimes \mathcal{O}(d - p))$.*

Notice that hypotheses (R) are never satisfied, since $S^p Q$ is not rigid.

8 Stability

In this last section we present some results about stability: first we consider the exceptional Steiner bundles on $\mathbb{P}^N$ with $N \geq 2$, then we restrict our attention to bundles on $\mathbb{P}^2$ and we utilize some important results of Drézet and Le Potier. Recall that on $\mathbb{P}^N$, with $N > 3$, the general problem of the stability of exceptional bundles is still open.

8.1 Stability of exceptional Steiner bundles on $\mathbb{P}^N$

The next result, which concerns the exceptional Steiner bundles, is based on the properties of the Fibonacci sequences. As in Section 3, we denote $a_k = a_{N+1,k}$. Recall that any exceptional Steiner bundle S_k has resolution

$$0 \longrightarrow \mathcal{O}^{a_{k-1}} \longrightarrow \mathcal{O}(1)^{a_k} \longrightarrow S_k \longrightarrow 0. \tag{8.1}$$

Theorem 8.1 *Any exceptional Steiner bundle S_k on $\mathbb{P}^N$ is stable for all $k \geq 0$ and for any $N \geq 2$.*

P r o o f. We prove the theorem by induction on k. If $k = 0, 1$, we get $S_0 = \mathcal{O}$ and $S_1 = \mathcal{O}(1)$, which are stable, since they are line bundles. Let us suppose that S_h is stable for all $h \leq k$ and let us prove the stability of S_{k+1}. Assume by contradiction that S_{k+1} is not stable. Then there exists a quotient Q such that

$$\mu(Q) \leq \mu(S_{k+1}).$$

We can suppose that Q is stable. From Theorem 5.1 we know that there exists the sequence

$$0 \longrightarrow S_{k-1} \longrightarrow S_k \otimes U_k \longrightarrow S_{k+1} \longrightarrow 0,$$

where $U_k = W$ if k is even, $U_k = W^*$ if k is odd. It follows that Q is also a quotient of $S_k \otimes U_k$ and so, from the stability of S_k, we obtain

$$\mu(Q) \geq \mu(S_k).$$

From resolution (8.1) we get $\mu(S_k) = \frac{a_k}{a_k - a_{k-1}}$. It is easy to check that

$$\mu(S_k) = \frac{a_k}{a_k - a_{k-1}} < \frac{a_{k+1}}{a_{k+1} - a_k} = \mu(S_{k+1})$$

and, denoting $r_h = a_h - a_{h-1}$ for any h, we compute

$$\frac{a_{k+1}}{r_{k+1}} - \frac{a_k}{r_k} = \frac{1}{r_{k+1}r_k}.$$

Hence, denoting by $\frac{c}{r}$ the slope of Q, we have to find two positive integer c, r such that $r < r_{k+1}$ and

$$\frac{a_k}{r_k} \le \frac{c}{r} \le \frac{a_k}{r_k} + \frac{1}{r_{k+1}r_k}.$$

With simple computations we get

$$0 \le \frac{r_k c - a_k r}{r} \le \frac{1}{r_{k+1}}$$

and, since $r < r_{k+1}$, the only possibility is $r_k c - a_k r = 0$, i.e. $\mu(Q) = \mu(S_k)$. Now since $S_k \otimes U_k$ is polystable (in fact it is the direct sum of $N+1$ copies of the stable bundle S_k) and Q is stable with the same slope, it follows that $Q = S_k$. Then S_k has to be a quotient of S_{k+1} and this is impossible because $\mathrm{Hom}(S_{k+1}, S_k) = 0$. This completes the proof. $\square$

8.2 Stability of bundles on $\mathbb{P}^2$

The problem of the stability of vector bundles on $\mathbb{P}^2$ has been studied by Drézet and Le Potier. In particular, in [7] they found a criterion to check the existence of a stable bundle with given rank and Chern classes, but this criterion is complicated to apply even for Steiner bundles.

Moreover, from another result of Drézet (see Theorem 3.1 of [6]) we know that if there exist no semi-stable bundles with given rank and Chern classes, then the generic bundle in the space of *prioritary bundles* with these rank and Chern classes is decomposable, hence non-simple.

A vector bundle P on $\mathbb{P}^2$ (or a coherent torsionfree sheaf) is called *prioritary* when

$$\mathrm{Ext}^2(P, P(-1)) = 0.$$

Prioritary bundles were introduced by Hirschowitz and Laszlo in [9].

It is easily seen that if E and F are priority and $\mathrm{Ext}^1(E, F(-1)) = 0$ then any bundle C on $\mathbb{P}^2$ with resolution (1.1) is priority. On the other hand, if the pair (E, F) satisfies hypotheses (R), by Proposition 4.4 we get that a generic cokernel bundle C in (1.1) is also generic in the space of prioritary bundles.

This implies that our Theorem 4.3 provides a criterion for the stability of generic bundles C on $\mathbb{P}^2$ with resolution (1.1), for any (E, F) satisfying hypotheses (R). Precisely we get the following result:

Theorem 8.2 *Let E and F be two prioritary bundles on $\mathbb{P}^2$ satisfying (R) and such that $\mathrm{Ext}^1(E, F(-1))$. Let C be defined by (1.1). Then the following statements are equivalent:*

(i) *C is stable,*

(ii) *$s^2 - wst + t^2 \le 1$.*

Remark 8.3 In particular the previous theorem implies that any Fibonacci bundle on $\mathbb{P}^2$ with respect to (E, F) is stable, if (E, F) satisfies (R) and $\mathrm{Ext}^1(E, F(-1))$.

Remark 8.4 If the pair (E, F) satisfies only the basic hypotheses (1.2) and (1.3), then we obtain the stability of a generic deformation of a corresponding Fibonacci bundle in the space of prioritary bundles.

The next result is a straightforward consequence of Theorem 8.2 in the case $E = \mathcal{O}$, $F = \mathcal{O}(d)$ and $d = 1, 2$.

Theorem 8.5 *Let C be a generic bundle on $\mathbb{P}^2$ with resolution (2.1) with $d = 1, 2$. Then the following statements are equivalent:*

(i) *C is stable,*

(ii) *$s^2 - \binom{d+2}{2}st + t^2 \le 1$,*

(iii) *either C is a Fibonacci (and exceptional) bundle or $t \le \frac{1}{2}\left(\binom{d+2}{2} + \sqrt{\binom{d+2}{2}^2 - 4}\right)s$.*

We stress that this criterion is equivalent to the Drézet–Le Potier criterion in the case of bundles with resolution (2.1) and $d = 1, 2$. Nevertheless our proof is completely independent, and it seems difficult to deduce it directly from [7].

From the description of non-simple bundles (Theorem 6.3), we can classify all the strictly semi-stable bundles on $\mathbb{P}^2$ with resolution (2.1) and we get the following result.

Corollary 8.6 *Let C be a generic bundle on $\mathbb{P}^2$ with resolution* (2.1) *with $d = 1, 2$. Then C is strictly semi-stable if and only if it is the sum of $n > 1$ copies of a Fibonacci bundle, if and only if it is semi-exceptional.*

Finally we remark that the results of this section allow us to improve a theorem of Hein, contained in the appendix of [5], about the stability of a generic *syzygy bundle*, i.e. of a bundle G on $\mathbb{P}^N$ with resolution

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(d)^t \longrightarrow G \longrightarrow 0. \tag{8.2}$$

In fact Theorem $A.1$ of [5] gives a sufficient condition for the semi-stability of syzygy bundles on $\mathbb{P}^N$ ($t \leq d(N + 1)$) and Theorem $A.2$ for the stability of syzygy bundles on $\mathbb{P}^2$. In particular he proves that a sufficient condition for the stability of a syzygy bundle with resolution (8.2) is

$$t \leq \frac{4}{5}d + 1.$$

The following improvement is a consequence of our Theorem 8.5.

Corollary 8.7 *A generic bundle G with resolution* (8.2) *on $\mathbb{P}^2$ for $d = 1, 2$ is stable if and only if $t \leq 3d$.*

References

[1] V. Ancona and G. Ottaviani, Some applications of Beĭlinson's theorem to projective spaces and quadrics, Forum Math. **3**, No. 2, 157–176 (1991).

[2] M. Atiyah, On the Krull–Schmidt theorem with application to sheaves, Bull. Soc. Math. France **84**, 307–317 (1956).

[3] M. C. Brambilla, Simplicity of generic Steiner bundles, Boll. Unione Mat. Ital. Sez. B, Artic. Ric. Mat. (8) **8**, No. 3, 723–735 (2005).

[4] M. C. Brambilla, Simplicity of vector bundles on $\mathbb{P}^n$ and exceptional bundles, Ph.D. thesis, University of Florence (2004).

[5] H. Brenner, Looking out for stable syzygy bundles, arXiv: math.AG/0311333 (2005).

[6] J.-M. Drézet, Variétés de modules alternatives, Ann. Inst. Fourier (Grenoble) **49**, No. 1, v–vi, ix, 57–139 (1999).

[7] J.-M. Drézet and J. Le Potier, Fibrés stables et fibrés exceptionnels sur $\mathbb{P}^2$, Ann. Sci. École Norm. Sup. (4) **18**, No. 2, 193–243 (1985).

[8] A. L. Gorodentsev and A. N. Rudakov, Exceptional vector bundles on projective spaces, Duke Math. J. **54**, No. 1, 115–130 (1987).

[9] A. Hirschowitz and Y. Laszlo, Fibrés génériques sur le plan projectif, Math. Ann. **297**, No. 1, 85–102 (1993).

[10] V. G. Kac, Infinite root systems, representations of graphs and invariant theory, Invent. Math. **56**, No. 1, 57–92 (1980).

[11] L. Kronecker, Algebraische Reduktion der Scharen bilinearer Formen, Sitzungsber. Akad. Berlin, 1225–1237 (1890)

[12] A. N. Rudakov, Helices and Vector Bundles, London Mathematical Society Lecture Note Series Vol. 148 (Cambridge Univ. Press, Cambridge, 1990).

[13] S. K. Zube, The stability of exceptional bundles on three-dimensional projective space, in: Helices and Vector Bundles, London Mathematical Society Lecture Note Series Vol. 148 (Cambridge Univ. Press, Cambridge, 1990), pp. 115–117.