RANK 2 STABLE SHEAVES WITH ODD DETERMINANT ON FANO THREEFOLDS OF GENUS 9.

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Abstract. A smooth prime Fano threefold X of genus 9 is associated to a surface $\mathbb{P}(\mathcal{V})$, ruled over a smooth plane quartic Γ. We consider the natural integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$.

We prove that for every $c_2 \geq 7$, the functor $\mathbf{\Phi}^!$ gives a birational map from a component of the Maruyama moduli space $\mathsf{M}(2, 1, c_2)$ of stable rank 2 sheaves F with $c_1(F) = 1$, $c_2(F) = c_2$ to a generically smooth component of the Brill-Noether locus of stable bundles $\mathcal{F}$ on Γ of rank $c_2 - 6$ and degree $c_2 - 5$, with $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \geq c_2 - 6$.

Moreover if $c_2 = 7$, we prove that the moduli space $\mathsf{M}_X(2, 1, 7)$ is isomorphic to the blowing-up of the Picard variety $\mathrm{Pic}^2(\Gamma)$ along the curve parameterizing lines contained in X.

1. Introduction

Let X be a smooth projective threefold, whose Picard group is generated by an ample divisor H_X. We consider Maruyama's coarse moduli scheme $\mathsf{M}_X(r, c_1, c_2)$ of H_X-semistable rank r sheaves F on X with $c_i(F) = c_i$.

Little is known about this space in general, but many results are available in special cases. For instance, rank 2 bundles on $\mathbb{P}^3$ have been intensively studied since [Bar77].

Since [AHDM78] and [AW77], the case which has attracted most attention is that of *instanton bundles*, i.e. stable rank 2 bundles F with $c_1(F) = 0$, $\mathrm{H}^2(\mathbb{P}^3, F(-2)) = 0$. Their moduli space is known to be smooth and irreducible for $c_2(F) \leq 5$, see [KO03], [CTT03] and references therein. The starting points in the investigation of this case are Beilinson's theorem and the notion of monad, see [BH78], [OSS80].

Now, if one desires to set up a similar analysis over a threefold X other than $\mathbb{P}^3$, one direction is to look at Fano threefolds. Recall that if the anticanonical divisor $-K_X$ is linearly equivalent to $i_X H_X$, for some positive integer i_X, then the variety X is called a *Fano threefold* of index i_X. These varieties are in fact completely classified by Iskovskih and later by Mukai, see [IP99] and references therein.

Our aim is to study the moduli space $\mathsf{M}_X(2, c_1, c_2)$ on a Fano threefold X of index $i_X = 1$. Recall that the genus of a Fano threefold X of index 1 is defined as $g = H_X^3/2 + 1$. Notice that, since the rank of a sheaf F

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in $\mathsf{M}_X(2, c_1, c_2)$ is 2, one can assume $c_1 \in \{0, 1\}$. Accordingly, we speak of bundles with odd or even determinant. If $c_1 = 1$, one sees that $\mathsf{M}_X(2, 1, c_2)$ is empty for $c_2 < m_g = \lceil g/2 + 1 \rceil$. The case of minimal $c_2 = m_g$ is well understood (see for instance [IM04a] for genus 7, [IR05] for genus 9, [Kuz96] for genus 12). For higher c_2, we are aware of the results contained in [IM07b], [AF06], [IM07a], [BF07], where only the last two papers study also the boundary of $\mathsf{M}_X(2, 1, c_2)$. Even less is known in the case of even determinant. We refer to [BF08b], where the space $\mathsf{M}_X(2, 0, 4)$ is studied for X of genus 7.

This paper, together with [BF07] and [BF08a] is devoted to the study of the space $\mathsf{M}_X(2, 1, c_2)$ for $c_2 > m_g$, with a special emphasis on $c_2 = m_g + 1$. Our main idea is to make use of Kuznetsov's semiorthogonal decomposition of the derived category of X (see [Kuz06]), to develop a suitable homological method, thus rephrasing the language of monads and Beilinson's theorem.

More precisely, in this paper we focus on Fano threefolds X of genus 9. Recall that, by a result of Mukai, [Muk88], [Muk89], the variety X is a linear section of the Lagrangian Grassmannian sixfold Σ. We consider the orthogonal plane quartic Γ, and the integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$, according to Kuznetsov's theorem, [Kuz06]. The functor is the right adjoint to the fully faithful functor $\mathbf{\Phi}$, provided by the universal sheaf $\mathscr{E}$ on $X \times \Gamma$ for the fine moduli space $\Gamma \cong \mathsf{M}_X(2, 1, 6)$. Recall that the threefold X is associated to a rank 2 stable bundle $\mathcal{V}$ on Γ, in such a way that $\mathbb{P}(\mathcal{V})$ is isomorphic to the Hilbert scheme $\mathscr{H}_2^0(X)$ of conics contained in X, see [Ili03].

For any $d \geq 7$, we proved in [BF07] that there exists a component $\mathsf{M}(d)$ of $\mathsf{M}_X(2, 1, d)$, whose general element is a vector bundle F with $\mathrm{H}^k(X, F(-1)) = 0$, for all k. Here we investigate in details the properties of $\mathsf{M}(d)$.

The main result of this paper is the following.

Theorem. *The map $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ gives:*

A) for any $d \geq 8$, a birational map of $\mathsf{M}(d)$ to a generically smooth $(2d-11)$-dimensional component of the Brill-Noether locus:

$$\{\mathcal{F} \in \mathsf{M}_\Gamma(d-6, d-5) \,|\, \mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \geq d-6\};$$

B) an isomorphism of $\mathsf{M}_X(2, 1, 7)$ with the blowing up of $\mathrm{Pic}^2(\Gamma)$ along a curve isomorphic to the Hilbert scheme $\mathscr{H}_1^0(X)$ of lines contained in X. The exceptional divisor consists of the sheaves in $\mathsf{M}_X(2, 1, 7)$ which are not globally generated.

Note that this result closely resembles that of [Dru00], [IM00] [MT01], regarding rank 2 sheaves on a smooth cubic threefold in $\mathbb{P}^4$. Their method relies on the Abel-Jacobi mapping.

The paper is organized as follows. In the following section we set up some notation. Then, in Section 3, we review the geometry of prime Fano threefolds X of genus 9, and we interpret some well-known facts concerning lines and conics contained in X in the language of vector bundles. In Section 4, we state and prove part (A) of the theorem above. Section 5 is devoted to part (B).

2. Preliminary definitions and results

Given a smooth complex projective n-dimensional polarized variety (X, H_X) and a sheaf F on X, we write $F(t)$ for $F \otimes \mathscr{O}_X(tH_X)$. Given a subscheme Z of X, we write F_Z for $F \otimes \mathscr{O}_Z$ and we denote by $\mathcal{I}_{Z,X}$ the ideal sheaf of Z in X, and by $N_{Z,X}$ its normal sheaf. We will frequently drop the second subscript. Given a pair of sheaves (F, E) on X, we will write $\mathrm{ext}^k_X(F, E)$ for the dimension of the Čech cohomology group $\mathrm{Ext}^k_X(F, E)$, and similarly $\mathrm{h}^k(X, F) = \dim \mathrm{H}^k(X, F)$. The Euler characteristic of (F, E) is defined as $\chi(F, E) = \sum_k (-1)^k \mathrm{ext}^k_X(F, E)$ and $\chi(F)$ is defined as $\chi(\mathscr{O}_X, F)$. We denote by $p(F, t)$ the Hilbert polynomial $\chi(F(t))$ of the sheaf F. The degree $\deg(L)$ of a divisor class L is defined as the degree of $L \cdot H_X^{n-1}$. The dualizing sheaf of X is denoted by ω_X.

If X is an smooth n-dimensional subvariety of $\mathbb{P}^m$, whose coordinate ring is Cohen-Macaulay, then X is said to be arithmetically Cohen-Macaulay (ACM). A locally free sheaf F on an ACM variety X is called *ACM* (*arithmetically Cohen-Macaulay*) if it has no intermediate cohomology, i.e. if $\mathrm{H}^k(X, F(t)) = 0$ for all integer t and for any $0 < k < n$. The corresponding module over the coordinate ring of X is thus a maximal Cohen-Macaulay module.

Let us now recall a few well-known facts about semistable sheaves on projective varieties. We refer to the book [HL97] for a more detailed account of these notions. We recall that a torsionfree coherent sheaf F on X is (Gieseker) *semistable* if for any coherent subsheaf E, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $p(E, t)/\mathrm{rk}(E) \leq p(F, t)/\mathrm{rk}(F)$ for $t \gg 0$. The sheaf F is called *stable* if the inequality above is always strict.

The *slope* of a sheaf F of positive rank is defined as $\mu(F) = \deg(c_1(F))/\mathrm{rk}(F)$, where $c_1(F)$ is the first Chern class of F. We recall that a torsionfree coherent sheaf F is *μ-semistable* if for any coherent subsheaf E, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $\mu(E) < \mu(F)$. The sheaf F is called *μ-stable* if the above inequality is always strict. We recall that the *discriminant* of a sheaf F is $\Delta(F) = 2rc_2(F) - (r-1)c_1(F)^2$, where the k-th Chern class $c_k(F)$ of F lies in $\mathrm{H}^{k,k}(X)$. Bogomolov's inequality, see for instance [HL97, Theorem 3.4.1], states that if F is also μ-semistable, then we have:

$$\Delta(F) \cdot H_X^{n-2} \geq 0. \tag{2.1}$$

Recall that by Maruyama's theorem, see [Mar80], if $\dim(X) = n \geq 2$ and F is a μ-semistable sheaf of rank $r < n$, then its restriction to a general hypersurface of X is still μ-semistable.

We introduce here some notation concerning moduli spaces. We denote by $\mathsf{M}_X(r, c_1, \ldots, c_n)$ the moduli space of S-equivalence classes of rank r torsionfree semistable sheaves on X with Chern classes $c_1, \ldots, c_n$. The Chern class c_k will be denoted by an integer as soon as $\mathrm{H}^{k,k}(X)$ has dimension 1. We will drop the last values of the classes c_k when they are zero. The moduli space of μ-semistable sheaves is denoted by $\mathsf{M}^{\mu}_X(r, c_1, \ldots, c_n)$.

Let us review some notation concerning the Hilbert scheme. Given a numerical polynomial $p(t)$, we let $\mathrm{Hilb}_{p(t)}(X)$ be the *Hilbert scheme* of closed subschemes of X with Hilbert polynomial $p(t)$. In case $p(t)$ has degree one, we let $\mathscr{H}_d^g(X)$ be the union of components of $\mathrm{Hilb}_{p(t)}(X)$ containing integral curves of degree d and arithmetic genus g.

As a basic technical tool, we will use the bounded derived category. Namely, given a smooth complex projective variety X, we will consider the derived category $\mathbf{D^b}(X)$ of complexes of sheaves on X with bounded coherent cohomology. For definitions and notation we refer to [GM96] and [Wei94]. In particular we write $[j]$ for the j-th shift to the right in the derived category.

We use the following terminology. Any claim referring to a *general* element in a given parameter space P, shall mean that the claim holds true for all elements of P, but for those who lie in a Zariski closed subset of P.

Let now X be a smooth projective variety of dimension 3. Recall that X is called *Fano* if its anticanonical divisor class $-K_X$ is ample. A Fano threefold X is *prime* if its Picard group is generated by the class of K_X. These varieties are classified up to deformation, see for instance [IP99, Chapter IV]. The number of deformation classes is 10, and they are characterized by the *genus*, which is the integer g such that $\deg(X) = -K_X^3 = 2\,g - 2$. Recall that the genus of a prime Fano threefold take values in $\{2, \ldots, 10, 12\}$.

If X is a prime Fano threefold of genus g, the Hilbert scheme $\mathscr{H}_1^0(Y)$ of lines contained in X is a scheme of pure dimension 1. It contains a nonreduced irreducible component if and only if the normal bundle of a general line in that component $L \subset X$ splits as $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$. The threefold X is said to be *exotic* if the Hilbert scheme $\mathscr{H}_1^0(Y)$ has a nonreduced component. It turns out that any threefold of genus 9 is not exotic, see [GLN06].

Recall also that a smooth projective surface S is a *K3 surface* if it has trivial canonical bundle and irregularity zero.

Remark that the cohomology groups $\mathrm{H}^{k,k}(X)$ of a prime Fano threefold X of genus g are generated by the divisor class H_X (for $k = 1$), the class L_X of a line contained in X (for $k = 2$), the class P_X of a closed point of X (for $k = 3$). Hence we will denote the Chern classes of a sheaf on Y by the integral multiple of the corresponding generator. Recall that $H_X^2 = (2\,g-2)L_X$. We use an analogous notation on a K3 surface S of genus g.

We recall by [HL97, Part II, Chapter 6] that, given a stable sheaf F of rank r on a K3 surface S of sectional genus g, with Chern classes c_1, c_2, the dimension at $[F]$ of the moduli space $\mathsf{M}_S(r, c_1, c_2)$ is:

$$\Delta(F) - 2\,(r^2 - 1). \tag{2.2}$$

We recall finally the formula of Hirzebruch-Riemann-Roch, in the case of prime Fano threefolds of genus 9. Let F be a rank r sheaf on a prime Fano threefold X of genus 9 with Chern classes c_1, c_2, c_3. Then we have:

$$\chi(F) = r + \frac{10}{3}\,c_1 + 4\,c_1^2 - \frac{1}{2}c_2 + \frac{8}{3}\,c_1^3 - \frac{1}{2}\,c_1\,c_2 + \frac{1}{2}\,c_3,$$
$$\chi(F, F) = r^2 - \frac{1}{2}\Delta(F).$$

3. Geometry of prime Fano 3-folds of genus 9

Throughout the paper we will denote by X a smooth prime Fano threefold of genus 9. In this section, we briefly sketch some of the basic features of

X. For detailed account on the geometry of these varieties, and the related $\mathsf{Sp}(3)$-geometry, we refer to the papers [Muk88], [Muk89], [Ili03], [IR05].

By a result of Mukai, the threefold X is isomorphic to a 3-codimensional linear section of the Lagrangian Grassmannian Σ of 3-dimensional subspaces of a 6-dimensional vector space V which are isotropic with respect to a skew-symmetric 2-form ω. The divisor class H_X embeds X in $\mathbb{P}^{10}$ as an ACM variety. It is well known that a general hyperplane section S of X is a smooth K3 surface polarized by the restriction H_S of H_X to S, with Picard number 1 and sectional genus 9.

The manifold Σ is homogeneous for the complex Lie group $\mathsf{Sp}(3)$, which acts on V preserving ω. The Lie algebra of this group has dimension 21, its Dynkin diagram is of type C_3 and the manifold Σ is $\mathsf{Sp}(3)/\mathsf{P}(\alpha_3)$. In fact, Σ is a Hermitian symmetric space. It is equipped with a universal homogeneous rank 3 subbundle $\mathcal{U}$, fitting in the universal exact sequence:

$$0 \to \mathcal{U} \to V \otimes \mathscr{O}_X \to \mathcal{U}^* \to 0. \tag{3.1}$$

Let us review the properties of the vector bundle $\mathcal{U}$. Its Chern classes satisfy $c_1(\mathcal{U}) = -1$, $c_2(\mathcal{U}) = 8$, $c_3(\mathcal{U}) = -2$. The bundle $\mathcal{U}$ is exceptional by [Kuz06]. Moreover, we have the following lemma.

Lemma 3.1. *The bundle $\mathcal{U}$ is stable and ACM. The same is true for its restriction $\mathcal{U}_S$ to a smooth hyperplane section surface S with* $\mathrm{Pic}(S) = \langle H_S \rangle$.

Proof. Consider the Koszul complex:

$$0 \to \wedge^3 B \otimes \mathscr{O}_\Sigma(-3) \to \cdots \to B \otimes \mathscr{O}_\Sigma(-1) \to \mathscr{O}_\Sigma \to \mathscr{O}_X \to 0,$$

and tensor it with $\mathcal{U}$.

By Bott's theorem we know that, for any t, the homogeneous vector bundles $\mathcal{U}(t)$ on Σ have natural cohomology. Using Riemann-Roch's formula on Σ, we get $\chi(\mathcal{U}(-t)) = 0$, for $t = 0, \ldots, 3$. We obtain:

$$\mathrm{H}^k(\Sigma, \mathcal{U}(-t)) = 0, \qquad \text{for} \qquad \begin{cases} \text{all } k \text{ and } t = 0, \ldots, 3, \\ k \neq 0 \text{ and } t < 0, \\ k \neq 6 \text{ and } t > 3. \end{cases}$$

It easily follows that $\mathcal{U}$ is ACM on X. Since $\wedge^2 \mathcal{U} \cong \mathcal{U}^*(-1)$, by Serre duality we get $\mathrm{H}^0(X, \wedge^2 \mathcal{U}) = 0$, so $\mathcal{U}$ is stable by Hoppe's criterion, see [Hop84, Lemma 2.6], or [AO94, Theorem 1.2].

To check the statement on S, consider the defining exact sequence:

$$0 \to \mathscr{O}_X(-1) \to \mathscr{O}_X \to \mathscr{O}_S \to 0. \tag{3.2}$$

Since the bundle $\mathcal{U}$ is ACM on X, tensoring (3.2) by $\mathcal{U}(-t)$, and using $\mathrm{H}^0(X, \mathcal{U}) = 0$, we get:

$$\mathrm{H}^1(S, \mathcal{U}(t)) = 0, \qquad \text{for } t \geq 0, \qquad \text{and } \mathrm{H}^0(S, \mathcal{U}) = 0.$$

Tensoring (3.2) by $\mathcal{U}^*(-t)$, since we have proved $\mathrm{H}^0(X, \mathcal{U}^*(-1)) = 0$, and since $\mathcal{U}$ is ACM on X, making use of Serre duality we obtain:

$$\mathrm{H}^1(S, \mathcal{U}(t)) = 0, \qquad \text{for } t \geq 1, \qquad \text{and } \mathrm{H}^0(S, \mathcal{U}^*(-1)) = 0.$$

This proves that the bundle $\mathcal{U}_S$ is ACM and that it is stable again by Hoppe's criterion. ∎

3.1. **Universal bundles and the decomposition of $\mathbf{D^b}(X)$.** Here we review the structure of the derived category of a smooth prime Fano threefold X of genus 9, in terms of the semiorthogonal decomposition provided by [Kuz06]. We will need to interpret this decomposition in terms of the universal vector bundle of the moduli space $\mathsf{M}_X(2,1,6)$. In view of the results of [IR05], and recalling [BF08a, Lemma 3.4], the moduli space $\mathsf{M}_X(2,1,6)$ is fine and isomorphic to a smooth plane quartic curve Γ. This curve can be obtained as an orthogonal linear section of Σ and is also called the homologically projectively dual curve to X. Let us denote by $\mathscr{E}$ the universal vector bundle on $X \times \Gamma$, and by p and q respectively the projections to X and Γ.

We have the integral functor $\mathbf{\Phi}$ associated to $\mathscr{E}$, and its right and left adjoint functors $\mathbf{\Phi}^!$ and $\mathbf{\Phi}^*$, which are defined by the formulas:

$$\mathbf{\Phi} : \mathbf{D^b}(\Gamma) \to \mathbf{D^b}(X), \qquad \mathbf{\Phi}(-) = \mathbf{R}p_*(q^*(-) \otimes \mathscr{E}), \tag{3.3}$$

$$\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^!(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(\omega_\Gamma))[1], \tag{3.4}$$

$$\mathbf{\Phi}^* : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^*(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(-H_X))[3]. \tag{3.5}$$

The topological invariants of $\mathscr{E}$ are the following:

$$c_1(\mathscr{E}) = H_X + N, \qquad c_2(\mathscr{E}) = 6L_X + H_X M + \eta,$$

where N and M are divisor classes on Γ, and η sits in $\mathrm{H}^3(X,\mathbb{C}) \otimes \mathrm{H}^1(\Gamma,\mathbb{C})$.

Lemma 3.2. *We have $\eta^2 = 6$ and $\deg(N) = 2\deg(M) - 1$.*

Proof. Recall that the bundle $\mathscr{E}$ is the universal for $\mathsf{M}_X(2,1,6)$, and write $\mathscr{E}_y$ for the bundle on X corresponding to the point $y \in \Gamma$. By [BF08a, Lemma 3.3], we have $\mathrm{Ext}^k_X(\mathscr{E}_y,\mathscr{E}_z) = 0$ if $k \geq 2$, for all $y, z \in \Gamma$. Moreover we have:

$$\hom_X(\mathscr{E}_y,\mathscr{E}_y) = \mathrm{ext}^1_X(\mathscr{E}_y,\mathscr{E}_y) = 1, \qquad \text{for all } y \in \Gamma,$$
$$\hom_X(\mathscr{E}_y,\mathscr{E}_z) = \mathrm{ext}^1_X(\mathscr{E}_y,\mathscr{E}_z) = 0, \qquad \text{for all } y \neq z \in \Gamma.$$

This gives $\mathbf{\Phi}^!(\mathscr{E}_y) \cong \mathscr{O}_y$. By [BF07, Proposition 3.4], for any $y \in \Gamma$, the bundle $\mathscr{E}_y$ satisfies:

$$\mathrm{H}^k(X,\mathscr{E}_y^*) = 0, \qquad \text{for all } k \in \mathbb{Z},$$

hence we have $\mathbf{\Phi}^!(\mathscr{O}_X) = 0$. Plugging the equations $\chi(\mathbf{\Phi}^!(\mathscr{O}_X)) = 0$ and $\chi(\mathbf{\Phi}^!(\mathscr{E}_y)) = 1$ into Grothendieck-Riemann-Roch's formula, we get our claim. □

By Kuznetsov's theorem, [Kuz06], we have the semiorthogonal decomposition:

$$\mathbf{D^b}(X) = \langle \mathscr{O}_X, \mathcal{U}^*, \mathbf{\Theta}(\mathbf{D^b}(\Gamma)) \rangle,$$

where $\mathbf{\Theta}$ is the integral functor associated to a sheaf $\mathscr{F}$ on $X \times \Gamma$, flat over Γ. We would like to see that $\mathbf{\Theta}$ actually agrees with $\mathbf{\Phi}$. We do this in a rather indirect way, in the following lemma.

Lemma 3.3. *The sheaf $\mathscr{F}$ is isomorphic to (a twist) of $\mathscr{E}$.*

Proof. It follows by [Kuz06, Appendix A] that $\mathscr{F}_y$ fits into a long exact sequence:

$$0 \to \mathscr{O}_X \to \mathcal{U}^* \to \mathscr{F}_y \to \mathscr{O}_Z \to 0,$$

where Z is the intersection of a 3-dimensional quadric contained in Σ with a codimension 2 linear section of Σ. Note that $\mathscr{F}_y$ is torsionfree of rank 2. Since X does not contain planes or 2-dimensional quadrics, Z must be a conic. Therefore, we have $c_1(\mathscr{O}_Z) = 0$, $c_2(\mathscr{O}_Z) = -2$, $c_3(\mathscr{O}_Z) = 0$. Thus we calculate $c_1(\mathscr{F}_y) = 1$, $c_2(\mathscr{F}_y) = 6$, $c_3(\mathscr{F}_y) = 0$, and we easily check that $\mathscr{F}_y$ is a stable sheaf, i.e. $\mathscr{F}_y$ sits in $\mathsf{M}_X(2,1,6)$. Note that, by [BF07, Proposition 3.4], $\mathscr{F}_y$ must be a vector bundle. Since $\mathscr{E}$ is a universal vector bundle for the fine moduli space $\Gamma = \mathsf{M}_X(2,1,6)$, we have thus that $\mathscr{F}$ is the twist by a line bundle on Γ of a pull-back of $\mathscr{E}$ via a map $f : \Gamma \to \Gamma$.

Note that if f is not constant, then it is an isomorphism and we are done. Now, in view of [Bri99], it is easy to prove that f is not constant since $\mathbf{\Theta}$ is fully faithful. Indeed, the sheaf $\mathscr{F}$ must satisfy:

$$\operatorname{Ext}^k_X(\mathscr{F}_y, \mathscr{F}_z) = 0, \qquad \text{for all } k \text{ if } y \neq z \in \Gamma.$$

But if f was constant, we would have $\hom_X(\mathscr{F}_y, \mathscr{F}_z) = 1$, for any $y, z \in \Gamma$. □

The semiorthogonal decomposition of $\mathbf{D^b}(X)$ can be thus rewritten as:

$$\mathbf{D^b}(X) = \langle \mathscr{O}_X, \mathcal{U}^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma)) \rangle. \tag{3.6}$$

Then, given a sheaf F over X, we have a functorial exact triangle:

$$\mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F)), \tag{3.7}$$

where $\mathbf{\Psi}$ is the inclusion of the subcategory $\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$ in $\mathbf{D^b}(X)$ and $\mathbf{\Psi}^*$ is the left adjoint functor to $\mathbf{\Psi}$. The k-th term of the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ can be written as follows:

$$(\mathbf{\Psi}(\mathbf{\Psi}^*(F)))^k \cong \operatorname{Ext}^{-k}_X(F, \mathscr{O}_X)^* \otimes \mathscr{O}_X \oplus \operatorname{Ext}^{1-k}_X(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^*. \tag{3.8}$$

Remark 3.4. The universal bundle $\mathscr{E}$ is determined up to twisting by the pull-back of a line bundle on Γ. In order to simplify some computations, we adopt the convention:

$$\deg(N) = \deg(\mathscr{E}_x) = 5.$$

Remark 3.5. Making use of mutations, one can easily write down the following semiorthogonal decomposition of $\mathbf{D^b}(X)$:

$$\mathbf{D^b}(X) = \langle \mathbf{\Phi_0}(\mathbf{D^b}(\Gamma)), \mathcal{U}, \mathscr{O}_X \rangle, \tag{3.9}$$

where $\mathbf{\Phi_0} : \mathbf{D^b}(\Gamma) \to \mathbf{D^b}(X)$ is defined as $\mathbf{\Phi_0} = \mathbf{R}p_*(q^*(-) \otimes \mathscr{E}(-H_X))$. Let $\mathbf{\Phi_0^*}$ be the left adjoint of the functor $\mathbf{\Phi_0}$.

Let q_1 and q_2 be the projections of $X \times X$ onto the two factors, and denote by $\mathbf{U}$ the complex on $X \times X$ defined by the natural map $\mathcal{U} \boxtimes \mathcal{U} \to \mathscr{O}_{X\times X}$, where $\mathscr{O}_{X\times X}$ has cohomological degree 0. Then the projection onto the subcategory $\langle \mathcal{U}, \mathscr{O}_X \rangle$ is given by the functor $\mathbf{R}q_{2*}(q_1^*(-) \otimes \mathbf{U})$.

Lemma 3.6. *We have the natural isomorphisms:*

$$\mathcal{H}^0(\mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*))) \cong \mathcal{U}^*,$$
$$\mathcal{H}^1(\mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*))) \cong \mathcal{U}(1).$$

Proof. Note that, for any object F of $\mathbf{D^b}(X)$, we have $\mathbf{\Phi_0^*}(F(-1)) \cong \mathbf{\Phi}^*(F)$, and for any object $\mathcal{F}$ of $\mathbf{D^b}(\Gamma)$, we have $\mathbf{\Phi}(\mathcal{F})(-1) \cong \mathbf{\Phi_0}(\mathcal{F})$. In particular, we get a natural isomorphism $\mathbf{\Phi_0}(\mathbf{\Phi_0^*}(\mathcal{U}^*(-1)))(1) \cong \mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*))$.

By the decomposition (3.9), we get a distinguished triangle:

$$\mathbf{R}q_{2*}(q_1^*(\mathcal{U}^*(-1)) \otimes \mathbf{U}) \to \mathcal{U}^*(-1) \to \mathbf{\Phi_0}(\mathbf{\Phi_0^*}(\mathcal{U}^*(-1))). \tag{3.10}$$

Since we have $\mathrm{H}^k(X,\mathcal{U}^*(-1)) = 0$ for all k, and $\mathrm{H}^k(X,\mathcal{U}^* \otimes \mathcal{U}(-1)) = 0$ for $k \neq 3$, $\mathrm{h}^3(X,\mathcal{U}^* \otimes \mathcal{U}(-1)) = 1$, the lefthandside in (3.10) is isomorphic to $\mathcal{U}[-2]$. Thus we have $\mathcal{H}^0(\mathbf{\Phi_0}(\mathbf{\Phi_0^*}(\mathcal{U}^*(-1)))) \cong \mathcal{U}^*(-1)$ and $\mathcal{H}^1(\mathbf{\Phi_0}(\mathbf{\Phi_0^*}(\mathcal{U}^*(-1)))) \cong \mathcal{U}$. This finishes the proof. □

3.2. **Lines and conics contained in X.** In this section we review some facts concerning the geometry of lines and conics contained in X. Along the way (Proposition 3.11), we reprove here a result of Iliev, see [Ili03]. We outline a different proof, since some of the arguments will be used further on. This proof is valid for all smooth prime Fano threefolds of genus 9.

Lemma 3.7. *Let C be any conic contained in X. Then we have:*

$$\mathrm{h}^0(X,\mathcal{U} \otimes \mathscr{O}_C) = 1, \qquad \mathrm{h}^1(X,\mathcal{U} \otimes \mathscr{O}_C) = 0, \tag{3.11}$$

$$\hom_X(\mathcal{U},\mathcal{I}_C) = 1, \qquad \mathrm{ext}_X^k(\mathcal{U},\mathcal{I}_C) = 0, \qquad \textit{for } k \neq 1 \tag{3.12}$$

Proof. By Riemann-Roch we have $\chi(\mathcal{U}^* \otimes \mathcal{I}_C) = 1$, and one can easily prove $\mathrm{Ext}_X^k(\mathcal{U},\mathcal{I}_C) = 0$, for $k \geq 2$. So there is at least a nonzero global section s of $\mathcal{U}^*$ which vanishes on the curve C. Note that s lifts to a section $\tilde{s}$ of $\mathcal{U}^*$ on Σ, and C is contained in the vanishing locus of $\tilde{s}$. This locus is a smooth 3-dimensional quadric $Q \subset \Sigma$.

It is easy to see that the restriction of $\mathcal{U}$ to Q splits as $\mathscr{O}_Q \oplus \mathscr{S}$, where $\mathscr{S}$ is the spinor bundle on Q. It is well-known that $\mathscr{S}$ is a stable bundle on Q with $\mathrm{rk}(\mathscr{S}) = 2$ and $c_1(\mathscr{S}) = -H_Q$. Moreover, the bundle $\mathscr{S}$ is ACM on Q. See for instance [Ott88].

The conic C is the complete intersection of two hyperplanes in Q, hence we have the Koszul complex:

$$0 \to \mathscr{O}_Q(-2\,H_Q) \to \mathscr{O}_Q(-H_Q)^2 \to \mathscr{O}_Q \to \mathscr{O}_C \to 0. \tag{3.13}$$

Tensoring (3.13) by $\mathscr{S}$, since $\mathscr{S}$ is stable and ACM on Q, we get $\mathrm{H}^k(C,\mathscr{S}) = 0$ for all k. This implies (3.11). Using (3.1), one easily gets (3.12). □

Lemma 3.8. *Let F be a sheaf in $\mathsf{M}_X(2,1,6)$, and let α be any nonzero element in $\mathrm{Hom}_X(\mathcal{U}^*,F)$. Then α gives the long exact sequence:*

$$0 \to \mathscr{O}_X \xrightarrow{\beta} \mathcal{U}^* \xrightarrow{\alpha} F \to \mathscr{O}_C \to 0, \tag{3.14}$$

for some conic C contained in X and β is a global section of $\mathcal{U}^*$.

Proof. Let I be the image of a nonzero map $\alpha : \mathcal{U}^* \to F$. Recall by Lemma 3.1 that $\mathcal{U}$ is stable. Thus, by stability of F we get $\mathrm{rk}(\ker\alpha) = 1$ and $c_1(\ker\alpha) = 0$. Since $\ker\alpha$ is reflexive, it must be invertible and we get an exact sequence of the form:

$$0 \to \mathscr{O}_X \to \mathcal{U}^* \to I \to 0. \tag{3.15}$$

Note that I is easily proved to be stable. To get (3.14), observe that the cokernel T of $I \hookrightarrow F$ satisfies $c_1(T) = 0$, $c_2(T) = -2$, $c_3(T) = 0$. Hence T agrees with $\mathscr{O}_C$, for some conic $C \subset X$, as soon as it has no isolated or embedded points. But from (3.15) we get $\mathrm{H}^1(X, I(-1)) = 0$ and, since $\mathrm{H}^0(X, F(-1)) = 0$ by stability, it follows $\mathrm{H}^0(X, T(-1)) = 0$ which implies our claim. □

Lemma 3.9. *Let F be a sheaf in $\mathsf{M}_X(2,1,6)$. Then we have:*

$$\hom_X(\mathcal{U}^*, F) = 2,$$
$$\mathrm{ext}^k_X(\mathcal{U}^*, F) = 0, \qquad \text{for all } k \geq 1. \tag{3.16}$$

Proof. Let us prove (3.16). For $k = 3$, in view of Serre duality, the vanishing of $\mathrm{Ext}^3_X(\mathcal{U}^*, F)$ is easily obtained by stability of $\mathcal{U}^*$ and F.

For $k = 2$, recall by [BF07, Proposition 3.4] that F is a globally generated vector bundle, and we have thus an exact sequence:

$$0 \to K \to \mathscr{O}_X^6 \to F \to 0. \tag{3.17}$$

We can prove, as in the proof of [BF08a, Lemma 3.3], that K is a stable vector bundle. This gives, since $\mathcal{U}^*$ is ACM:

$$\mathrm{Ext}^2_X(\mathcal{U}^*, F) \cong \mathrm{Ext}^3_X(\mathcal{U}^*, K) \cong \mathrm{Hom}_X(K, \mathcal{U}^*(-1))^* = 0,$$

where the last vanishing takes place by stability.

Let us now consider the case $k = 1$. Observe that $\mathrm{Hom}_X(\mathcal{U}^*, F) \neq 0$ since by Riemann-Roch we have $\chi(\mathcal{U}^*, F) = 2$ and we have proved (3.16) for $k = 2$. A nonzero map $\alpha : \mathcal{U}^* \to F$ must give rise to (3.14) by Lemma 3.8. Tensoring (3.14) by $\mathcal{U}$, since $\mathcal{U}$ is an exceptional ACM bundle by Lemma 3.1, we obtain (3.16) for $k = 1$, by virtue of (3.11). □

Lemma 3.10. *Let F be a sheaf in $\mathsf{M}_X(2,1,6)$. Then we have $\mathrm{Ext}^k_X(\mathcal{U}, F^*) = 0$ for all k.*

Proof. Recall that F is a globally generated ACM bundle. Clearly, we have $\mathrm{H}^0(F^* \otimes \mathcal{U}) = 0$. Now, dualize the exact sequence (3.17), and tensor it by $\mathcal{U}$. Note that $\mu(K^* \otimes \mathcal{U}) = -1/12$, so $\mathrm{H}^0(X, K^* \otimes \mathcal{U}) = \mathrm{H}^1(X, F^* \otimes \mathcal{U}) = 0$ by stability. Similarly, we obtain $\mathrm{H}^3(X, F^* \otimes \mathcal{U}) = 0$. By Riemann-Roch we compute $\chi(F^* \otimes \mathcal{U}) = 0$, so the group $\mathrm{H}^2(X, F^* \otimes \mathcal{U})$ vanishes too, and our statement is proved. □

The following result was already proved by Iliev, [Ili03].

Proposition 3.11 (Iliev)**.** *Let X be a smooth prime Fano threefold of genus 9. Then the sheaf $\mathcal{V} = q_*(p^*(\mathcal{U}) \otimes \mathscr{E})$ is a rank 2 vector bundle on Γ with $\deg(\mathcal{V}) = 1$, and we have a natural isomorphism:*

$$\mathcal{V}^* \cong \mathbf{\Phi}^*(\mathcal{U}^*). \tag{3.18}$$

The Hilbert scheme $\mathscr{H}^0_2(X)$ is isomorphic to the projective bundle $\mathbb{P}(\mathcal{V})$ over the curve Γ.

Proof. In view of Lemma 3.9, we have $\mathbf{R}^k q_*(p^*(\mathcal{U}) \otimes \mathscr{E}) = 0$, for $k \geq 1$, and $\mathcal{V}$ is a locally free sheaf on Γ of rank $\mathrm{h}^0(X, \mathcal{U} \otimes \mathscr{E}_y) = 2$.

By an instance of Grothendieck duality, see [Har66, Chapter III], given a sheaf $\mathscr{P}$ on $X \times \Gamma$, we have:

$$\mathbf{R}\mathcal{H}om_\Gamma(\mathbf{R}q_*(\mathscr{P}), \mathscr{O}_\Gamma) \cong \mathbf{R}q_*(\mathscr{O}_X(-1) \otimes \mathbf{R}\mathcal{H}om_{X\times\Gamma}(\mathscr{P}, \mathscr{O}_{X\times\Gamma}))[3], \tag{3.19}$$

and the isomorphism is functorial. Setting $\mathscr{P} = p^*(\mathcal{U}) \otimes \mathscr{E}$ in (3.19), we get (3.18).

Consider now an element ξ of the projective bundle $\mathbb{P}(\mathcal{V})$. It is uniquely represented by a pair $([\alpha], y)$, where y is a point of Γ, and $[\alpha]$ is an element of $\mathbb{P}(\mathrm{H}^0(X, \mathcal{U} \otimes \mathscr{E}_y))$. By Lemma 3.8, the morphism α gives (3.14). Applying the functor $\mathscr{H}om_X(-, \mathscr{O}_X)$ to (3.14), one can easily write down the exact sequence:

$$0 \to \mathscr{E}_y^* \xrightarrow{\alpha^\top} \mathcal{U} \xrightarrow{\beta^\top} \mathcal{I}_C \to 0. \tag{3.20}$$

Given two elements $\xi_1 = ([\alpha_1], y_1)$ and $\xi_2 = ([\alpha_2], y_2)$ we have thus two ideal sheaves $\mathcal{I}_{C_1}$ and $\mathcal{I}_{C_2}$. We want to show that if we have $\mathcal{I}_{C_1} \cong \mathcal{I}_{C_2}$, then $\xi_1 = \xi_2$. Note that an isomorphism $\gamma : \mathcal{I}_{C_1} \to \mathcal{I}_{C_2}$ lifts to a nontrivial map $\tilde{\gamma} : \mathcal{U} \to \mathcal{U}$ as soon as:

$$\mathrm{Ext}^1_X(\mathcal{U}, \mathscr{E}^*_{y_2}) = 0, \tag{3.21}$$

which in turn is given by Lemma 3.10. Thus, by the simplicity of $\mathcal{U}$, the map $\tilde{\gamma}$ must be a multiple of the identity, and we have an isomorphism $\hat{\gamma} : \mathscr{E}_{y_1} \to \mathscr{E}_{y_2}$ with $\tilde{\gamma} \circ \alpha_1 = \alpha_2 \circ \hat{\gamma}$.

Summing up, we have an injective map $\vartheta : \mathbb{P}(\mathcal{V}) \hookrightarrow \mathscr{H}_2^0(X)$. Since the variety $\mathbb{P}(\mathcal{V})$ is a projective surface and $\mathscr{H}_2^0(X)$ is an irreducible surface, we conclude that ϑ is surjective.

To prove that ϑ is a local isomorphism, we show that the tangent space $\mathrm{Ext}^1_X(\mathcal{I}_C, \mathcal{I}_C)$ is naturally identified with the tangent space to $\mathcal{T}_\xi(\mathbb{P}(\mathcal{V}))$ to $\mathbb{P}(\mathcal{V})$ at the point $\xi = \vartheta^{-1}([C])$. Applying $\mathrm{Hom}_X(-, \mathcal{I}_C)$ to (3.20), we obtain for each k an isomorphism:

$$\mathrm{Ext}^{k+1}_X(\mathcal{I}_C, \mathcal{I}_C) \cong \mathrm{Ext}^k_X(\mathscr{E}_y^*, \mathcal{I}_C).$$

Therefore, tensoring now (3.20) by $\mathscr{E}_y$ and taking global sections we get $\mathrm{Ext}^1_X(\mathcal{I}_C, \mathcal{I}_C) = 0$. We also obtain the top row in the following commutative exact diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{H}^0(X, \mathscr{E}_y^* \otimes \mathscr{E}_y) & \to & \mathrm{H}^0(X, \mathcal{U} \otimes \mathscr{E}_y) & \to & \mathrm{Ext}^1_X(\mathcal{I}_C, \mathcal{I}_C) & \to & \mathrm{H}^1(X, \mathscr{E}_y^* \otimes \mathscr{E}_y) & \to 0 \\ & \downarrow & & \downarrow & & \vdots & & \downarrow & \\ 0 \to & \mathscr{O}_{\Gamma, y} & \to & \mathcal{V}_y & \to & \mathcal{T}_\xi(\mathbb{P}(\mathcal{V})) & \to & \mathcal{T}_y(\Gamma) & \to 0. \end{array} \tag{3.22}$$

Here the bottom row is the natural exact sequence of the tangent spaces for the $\mathbb{P}^1$-bundle $\mathbb{P}(V) \to \Gamma$, and the first, second and fourth vertical maps are clearly isomorphisms. Hence we have $\mathrm{Ext}^1_X(\mathcal{I}_C, \mathcal{I}_C) \cong \mathcal{T}_\xi(\mathbb{P}(\mathcal{V}))$ and we are done. □

Lemma 3.12. *We have a natural isomorphism* $\mathbf{\Phi}^!(\mathcal{U}(1))[-1] \cong \mathbf{\Phi}^*(\mathcal{U}^*)$. *In particular, we get* $\det(\mathcal{V}^*) \cong \omega_\Gamma(-N)$, *where* $c_1(\mathscr{E}) = H_X + N$.

Proof. By Grothendieck duality (3.19), we get a natural isomorphism:

$$\mathcal{V} \cong \mathbf{\Phi}^*(\mathcal{U}^*)^* \cong \mathbf{\Phi}^!(\mathcal{U}(1)) \otimes \omega_\Gamma^*(N)[-1],$$

and since $\mathcal{V}$ has rank 2, the second statement thus follows from the first one.

In view of Proposition 3.11, the rank 2 bundle $\mathbf{\Phi}^*(\mathcal{U}^*)$ is stable. Thus we only need to show that there is a nonzero morphism from $\mathbf{\Phi}^!(\mathcal{U}(1))[-1]$ to $\mathbf{\Phi}^*(\mathcal{U}^*)$. Thus we compute:

$$\begin{aligned}\mathrm{Hom}_\Gamma(\mathbf{\Phi}^!(\mathcal{U}(1))[-1], \mathbf{\Phi}^*(\mathcal{U}^*)) \cong \mathrm{Hom}_X(\mathcal{U}(1), \mathbf{\Phi}(\mathbf{\Phi}^*(\mathcal{U}^*))[1]) \cong \\ \cong \mathrm{Hom}_X(\mathcal{U}(1), \mathcal{U}(1)),\end{aligned}$$

where the last isomorphism follows from Lemma 3.6. This concludes the proof. □

Remark 3.13. In view of the previous results, we can identify $\mathcal{V}$ with a twist of the stable rank 2 bundle of degree 3 defined by Iliev in [Ili03, Section 5]. Let $K_\Gamma = c_1(\omega_\Gamma)$ and recall that by Mukai's theorem ([Muk01]) X is isomorphic to the *type II Brill-Noether locus*:

$$\mathsf{M}_\Gamma(2, K_\Gamma, 3\mathcal{V}) = \{\mathcal{F} \in \mathsf{M}_\Gamma(2, c_1(\mathcal{V}) + K_\Gamma) \,|\, \mathrm{h}^0(\Gamma, \mathcal{F} \otimes \mathcal{V}^*) \geq 3\}.$$

Therefore, the bundle $\mathscr{E}$ is universal also for the moduli space $X \cong \mathsf{M}_\Gamma(2, K_\Gamma, 3\mathcal{V})$.

Lemma 3.14. *Let L be a line contained in X. Then we have a functorial exact sequence:*

$$0 \to \mathscr{O}_X \to A_L \otimes \mathcal{U}^* \xrightarrow{\zeta_L} \mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_L(-1))) \to \mathscr{O}_L(-1) \to 0, \tag{3.23}$$

where $A_L = \mathrm{H}^1(L, \mathcal{U}^*(-2))$ has dimension 2. Moreover, the map

$$\psi : L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L(-1))$$

gives an isomorphism of the Hilbert scheme $\mathscr{H}_1^0(X)$ onto a component W of the locus:

$$\{\mathcal{L} \in \mathrm{Pic}^2(\Gamma) \,|\, \mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{L}) \geq 2\}. \tag{3.24}$$

Proof. Recall that, for each $y \in \Gamma$, the sheaf $\mathscr{E}_y$ is a globally generated bundle with $c_1(\mathscr{E}_y) = 1$. Thus, it splits over L as $\mathscr{O}_L \oplus \mathscr{O}_L(1)$. It follows that $\mathbf{\Phi}^!(\mathscr{O}_L(-1))$ is a sheaf concentrated in degree 0, and its rank equals $\mathrm{h}^0(L, \mathscr{E}_y^*) = 1$. Its degree is computed by Grothendieck-Riemann-Roch formula.

To get (3.23), we use (3.7) and (3.8). We have thus to compute the cohomology groups:

$$\mathrm{Ext}_X^k(\mathscr{O}_L(-1), \mathscr{O}_X), \tag{3.25}$$

$$\mathrm{Ext}_X^k(\mathscr{O}_L(-1), \mathcal{U}), \tag{3.26}$$

and note that $\mathcal{U}^*$ splits over L as $\mathscr{O}_L^2 \oplus \mathscr{O}_L(1)$. So, using Serre duality, we see that both (3.25) and (3.26) vanish for $k \neq 2$, while for $k = 2$ (3.25) has dimension 1 and (3.26) has dimension 2. Setting $A_L = \mathrm{H}^1(L, \mathcal{U}^*(-2)) \cong \mathrm{Ext}_X^2(\mathscr{O}_L(-1), \mathcal{U})^*$, we obtain the functorial resolution (3.23) and $\dim(A_L) = 2$.

Set $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_L(-1))$, and recall the isomorphism (3.18). Applying the functor $\mathrm{Hom}_X(\mathcal{U}^*, -)$ to the long exact sequence (3.23), since $\mathcal{U}$ is exceptional, and both $\mathrm{Hom}_X(\mathcal{U}^*, \mathscr{O}_X)$ and $\mathrm{Hom}_X(\mathcal{U}^*, \mathscr{O}_L(-1))$ vanish, we get a natural isomorphism:

$$\mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{L}) \cong \mathrm{Hom}_X(\mathcal{U}^*, \mathbf{\Phi}(\mathcal{L})) \cong A_L.$$

Therefore, the line bundle $\mathcal{L}$ lies in the locus defined by (3.24), and actually we have $\mathrm{h}^0(\Gamma, \mathcal{V}\otimes\mathcal{L}) = 2$. Moreover, up to multiplication by a nonzero scalar, the morphism ζ_L coincides with the natural evaluation of maps from $\mathcal{U}^*$ to $\mathbf{\Phi}(\mathcal{L})$. Thus, the mapping $L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L(-1))$ is injective, since $\mathscr{O}_L(-1)$ can be recovered as $\mathrm{cok}(\zeta_L)$.

Now we shall identify the tangent space of $\mathscr{H}_1^0(X)$ at the point $[L]$ with that of the component W, at the point $[\mathcal{L}]$. Note that the morphism $\mathbf{\Phi}^*(\zeta_L)$ must agree with the natural evaluation

$$A_L \otimes \mathcal{V}^* \to \mathcal{L} \tag{3.27}$$

of maps from $\mathcal{V}^*$ to $\mathcal{L}$. Remark also that the tangent space to W at the point $[\mathcal{L}]$ is computed as the kernel of the map obtained applying the functor $\mathrm{Ext}^1_\Gamma(-,\mathcal{L})$ to (3.27).

Applying the functor $\mathrm{Hom}_X(-,\mathscr{O}_L(-1))$ to (3.23), and using the obvious vanishing $\mathrm{H}^k(L,\mathscr{O}_X(-1)) = 0$ for all k, we obtain a commutative exact diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathcal{L}),\mathscr{O}_L(-1)) & \xrightarrow{\mathrm{Ext}^1(\zeta_L,\mathscr{O}_L(-1))} & A_L^*\otimes \mathrm{Ext}^1_X(\mathcal{U}^*,\mathscr{O}_L(-1)) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{Ext}^1_\Gamma(\mathcal{L},\mathcal{L}) & \xrightarrow{\mathrm{Ext}^1(\mathrm{ev},\mathcal{L})} & A_L^*\otimes \mathrm{Ext}^1_\Gamma(\mathcal{V}^*,\mathcal{L}). \end{array} \tag{3.28}$$

Here, the kernel (respectively, the cokernel) of $\mathrm{Ext}^1(\zeta_L,\mathscr{O}_L(-1))$ is naturally identified with the tangent space $T_{[L]}\mathscr{H}_1^0(X) \cong \mathrm{Ext}^1_X(\mathscr{O}_L,\mathscr{O}_L)$, (respectively, with the obstruction space $\mathrm{Ext}^2_X(\mathscr{O}_L,\mathscr{O}_L)$). Thus, the diagram (3.28) allows to identify the tangent space (and the obstruction space) of $\mathscr{H}_1^0(X)$ at $[L]$ with those of W at $\mathcal{L}$. □

Remark 3.15. Let L be a line contained in X and set $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_L(-1))$. Note that the normal sheaf $\mathcal{N}_W$ at the point $[\mathcal{L}]$ to the subscheme W of $\mathrm{Pic}^2(\Gamma)$ is naturally identified with $A_L^*\otimes\mathrm{Ext}^1_\Gamma(\mathcal{V}^*,\mathcal{L})$, where A_L is canonically isomorphic to $\mathrm{Hom}_\Gamma(\mathcal{V}^*,\mathcal{L})$. Since $\dim(A_L) = 2$ and since $ext^1_\Gamma(\mathcal{V}^*,\mathcal{L}) = \mathrm{h}^1(L,\mathcal{U}^*(-1)) = 1$, the sheaf $\mathcal{N}_W$ is in fact locally free of rank 2, and its fibre over $[\mathcal{L}]$ can be identified (up to twist by a line bundle on W) with A_L^*.

Remark 3.16. It is well-known that, if X is general, then the scheme $\mathscr{H}_1^0(X)$ is a smooth irreducible curve, and hence so is W.

Lemma 3.17. *Let L be a line contained in X. Then we have a natural isomorphism:*

$$\mathrm{Hom}_X(\mathcal{U},\mathcal{I}_L) \cong A_L^*. \tag{3.29}$$

The set S_L of surjective morphisms $\gamma : \mathcal{U} \to \mathcal{I}_L$ is open and dense in $\mathbb{P}(A_L)$. The subscheme $\mathbb{P}(A_L)\setminus S_L$ is in natural bijection with the length 5 scheme of reducible conics $D \subset X$ which contain L. For a map γ with $[\gamma] \in \mathbb{P}(A_L)\setminus S_L$, we have $\mathrm{Im}(\gamma) = \mathcal{I}_D$.

Proof. To get the first statement, we use (3.1) and we obtain the following natural isomorphisms:

$$\begin{aligned}\mathrm{Hom}_X(\mathcal{U},\mathcal{I}_L) \cong \mathrm{H}^0(X,\mathcal{I}_L\otimes\mathcal{U}^*) \cong \mathrm{H}^1(X,\mathcal{I}_L\otimes\mathcal{U}) \cong \\ \cong \mathrm{H}^0(L,\mathcal{U}) \cong \mathrm{H}^1(L,\mathcal{U}^*(-2))^* = A_L^*.\end{aligned}$$

Let now γ be a map in $\mathrm{Hom}_X(\mathcal{U},\mathcal{I}_L)$. By stability, $\mathrm{Im}(\gamma)$ must be a subsheaf of $\mathcal{I}_L$ with trivial determinant. Thus $\ker(\gamma)$ is a reflexive sheaf of rank 2 with $c_1 = -1$, hence $c_3(\ker(\gamma)) \geq 0$. It is easy to see that $\ker(\gamma)$ is stable, so $c_2(\ker(\gamma)) \geq 6$. On the other hand, we have $c_2(\ker(\gamma)) = 8 - c_2(\mathrm{Im}(\gamma)) \leq 7$, so $c_2(\ker(\gamma))$ equals 6 or 7. If $c_2(\ker(\gamma)) = 7$ implies $c_3(\mathrm{Im}(\gamma)) \leq -1$ so γ is surjective. Then we can assume $c_2(\ker(\gamma)) = 6$ and, by [BF07, Proposition 3.4], we have that $\ker(\gamma)$ is a locally free sheaf, so $c_3(\ker(\gamma)) = 0$. This gives $c_3(\mathrm{Im}(\gamma)) = 0$, so $\mathrm{Im}(\gamma) \cong \mathcal{I}_D$, for some conic D. This proves the last statement.

Given two non proportional maps γ_1, γ_2 in $\mathrm{Hom}_X(\mathcal{U},\mathcal{I}_L)$, assuming that neither is surjective, we get $\mathrm{Im}(\gamma_1) \ncong \mathrm{Im}(\gamma_2)$ in view of the vanishing (3.21). Therefore, up to a nonzero scalar, each non surjective map γ determines uniquely a conic $D \supset L$. The converse is obvious, so it only remains to check that the subscheme of these maps has length 5. This is true if L is general, see [Isk78], so we only need to check that the length is always finite. But $\mathbb{P}(A_L)$ contains no infinite proper subschemes, so all elements γ of $\mathrm{Hom}(\mathcal{U},\mathcal{I}_L)$ should give $\mathrm{Im}(\gamma) = \mathcal{I}_D$, so $\mathrm{hom}(\mathcal{U},\mathcal{I}_D) = 2$, contradicting Lemma 3.7. □

Lemma 3.18. *Let L be a line contained in X. Then $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle of degree 1 on Γ.*

Proof. Recall that, for each $y \in \Gamma$, the sheaf $\mathscr{E}_y$ is a globally generated bundle with $c_1(\mathscr{E}_y) = 1$. Thus, it splits over L as $\mathscr{O}_L \oplus \mathscr{O}_L(1)$. It follows that $\mathbf{\Phi}^!(\mathscr{O}_L)$ is a sheaf concentrated in degree -1, and its rank equals $\mathrm{h}^0(L, \mathscr{E}_y^*) = 1$. Its degree is computed by Grothendieck-Riemann-Roch. □

4. Stable sheaves of rank 2 with odd determinant

Recall from [BF07] that, for each $c_2 \geq 7$, there exists a component $\mathsf{M}(c_2)$ of $\mathsf{M}_X(2,1,c_2)$ containing a locally free sheaf F which satisfies:

$$\mathrm{H}^1(X, F(-1)) = 0. \tag{4.1}$$

$$\mathrm{Ext}^2_X(F,F) = 0, \tag{4.2}$$

and the extra assumption $\mathrm{H}^0(X, F \otimes \mathscr{O}_L(-1)) = 0$, for some line $L \subset X$ having normal bundle $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$. For $c_2 = 6$, we have $\mathsf{M}(6) = \mathsf{M}_X(2,1,6) \cong \Gamma$. For $c_2 \geq 7$, $\mathsf{M}(c_2)$ is defined recursively as the unique component of $\mathsf{M}_X(2,1,c_2)$ which contains a sheaf F fitting into:

$$0 \to F \to G \to \mathscr{O}_L \to 0, \tag{4.3}$$

where G is a general sheaf lying in $\mathsf{M}(c_2 - 1)$. Here we are going to prove the following main result.

Theorem 4.1. *For any integer $c_2 \geq 7$, there is a birational map φ, generically defined by $F \mapsto \mathbf{\Phi}^!(F)$, from $\mathsf{M}(c_2)$ to a generically smooth $(2c_2 - 11)$-dimensional component $\mathsf{B}(c_2)$ of the locus:*

$$\{\mathcal{F} \in \mathsf{M}_\Gamma(c_2 - 6, c_2 - 5) \,|\, \mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \geq c_2 - 6\}. \tag{4.4}$$

We begin with a series of lemmas.

Lemma 4.2. *Let $c_2 \geq 7$, and let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$, satisfying (4.1). Then $\mathbf{\Phi}^!(F)$ is a vector bundle on Γ, of rank $c_2 - 6$ and degree $c_2 - 5$.*

Proof. Using stability of F and Riemann-Roch's formula we get:

$$\mathrm{H}^k(X, F(-1)) = 0, \qquad \text{for all } k. \tag{4.5}$$

By the definition (3.4) of $\mathbf{\Phi}^!$, the stalk of $\mathcal{H}^k(\mathbf{\Phi}^!(F))$ over the point $y \in \Gamma$ is given by:

$$\mathrm{H}^{k+1}(X, \mathscr{E}_y^* \otimes F) \otimes \omega_{\Gamma,y}. \tag{4.6}$$

Let us check that (4.6) vanishes for all $y \in \Gamma$ and for $k \neq 0$. For $k = -1$, the statement is clear. Indeed, by stability, any nonzero morphism $\mathscr{E}_y \to F$ would be an isomorphism for $\mathscr{E}_y$ is locally free. But $c_2(\mathscr{E}_y) \neq c_2(F)$.

To check the case $k = 1$, by Serre duality we can show $\mathrm{Ext}^1_X(F, \mathscr{E}_y) = 0$. Setting $E = \mathscr{E}_y$ in (3.17), and applying $\mathrm{Hom}_X(F, -)$, in view of (4.5) we get:

$$\mathrm{Ext}^1_k(F, \mathscr{E}_y^*) = \mathrm{Hom}_X(F, K^*) = 0,$$

where the last equality holds by stability. Finally, (4.6) holds for $k = 2$ again by stability.

We have thus proved that $\mathbf{\Phi}^!(F)$ is a vector bundle on Γ. By Riemann-Roch we compute its rank as $\mathrm{rk}(\mathbf{\Phi}^!(F)) = \chi(F \otimes \mathscr{E}_y) = c_2 - 6$. Using Grothendieck-Riemann-Roch's formula, one can easily compute the degree of $\mathbf{\Phi}^!(F)$. □

Lemma 4.3. *Let $d \geq 7$, and let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$, satisfying (4.1). Then we have a functorial resolution of the form:*

$$0 \to A_F \otimes \mathcal{U}^* \xrightarrow{\zeta_F} \mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to 0, \tag{4.7}$$

where $A_F = \mathrm{Ext}^2_X(F, \mathcal{U})^*$ has dimension $c_2 - 6$.

Proof. To write down (4.7), we use the exact triangle (3.7). We must calculate the groups $\mathrm{Ext}^k_X(F, \mathscr{O}_X)$ and $\mathrm{Ext}^k_X(F, \mathcal{U})$ for all k. We have proved that the former vanishes for all k, see (4.5).

If $k = 0, 3$, we easily get $\mathrm{Ext}^k_X(F, \mathcal{U}_+) = 0$ by stability of the sheaves $\mathcal{U}_+$ and F. Applying the functor $\mathrm{Hom}_X(F, -)$ to (3.1) we get $\mathrm{Ext}^1_X(F, \mathcal{U}) \cong \mathrm{Hom}_X(F, \mathcal{U}^*) = 0$, where the vanishing follows from the stability of F and $\mathcal{U}$. By Riemann-Roch we get $\mathrm{ext}^2_X(F, \mathcal{U}) = c_2 - 6$. □

Lemma 4.4. *Let $c_2 \geq 8$, and let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$, satisfying (4.1). Then we a natural isomorphism:*

$$A_F \cong \mathrm{Hom}_X(\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))), \tag{4.8}$$

$$\mathrm{Ext}^1_X(\mathcal{U}^*, F) \cong \mathrm{Ext}^1_X(\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))). \tag{4.9}$$

In particular, the natural map ζ_F in (4.7) is uniquely determined up to a nonzero scalar.

Proof. In view of Lemma 4.3, we have the resolution (4.7). We apply to it the functor $\mathrm{Hom}_X(\mathcal{U}^*, -)$, and we recall that $\mathcal{U}^*$ is exceptional. In fact, we are going to show:

$$\mathrm{Ext}^k_X(\mathcal{U}^*, F) = 0, \qquad \text{for } k = 0, 2, 3, \tag{4.10}$$

where the case $k = 0$ proves the lemma. By contradiction, we consider a nonzero map $\gamma : \mathcal{U}^* \to F$. By the argument of Lemma 3.8 we have $\ker(\gamma) \cong \mathscr{O}_X$, so $c_2(\mathrm{Im}(\gamma)) = 8$, which is impossible for $c_2(F) \geq 9$. For $c_2(F) = 8$, note that $c_3(\mathrm{Im}(\gamma)) = 2$ gives $c_2(\mathrm{cok}(\gamma)) = 0$, $c_3(\mathrm{cok}(\gamma)) = -2$ which is also impossible. □

Note that it is now immediate to show (4.10) also for $k = 2, 3$. Indeed, for $k \geq 2$, we have:

$$\mathrm{Ext}^k_X(\mathcal{U}^*, F) \cong \mathrm{Ext}^k_X(\mathcal{U}^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))) \cong \mathrm{Ext}^k_\Gamma(\mathbf{\Phi}^*(\mathcal{U}^*), \mathbf{\Phi}^!(F)) = 0,$$

since $\mathbf{\Phi}^*(\mathcal{U}^*)$ and $\mathbf{\Phi}^!(F)$ are sheaves on a curve.

Lemma 4.5. *Let F be a sheaf in $\mathsf{M}_X(2, 1, c_2)$ satisfying* (4.1)*, and set $\mathcal{F} = \mathbf{\Phi}^!(F)$. Then $\mathcal{F}$ satisfies $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) = c_2 - 6$. Further, if F satisfies* (4.2)*, then the natural map:*

$$\mathrm{H}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) \otimes \mathrm{H}^0(\Gamma, \mathcal{V}^* \otimes \mathcal{F} \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma, \mathcal{F}^* \otimes \mathcal{F} \otimes \omega_\Gamma) \tag{4.11}$$

is injective.

Proof. Recall the notation $A_F = \mathrm{Ext}^2_X(F, \mathcal{U})^*$. Note that, by (4.8), (4.9) and (3.18) we have natural isomorphisms:

$$\begin{aligned} A_F &\cong \mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{F}), \\ \mathrm{Ext}^1_X(\mathcal{U}^*, F) &\cong \mathrm{Ext}^1_\Gamma(\mathcal{V}^*, \mathcal{F}), \end{aligned}$$

and we have seen that A_F has dimension $c_2 - 6$.

We have thus proved the first claim, and the map $\mathbf{\Phi}^*(\zeta_F)$ must agree up to a nonzero scalar with the natural evaluation:

$$\mathrm{ev} : \mathrm{Hom}_\Gamma(\mathcal{V}^*, \mathcal{F}) \otimes \mathcal{V}^* \to \mathcal{F}. \tag{4.12}$$

We have thus a commutative exact diagram:

$$\begin{array}{ccc} \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathcal{F}), F) & \xrightarrow{\mathrm{Ext}^1_X(\zeta_L, F)} & A_F^* \otimes \mathrm{Ext}^1_X(\mathcal{U}^*, F) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{F}) & \xrightarrow{\mathrm{Ext}^1_\Gamma(\mathrm{ev}, \mathcal{F})} & A_F^* \otimes \mathrm{Ext}^1_\Gamma(\mathcal{V}^*, \mathcal{F}). \end{array} \tag{4.13}$$

Therefore, we have the natural isomorphisms:

$$\begin{aligned} \mathrm{Ext}^1_X(F, F) &\cong \ker(\mathrm{Ext}^1_X(\zeta_L, F)) \cong \ker(\mathrm{Ext}^1_\Gamma(\mathrm{ev}, \mathcal{F})), \\ \mathrm{Ext}^2_X(F, F) &\cong \mathrm{cok}(\mathrm{Ext}^1_S(\zeta_L, F)) \cong \mathrm{cok}(\mathrm{Ext}^1_\Gamma(\mathrm{ev}, \mathcal{F})). \end{aligned}$$

Thus the map $\mathrm{Ext}^1_\Gamma(\mathrm{ev}, \mathcal{F})$ is surjective as soon as F satisfies (4.2). This implies our claim, since the map (4.11) is the transpose of $\mathrm{Ext}^1_\Gamma(\mathrm{ev}, \mathcal{F})$. □

We are now in position to prove the main result of this section.

Proof of Theorem 4.1. Recall that the variety $\mathsf{M}(c_2)$ contains a vector bundle F satisfying (4.1), hence by semicontinuity Lemma 4.2 applies to an open dense subset of $\mathsf{M}(c_2)$. Thus, for any sheaf F in this open set, $\mathcal{F} = \mathbf{\Phi}^!(F)$ is a vector bundle on Γ of rank $c_2 - 6$ and degree $c_2 - 5$, and it satisfies $\mathrm{h}^0(\Gamma, \mathcal{V} \otimes \mathcal{F}) = c_2 - 6$ by by Lemma 4.5.

Let us now prove that, if F is general in $\mathsf{M}(c_2)$, then the vector bundle $\mathbf{\Phi}^!(F)$ is stable over Γ. In fact we prove that, if F is a sheaf fitting into (4.3), and G is general in $\mathsf{M}(c_2 - 1)$, then $\mathcal{F} = \mathbf{\Phi}^!(F)$ is stable over Γ. Since stability is an open property by [Mar76], this will imply that $\mathbf{\Phi}^!(F)$ is stable for F general in $\mathsf{M}(c_2)$. By induction, we may assume that $\mathbf{\Phi}^!(G)$ is a stable vector bundle, of rank $c_2 - 7$ and degree $c_2 - 6$.

Applying $\mathbf{\Phi}^!$ to (4.3), we get an exact sequence of bundles on Γ:

$$0 \to \mathbf{\Phi}^!(\mathscr{O}_L)[-1] \to \mathcal{F} \to \mathbf{\Phi}^!(G) \to 0, \tag{4.14}$$

where $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle of degree 1 by Lemma 3.18. Note that this extension must be nonsplit, for it corresponds to a nontrivial element in:

$$\mathrm{Ext}^1_\Gamma(\mathbf{\Phi}^!(G), \mathbf{\Phi}^!(\mathscr{O}_L)[-1]) \cong \mathrm{Hom}_X(G, \mathscr{O}_L).$$

Assume by contradiction that $\mathcal{F}$ is not stable, so it contains a subsheaf $\mathcal{K}$ with $\mu(\mathcal{K}) \geq \mu(\mathcal{F})$ and $\mathrm{rk}(\mathcal{K}) < \mathrm{rk}(\mathcal{F})$. The sequence (4.14) induces an exact sequence:

$$0 \to \mathcal{K}' \to \mathcal{K} \to \mathcal{K}'' \to 0,$$

with $\mathcal{K}'' \subset \mathbf{\Phi}^!(G)$ and $\mathcal{K}' \subset \mathbf{\Phi}^!(\mathscr{O}_L[-1])$. If $\mathcal{K}' = 0$, then $\mu(\mathcal{K}) = \mu(\mathcal{K}'') < \mu(\mathbf{\Phi}^!(G))$ for $\mathbf{\Phi}^!(G)$ is stable and (4.14) is nonsplit. But since $\mu(\mathbf{\Phi}^!(G)) - \mu(\mathcal{F}) = \frac{1}{(c_2-5)(c_2-6)}$, we have that $\mu(\mathcal{K})$ cannot fit in the interval $[\mu(\mathcal{F}), \mu(\mathbf{\Phi}^!(G))[$. If $\mathrm{rk}(\mathcal{K}') = 1$, one can easily apply a similar argument.

We have thus proved that an open dense subset of $\mathsf{M}(c_2)$ maps into the locus defined by (4.4). This locus is equipped with a natural structure of a subvariety of the moduli space $\mathsf{M}_\Gamma(c_2 - 6, c_2 - 5)$. Its tangent space at the point $[\mathcal{F}]$ is thus $\ker(\mathrm{Ext}^1_\Gamma(\mathrm{ev}, \mathcal{F}))$, while the obstruction sits in $\mathrm{cok}(\mathrm{Ext}^1_\Gamma(\mathrm{ev}, \mathcal{F}))$, where ev is defined by (4.12). Notice that, by Lemma 4.5, the latter space vanishes if F satisfies (4.2), so $\mathsf{B}(c_2)$ is generically isomorphic to the $(2d-11)$-dimensional variety $\mathsf{M}(c_2)$. □

5. The moduli space $\mathsf{M}_X(2,1,7)$ as a blowing up of $\mathrm{Pic}^2(\Gamma)$

In this section, we set up a more detailed study of the moduli space $\mathsf{M}_X(2,1,7)$. This space can be analyzed under no generality assumptions. In fact, the map φ sends the space $\mathsf{M}_X(2,1,7)$ to the abelian variety $\mathrm{Pic}^2(\Gamma)$, In turn, $\mathrm{Pic}^2(\Gamma)$ contains a copy of the Hilbert scheme $\mathscr{H}^0_1(X)$, via the map ψ (see Lemma 3.14), as a subvariety of codimension 2. The relation between these varieties is given by the main result of this section.

Theorem 5.1. *The mapping $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ gives an isomorphism of the moduli space $\mathsf{M}_X(2,1,7)$ to the blowing up of $\mathrm{Pic}^2(\Gamma)$ along the subvariety $W = \psi(\mathscr{H}^0_1(X))$. The exceptional divisor consists of the sheaves in $\mathsf{M}_X(2,1,7)$ which are not globally generated.*

We will need some lemmas.

Lemma 5.2. *Let F be a sheaf in $\mathsf{M}_X(2,1,7)$. Then, we have:*

$$\mathrm{H}^k(X, F(-1)) = \mathrm{H}^k(X, F) = 0, \qquad \text{for } k = 1, 2. \tag{5.1}$$

Moreover, either F is a locally free, or there exists an exact sequence:

$$0 \to F \to E \to \mathscr{O}_L \to 0, \tag{5.2}$$

where E is a bundle in $\mathsf{M}_X(2,1,6)$ and L is a line contained in Y.

Furthermore, the following statements are equivalent:

i) the sheaf F is not globally generated;
ii) the group $\mathrm{Hom}_X(\mathcal{U}^*, F)$ is nontrivial;
iii) there exists a line $L \subset X$, a sheaf I in $\mathsf{M}_X(2,1,8,2)$ and two exact sequence:

$$0 \to \mathscr{O}_X \to \mathcal{U}^* \to I \to 0, \tag{5.3}$$

$$0 \to I \to F \to \mathscr{O}_L(-1) \to 0. \tag{5.4}$$

Proof. The first two statements are taken from [BF07, Proposition 3.5]. Clearly condition (iii) implies both conditions (ii) and (i).

Let us prove (ii) $\Rightarrow$ (iii). Consider a nonzero map $\gamma : \mathcal{U}^* \to F$. The argument of Lemma 3.8 implies $\ker \gamma \cong \mathscr{O}_X$ and the cokernel T of γ has $c_1(T) = 0$, $c_2(T) = -1$, $c_3(T) = -1$, so $T \cong \mathscr{O}_L(-1)$, for some line $L \subset X$, if T is supported on a Cohen-Macaulay curve. On the other hand, from (5.3) we get $\mathrm{H}^1(X, I(-1)) = 0$, so by $\mathrm{H}^0(X, F(-1)) = 0$, we have $\mathrm{H}^0(X, T(-1)) = 0$, and we are done.

It remains to show (i) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (5.4)**:** Assume that F is not globally generated, that is the evaluation map $\mathrm{ev} : \mathrm{H}^0(X, F) \otimes \mathscr{O}_X \to F$ is not surjective. Set $K = \ker(\mathrm{ev})$, $I = \mathrm{Im}(\mathrm{ev})$ and $T = \mathrm{cok}(\mathrm{ev})$. Now it is enough to prove the following facts:

$$c_2(T) = -1, \qquad c_3(T) = -1, \tag{5.5}$$

(5.6) the sheaf T has no isolated or embedded points.

The stability of F easily implies $\mathrm{rk}(I) = 2$ and $c_1(I) = 1$. Since T is a torsion sheaf with $c_1(T) = 0$, we have $c_2(T) = -\ell \leq 0$. Looking at the sheaf K, we see that it is reflexive of rank 3 with:

$$c_1(K) = -1, \quad c_2(K) = 9 - \ell, \quad c_3(K) = c_3(T) - 2 + \ell.$$

Thus, we are now reduced to prove $c_3(K) = -2$ and $\ell = 1$. By Riemann-Roch, we compute $\chi(K) = \frac{1}{2} c_3(K) + 1$. By definition of the evaluation map ev, taking global sections of the composition:

$$\mathrm{H}^0(X, F) \otimes \mathscr{O}_X \twoheadrightarrow I \hookrightarrow F,$$

we obtain an isomorphism. This implies:

$$\mathrm{H}^0(X, K) = \mathrm{H}^1(X, K) = 0, \tag{5.7}$$

$$\mathrm{H}^0(X, T) \cong \mathrm{H}^1(X, I) \cong \mathrm{H}^2(X, K), \tag{5.8}$$

and one can easily see $\mathrm{H}^3(X, K) = 0$.

We postpone the proof of the following claim, and we assume it for the time being.

Claim 5.3. *We have* $c_2(K) \in \{8, 9\}$ *and* $\mathrm{H}^2(X, K) = 0$.

Note that the second statement of the above claim proves that $\mathrm{H}^k(X, K) = 0$ for all k. Hence we have $\chi(K) = 0$, which implies $c_3(K) = -2$. Then, by the first statement of Claim 5.3, we obtain $\ell = 1$, for otherwise T would be zero. This proves (5.5). Note that (5.6) follows from the vanishing of (5.8). This finishes the proof.

(i) $\Rightarrow$ (5.3)**:** Note that $\chi(\mathcal{U}^*, K) = -1$. Since $\mathrm{Ext}^3_X(\mathcal{U}^*, K) = 0$ by stability, we get $\mathrm{Ext}^1_X(\mathcal{U}^*, K) \neq 0$. Applying the functor $\mathrm{Hom}(\mathcal{U}^*, -)$ to the sequence:

$$0 \to K \to \mathrm{H}^0(X, F) \otimes \mathscr{O}_X \to I \to 0,$$

one easily obtains that the group $\mathrm{Hom}_X(\mathcal{U}^*, I) \cong \mathrm{Ext}^1_X(\mathcal{U}^*, K)$ contains a nontrivial morphism α. Composing α with the injection of I in F, we see that α is in fact surjective, so we get (5.3).

□

Proof of Claim 5.3. We observe that the restriction of K to a general hyperplane section S is stable, using Hoppe's criterion. Indeed, we have $\mathrm{H}^0(S, K) = 0$ by (5.7), while the group $\mathrm{H}^0(S, \wedge^2 K)$ vanishes since it is a subgroup of $\mathrm{H}^0(S, K) \otimes \mathrm{H}^0(S, F) = 0$. Then from (2.2) it follows that $c_2(K) \geq 8$. This proves the first assertion.

Let us now show the second one. Tensoring (3.2) by $K(1)$, we are reduced to show the vanishing of the groups $\mathrm{H}^2(X, K(1))$ and $\mathrm{H}^1(S, K(1))$.

Looking at the first one, assume by contradiction that there exists a nontrivial extension of the form:

$$0 \to \mathscr{O}_X(-1) \to \tilde{K} \to K(1) \to 0,$$

where $\tilde{K}$ is a rank 4 vector bundle with $c_1(\tilde{K}) = 1$ and $c_2(\tilde{K}) < 0$. Then $\tilde{K}$ is not semistable by Bogomolov's inequality (2.1). By considering the possible values of the slope of a destabilizing subsheaf of $\tilde{K}$, one sees that Harder-Narasimhan filtration has the form $0 \subset K_1 \subset \tilde{K}$ and $Q = \tilde{K}/K_1$ is semistable, and $\mu(K_1)$ can be either $\frac{1}{2}$ or $\frac{1}{3}$. Then by Bogomolov's inequality we have $c_2(K_1) \geq 0$. In any case $c_1(Q) = 0$, so $c_2(Q) \geq 0$. This contradicts $c_2(\tilde{K}) < 0$.

Let us now turn to the group $\mathrm{H}^1(S, K(1))$, and observe that it is dual to $\mathrm{Ext}^1_S(K_S(1), \mathscr{O}_S)$. Assuming it to be nontrivial, we get a nonsplit exact sequence on S of the form:

$$0 \to \mathscr{O}_S \to \widetilde{K_S} \to K_S(1) \to 0, \tag{5.9}$$

where $\widetilde{K_S}$ is a rank 4 vector bundle on S with $c_1(\widetilde{K_S}) = 2$ and $c_2(\widetilde{K_S}) \leq 25$. Then $\widetilde{K_S}$ is not stable by (2.2). This time one can check that the only possible destabilizing subsheaf K_1 must have slope $\frac{1}{2}$. The same happens to $Q = \widetilde{K_S}/K_1$. By semistability of K_1 and Q one has

$$c_2(\widetilde{K_S}) = c_2(K_1) + c_2(Q) + 16 \geq 28,$$

a contradiction. □

The following lemma is now straightforward.

Lemma 5.4. *The map $\varphi : F \to \mathbf{\Phi}^!(F)$ sends $\mathsf{M}_X(2,1,7)$ to $\mathrm{Pic}^2(\Gamma)$. If the sheaf F is globally generated, then φ is a local isomorphism around F.*

Proof. Set $\mathcal{F} = \mathbf{\Phi}^!(F)$. In view of (5.1) and Lemma 4.2, the map φ takes values in $\mathrm{Pic}^2(\Gamma)$. Assume now F globally generated. By Lemma 5.2 we have $\mathrm{Hom}_X(\mathcal{U}^*, F) = 0$, so by applying the functor $\mathrm{Hom}_X(\mathcal{U}^*, -)$ to the resolution (4.7) we get (4.8), from which it follows that φ is injective at F.

Recall that $\mathrm{Ext}^k_X(\mathcal{U}^*, F) = 0$ for $k = 2, 3$, and by Riemann-Roch we have $\chi(\mathcal{U}^*, F) = 0$. Thus we must also have:

$$\mathrm{Ext}^1_X(\mathcal{U}^*, F) = 0,$$

and, by the infinitesimal analysis of Lemma 4.5, the differential of φ at $[F]$ induces an isomorphism:

$$\mathrm{Ext}^1_X(F, F) \cong \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathcal{F}), F) \cong \mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{F}) \cong \mathrm{H}^1(\Gamma, \mathscr{O}_\Gamma).$$

□

Recall that we denote by A_L the 2 dimensional vector space $\mathrm{Hom}_X(\mathcal{U}, \mathcal{I}_L)^*$.

Lemma 5.5. *Let L be a line contained in X. Then there is a natural injective map $\theta : \mathbb{P}(A_L) \mapsto \mathsf{M}_X(2,1,7)$ such that any sheaf F in the image of θ sits into* (5.4)*, for some sheaf I sitting in* (5.3)*.*

Proof. Let us define the map $\theta : \mathbb{P}(A_L) \to \mathsf{M}_X(2,1,7)$. In view of lemma 3.17, for any element $[\gamma] \in \mathbb{P}(A_L)$, we have two alternatives:

i) the map γ is surjective;
ii) the image of the map γ is isomorphic to $\mathcal{I}_C$, for some reducible conic $C \subset X$ which is the union of L and another line $L' \subset X$.

If (i) takes place, we define $\theta([\gamma])$ as the dual of $\ker(\gamma)$. Note that this correspondence is one to one. Indeed, assuming $\theta([\gamma_1]) = \theta([\gamma_2])$, we would have $G_1 = \ker(\gamma_1) \cong G_2 = \ker(\gamma_2)$. But the isomorphism $G_1 \cong G_2$ would then lift to an isomorphism $\mathcal{U} \to \mathcal{U}$, for $\mathrm{Ext}^1_X(\mathcal{I}_L, \mathcal{U}) = 0$. Since both $\mathcal{U}$ and $\mathcal{I}_L$ are simple, this would then mean that γ_1 is a multiple of γ_2.

Assume now that (ii) takes place. We have thus an exact sequence of the form (3.20), with $\beta^\top = \gamma$. Since C contains L, we have:

$$0 \to \mathscr{O}_L(-1) \to \mathscr{O}_C \to \mathscr{O}_{L'} \to 0, \tag{5.10}$$

for some line $L' \subset X$. Dualizing (3.20) one obtains (3.14). We define thus a surjective map as the composition $\ker(\gamma)^* \twoheadrightarrow \mathscr{O}_C \twoheadrightarrow \mathscr{O}_{L'}$, and we let $\theta([\gamma])$ be the kernel of this map.

To prove that θ is injective also in this case, observe that the map γ is determined up to a nonzero scalar by $\ker(\gamma)$ and by L', since $\mathrm{Ext}^1_X(\mathcal{I}_C, \mathcal{U}) = 0$. On the other hand, there is a unique (up to a scalar) surjection $\ker(\gamma)^* \to \mathscr{O}_{L'}$, so θ is injective.

Finally, it is clear by the definition that in both cases (i) and (ii), the sheaf defined by $\theta([\gamma])$ sits into (5.4). □

Lemma 5.6. *Let G be a sheaf in $\mathsf{M}_X(2,1,7)$, and assume that G is not globally generated. Then the set of sheaves F in $\mathsf{M}_X(2,1,7)$ such that $\varphi(F) = \varphi(G)$ is identified with $\theta(\mathbb{P}(A_L))$, for some line $L \subset X$.*

The subscheme of those sheaves F which are not locally free, and satisfy $\varphi(F) = \varphi(G)$, has length 5.

Proof. In view of Lemma 5.2, there exists a line $L \subset X$ such that G is not globally generated over L, i.e. we have the exact sequence (5.4), with F replaced by G. Applying the functor $\mathbf{\Phi}^!$ to this exact sequence, we get:

$$\varphi(G) = \mathbf{\Phi}^!(G) \cong \mathbf{\Phi}^!(\mathscr{O}_L(-1)) = \psi([L]). \tag{5.11}$$

Since φ is a local isomorphism on the set of globally generated sheaves, any sheaf F with $\varphi(F) = \varphi(G)$ must not be globally generated. Dualizing (5.4) and (5.3) we obtain $F^* \cong I^*$ and:

$$0 \to F^* \to \mathcal{U} \xrightarrow{\delta} \mathscr{O}_X \to \mathscr{E}xt^1_X(I, \mathscr{O}_X) \to 0$$

$$0 \to \mathscr{E}xt^1_X(F, \mathscr{O}_X) \to \mathscr{E}xt^1_X(I, \mathscr{O}_X) \to \mathscr{O}_L \to 0. \tag{5.12}$$

We have here the following two alternatives.

a) the sheaf F is locally free, and $\mathrm{Im}(\delta) \cong \mathcal{I}_L$;
b) we have $F/F^{**} \cong \mathscr{O}_{L'}$ for some line $L' \subset X$, and by (5.2) this implies:

$$F^*(1) \in \mathsf{M}_X(2,1,6), \qquad \mathrm{Im}(\delta) \cong \mathcal{I}_C,$$

for some reducible conic $C \subset X$, and (5.12) becomes of the form (5.10).

We let γ be the restriction of δ to its image $\mathcal{I}_L$. Clearly, if (a) takes place, then F is isomorphic to $\theta([\gamma])$, and γ is as in case (i) of Lemma 5.5.

Similarly, if (b) takes place, then γ is as in case (ii) of Lemma 5.5, and F is isomorphic to $\theta([\gamma])$. The set of sheaves F which are not locally free and with $\varphi(F) = \varphi(G)$ is thus in natural bijection with the set of elements $[\gamma]$ in $\mathbb{P}(A_L)$ such that γ is not surjective. By Lemma 3.17, this is identified with the set of reducible conics which contain L, which has length 5. □

We are now in position to prove our main result.

Proof of Theorem 5.1. We have seen in Lemma 5.4 that φ is a local isomorphism along the open set of globally generated sheaves.

On the other hand, the map φ equips the subscheme of sheaves which are not globally generated with a structure of $\mathbb{P}^1$ bundle over $\psi(\mathscr{H}^0_1(X))$. Indeed, if a sheaf G is not globally generated, by (5.11), $\varphi(G)$ lies in W. Moreover by Lemmas 5.5 and 5.6, if $\varphi(G) = \psi([L])$, then $\varphi(\theta(\mathbb{P}(A_L))) = \psi([L])$.

Thus, it only remains to provide a natural identification of the fibre of $\varphi(G)$ with the projectivized normal bundle of $\psi([L])$ in $\mathrm{Pic}^2(\Gamma)$. By Remark 3.15 and Lemma 3.14, the latter is functorially identified with $\mathbb{P}(A_L)$. On the other hand, by Lemmas 5.5 and 5.6, via the map θ the former is also naturally identified with the projective line $\mathbb{P}(A_L)$. This concludes the proof. □

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