VECTOR BUNDLES ON FANO THREEFOLDS OF GENUS 7 AND BRILL-NOETHER LOCI

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ABSTRACT. Given a smooth prime Fano threefold X of genus 7 we consider its homologically projectively dual curve Γ and the natural integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$.

We prove that, for $d \geq 6$, $\mathbf{\Phi}^!$ gives a birational map from a component of the moduli scheme $\mathsf{M}_X(2,1,d)$ of rank 2 stable sheaves on X with $c_1 = 1$, $c_2 = d$ to a generically smooth $(2\,d-9)$-dimensional component of the Brill-Noether variety $W^{2\,d-11}_{d-5,5\,d-24}$ of stable vector bundles on Γ of rank $d-5$ and degree $5\,d-24$ with at least $2\,d-10$ sections.

This map turns out to be an isomorphism for $d=6$, and the moduli space $\mathsf{M}_X(2,1,6)$ is fine. For general X, this moduli space is a smooth irreducible threefold.

1. Introduction

Let X be a smooth complex projective variety of dimension 3, with Picard number one, and assume that the anticanonical divisor K_X is ample. Then X is called a Fano threefold, and one defines the index i_X of X as the greatest integer i such that $-K_X/i$ lies in $\mathrm{Pic}(X)$. We are interested in the Maruyama moduli scheme $\mathsf{M}_X(2,c_1,c_2)$ of semistable sheaves of rank 2 and Chern classes c_1, c_2, and with $c_3 = 0$, defined on a Fano threefold X.

The maximum value of i_X is 4, and in this case X must be isomorphic to $\mathbb{P}^3$. The study of the moduli space $\mathsf{M}_{\mathbb{P}^3}(2,c_1,c_2)$ was pioneered by Barth in [Bar77], and pursued later by several authors. Roughly speaking, the main questions concern rationality, irreducibility and smoothness of these moduli spaces; many of them are still open. Among the main tools to study the problem, we recall monads and Beilinson's theorem, see [BH78], [Bei78] and [OSS80].

The next case is $i_X = 3$. Then X has to be isomorphic to a quadric hypersurface. This case was considered by Ein and Sols ([ES84]) and later by Ottaviani and Szurek, see [OS94].

In the case $i_X = 2$, there are 5 deformation classes of Fano threefolds as it results from Iskovskikh's classification, see [IP99]. Perhaps the most studied among them is the cubic hypersurface V_3 in $\mathbb{P}^4$. The geometry of these threefolds is deeply linked to the properties of the families of curves they contain. A cornerstone in this sense is the paper [CG72] of Clemens and Griffiths on V_3. For a survey of results about moduli spaces of vector bundles on V_3 we refer to [Bea02]. In particular we mention [Dru00], [MT01] and [BMR94].

In the case $i_X = 1$, we say that X is a *prime* Fano threefold. Then, one defines the genus of X as the integer $g = -K_X^3/2 + 1$. The genus satisfies $2 \leq g \leq 12$,

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2000 *Mathematics Subject Classification.* Primary 14J60. Secondary 14H30, 14F05, 14D20.

Key words and phrases. Fano threefolds of genus 7. Moduli space of vector bundles. Brill-Noether theory for vector bundles on curves. Semiorthogonal decomposition.

Both authors were partially supported by Italian MIUR funds. The second author was partially supported by Groupement de Recherche Européen Italo Français en Géométrie Algébrique - GRIFGA.

$g \neq 11$, and there are 10 deformation classes of prime Fano threefolds. Their birational geometry has been extensively studied as well, see [IP99]. The geometry of the moduli spaces of rank 2 vector bundles on X has been more recently investigated by several authors, for instance in the papers [IM00b] (for genus 3), [IM04a], [IM07c] (for genus 7), [IM07], [IM00a] (for genus 8), [IR05] (for genus 9), [AF06] (for genus 12). Among the main tools we mention the Abel-Jacobi map and Serre's correspondence between rank 2 vector bundles and curves contained in X.

The purpose of the present paper is to investigate the properties of the moduli spaces of rank 2 bundles on a smooth prime Fano threefold X, making use of homological methods. We first observe that (under a mild generality assumption on X), given any integer $d \geq g/2 + 1$, the moduli space $\mathsf{M}_X(2,1,d)$ contains a generically smooth component $\mathsf{M}(d)$ of dimension $2\,d - g - 2$, such that its general element F is a stable locally free sheaf with $\mathrm{H}^1(X, F(-1)) = 0$, see Theorem 3.12.

Then, we focus on genus 7, where an analogue of Beilinson's theorem is provided by the semiorthogonal decomposition of the bounded derived category $\mathbf{D^b}(X)$ obtained by Kuznetsov in [Kuz05]. We use this decomposition to study the component $\mathsf{M}(d)$. More precisely, we consider the homologically projectively dual curve Γ in the sense of [Kuz06], and the corresponding integral functor $\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma)$. Making use of the canonical resolution of a general element of $\mathsf{M}(d)$, we show that $\mathbf{\Phi}^!$ gives a birational map φ from $\mathsf{M}(d)$ to a component of $W^{2\,d-11}_{d-5,5\,d-24}$ (Theorem 5.8), where we denote by $W^s_{r,c}$ the Brill-Noether variety of stable vector bundles on Γ of rank r and degree c with at least $s+1$ independent global sections.

We prove that the map φ is in fact an isomorphism in the case $d = 6$. In particular the moduli space $\mathsf{M}_X(2,1,6)$ is fine and isomorphic to a connected threefold (Theorem 5.10, part A). If X is general enough, the moduli space $\mathsf{M}_X(2,1,6)$ is actually smooth and irreducible (5.10, part B). We also exhibit an involution of $\mathsf{M}_X(2,1,6)$ which interchanges the set of sheaves which are not globally generated with the one of those which are not locally free. Finally we show that, if S is a general hyperplane section surface, the space $\mathsf{M}_X(2,1,6)$ embeds as a Lagrangian subvariety of $\mathsf{M}_S(2,1,6)$ with respect to the Mukai form, away from finitely many double points (Theorem 5.18).

The paper is organized as follows. In Section 2 we review the geometry of Fano threefolds X of genus 7 and the structure of their derived category. In Section 3 we construct (under mild generality assumptions) a generically smooth component $\mathsf{M}(d)$ of $\mathsf{M}_Y(2,1,d)$, over a smooth prime Fano threefold Y, and we recall some basic facts concerning bundles with minimal c_2. In Section 4, we prove that the functor $\mathbf{\Phi}^!$ provides an isomorphism between the Hilbert scheme $\mathscr{H}^0_3(X)$ and the symmetric cube $\Gamma^{(3)}$, see Theorem 4.4). Section 5 contains our main results.

Acknowledgments. We are grateful to Kieran O'Grady for pointing out the paper by Tyurin, and to Dimitri Markushevich for helping us in making more precise the generality assumption in Theorem 5.10. We would like to thank the referee for several helpful comments, in particular for suggesting Corollary 2.9 and a shorter proof of Lemma 5.15.

2. Preliminaries

Let us introduce some basic material. The main notions we will need concern moduli spaces of semistable sheaves, smooth Fano threefolds, Brill-Noether varieties and a bit of homological algebra.

2.1. **Notation and preliminary results.** Given a smooth complex projective n-dimensional polarized variety (X, H_X) and a sheaf F on X, we write $F(t)$ for $F \otimes \mathscr{O}_X(tH_X)$. Given a subscheme Z of X, we write F_Z for $F \otimes \mathscr{O}_Z$ and we denote by $\mathcal{I}_{Z,X}$ the ideal sheaf of Z in X, and by $N_{Z,X}$ its normal sheaf. We will frequently drop the second subscript. Given a pair of sheaves (F, E) on X, we will write $\mathrm{ext}^k_X(F, E)$ for the dimension of the group $\mathrm{Ext}^k_X(F, E)$, and similarly $\mathrm{h}^k(X, F) = \dim \mathrm{H}^k(X, F)$. The Euler characteristic of (F, E) is defined as $\chi(F, E) = \sum_k (-1)^k \mathrm{ext}^k_X(F, E)$ and $\chi(F)$ is defined as $\chi(\mathscr{O}_X, F)$. We denote by $p(F, t)$ the Hilbert polynomial $\chi(F(t))$ of the sheaf F. The dualizing sheaf of X is denoted by ω_X. We define also the natural evaluation:

$$e_{E,F} : \mathrm{Hom}_Y(E, F) \otimes E \to F.$$

If X is a smooth projective subvariety of $\mathbb{P}^m$, we say that X is *ACM* (for *arithmetically Cohen-Macaulay*) if its coordinate ring is Cohen-Macaulay. A locally free sheaf F on an ACM variety X of dimension $n \geq 1$ is called *ACM* if it has no intermediate cohomology, i.e. if $\mathrm{H}^k(X, F(t)) = 0$ for all integer t and for any $0 < k < n$. This is equivalent to $\oplus_{t\in\mathbb{Z}} \mathrm{H}^0(X, F(t))$ being a Cohen-Macaulay module over the coordinate ring of X.

2.1.1. *Semistable sheaves and their moduli spaces.* We refer to the book [HL97] for a detailed account of all the notions introduced here. We recall that a torsion-free coherent sheaf F on X is (Gieseker) *semistable* if for any coherent subsheaf E, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $p(E, t)/\mathrm{rk}(E) \leq p(F, t)/\mathrm{rk}(F)$ for $t \gg 0$. The sheaf F is called *stable* if the inequality above is always strict.

The *slope* of a sheaf F of positive rank is defined as $\mu(F) = \deg(c_1(F) \cdot H_X^{n-1})/\mathrm{rk}(F)$, where $c_1(F)$ is the first Chern class of F. We recall that a torsion-free coherent sheaf F is *μ-semistable* if for any coherent subsheaf E, with $0 < \mathrm{rk}(E) < \mathrm{rk}(F)$, one has $\mu(E) \leq \mu(F)$. The sheaf F is called *μ-stable* if the above inequality is always strict. We recall that the *discriminant* of a sheaf F is $\Delta(F) = 2rc_2(F) - (r-1)c_1(F)^2$, where the k-th Chern class $c_k(F)$ of F lies in $\mathrm{H}^{k,k}(X)$. Bogomolov's inequality, see for instance [HL97, Theorem 3.4.1], states that if F is μ-semistable, then we have:

$$\Delta(F) \cdot H_X^{n-2} \geq 0. \tag{2.1}$$

Recall that by Maruyama's theorem, see [Mar80], if $\dim(X) = n \geq 2$ and F is a μ-semistable sheaf of rank $r < n$, then its restriction to a general hypersurface of X is still μ-semistable.

We introduce here some notation concerning moduli spaces. We denote by $\mathsf{M}_X(r, c_1, \ldots, c_n)$ the moduli space of S-equivalence classes of rank r torsion-free semistable sheaves on X with Chern classes $c_1, \ldots, c_n$. The Chern class c_k will be denoted by an integer as soon as $\mathrm{H}^{k,k}(X)$ has dimension 1. We will drop the last values of the classes c_k when they are zero. We denote by $\mathsf{M^s}_X(r, c_1, \ldots, c_n)$ the subset of stable sheaves of $\mathsf{M}_X(r, c_1, \ldots, c_n)$. The point of $\mathsf{M^s}_X(r, c_1, \ldots, c_n)$ represented by a sheaf F will be denoted by $[F]$ or sometimes simply by F.

We denote by $\mathscr{H}^g_d(X)$ the union of components of the *Hilbert scheme* of closed subschemes Z of X with Hilbert polynomial $p(\mathscr{O}_Z, t) = dt+1-g$, containing integral curves of degree d and arithmetic genus g.

We use the following terminology: any claim referring to a *general* element in a given parameter space P, will mean that the claim holds for all elements of P, except possibly for those who lie in a union of Zariski closed subsets of P.

2.1.2. *Homological algebra.* As a basic tool, we will use the bounded derived category of coherent sheaves. Namely, given a smooth complex projective variety X, we will consider the derived category $\mathbf{D^b}(X)$ of complexes of sheaves on X with bounded coherent cohomology. For definitions and notation we refer to [GM96] and [Wei94]. In particular we write $[j]$ for the j-th shift to the right in the derived category.

Let Z be a local complete intersection subvariety of a smooth projective variety X. In view of the Fundamental Local Isomorphism (see [Har66, Proposition III.7.2]), we have the natural isomorphisms:

$$\mathscr{E}xt^k_X(\mathscr{O}_Z, \mathscr{O}_Z) \cong \mathscr{E}xt^{k-1}_X(\mathcal{I}_Z, \mathscr{O}_Z) \cong \bigwedge^k N_Z, \tag{2.2}$$

$$\mathscr{T}or^X_k(\mathscr{O}_Z, \mathscr{O}_Z) \cong \mathscr{T}or^X_{k-1}(\mathcal{I}_Z, \mathscr{O}_Z) \cong \bigwedge^k N^*_Z. \tag{2.3}$$

We will also use the following spectral sequences:

$$E_2^{p,q} = \mathrm{Ext}^p_X(\mathcal{H}^{-q}(a), A) \Longrightarrow \mathrm{Ext}^{p+q}_X(a, A), \tag{2.4}$$

$$E_2^{p,q} = \mathrm{Ext}^p_X(B, \mathcal{H}^q(b)) \Longrightarrow \mathrm{Ext}^{p+q}_X(B, b), \tag{2.5}$$

$$E_2^{p,q} = \mathrm{H}^p(X, \mathscr{E}xt^q_X(A, B)) \Longrightarrow \mathrm{Ext}^{p+q}_X(A, B). \tag{2.6}$$

where a, b are complexes of sheaves on X, and A, B are sheaves on X. Recall that the maps in the E_2 term of these spectral sequences are differentials:

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}.$$

2.1.3. *Brill-Noether loci for vector bundles on a smooth projective curve.* We recall here some basic results in Brill-Noether theory, for definitions and notations we refer for instance to [TiB91]. Let Γ be a smooth projective curve of genus g. The Brill-Noether locus $W^s_{r,c} \subset \mathsf{M^s}_\Gamma(r, c)$ is defined to be the subvariety consisting of rank r stable bundles of degree c on Γ having at least $s + 1$ independent global sections. The expected dimension of this variety is:

$$\rho(r, c, s) = r^2(g - 1) - (s + 1)\,(s + 1 - c + r(g - 1)) + 1.$$

Consider a stable rank r vector bundle $\mathcal{F}$ on Γ, with $\deg(\mathcal{F}) = c$ and $\mathrm{h}^0(\Gamma, \mathcal{F}) = s + 1$. We define the Gieseker-Petri map as the natural linear application:

$$\pi_{\mathcal{F}} : \mathrm{H}^0(\Gamma, \mathcal{F}) \otimes \mathrm{H}^0(\Gamma, \mathcal{F}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma, \mathcal{F} \otimes \mathcal{F}^* \otimes \omega_\Gamma). \tag{2.7}$$

The map $\pi_{\mathcal{F}}$ is injective if and only if $[\mathcal{F}]$ is a non-singular point of a component of $W^s_{r,d}$ of dimension $\rho(r, d, s)$. We will use more frequently in the sequel the transpose of the Petri map which we write in the form:

$$\pi^\top_{\mathcal{F}} : \mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{F}) \to \mathrm{H}^0(\Gamma, \mathcal{F})^* \otimes \mathrm{H}^1(\Gamma, \mathcal{F}) \tag{2.8}$$

In fact the tangent space to $W^s_{r,d}$ at the point $[\mathcal{F}]$ can be interpreted as the kernel of $\pi^\top_{\mathcal{F}}$, while the space of obstructions at $[\mathcal{F}]$ is identified with the cokernel of $\pi^\top_{\mathcal{F}}$.

2.1.4. *Smooth prime Fano threefolds.* Let now X be a smooth projective variety of dimension 3. Recall that X is called *Fano* if its anticanonical divisor class $-K_X$ is ample. A Fano threefold X is said to be *prime* if its Picard group is generated by the class of K_X. These varieties are classified up to deformation, see for instance [IP99, Chapter IV]. The number of deformation classes is 10, and they are characterized by the *genus*, which is the integer g such that $\deg(X) = -K_X^3 = 2\,g - 2$. Recall that the genus of a prime Fano threefold take values in $\{2, \ldots, 10, 12\}$. If $-K_X$ is very ample, we say that X is *non-hyperelliptic*. In this case we have $g \geq 3$.

If X is a prime Fano threefold of genus g, the Hilbert scheme $\mathscr{H}^0_1(X)$ of lines contained in X is a scheme of dimension 1. It is known by [Isk78] that the normal bundle of a line $L \subset X$ splits either as $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$ or as $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-2)$. The Hilbert scheme $\mathscr{H}^0_1(X)$ contains a component which is non-reduced at any point if and only if the normal bundle of a general line L in that component splits as $\mathscr{O}_L(1) \oplus \mathscr{O}_L(-2)$. In this case, the threefold X is said to be *exotic* (see [Pro90]). On the other hand, we say that X is *ordinary* if it contains a line L with normal bundle $\mathscr{O}_L \oplus \mathscr{O}_L(-1)$, equivalently if $\mathscr{H}^0_1(X)$ has a generically smooth component.

Recall that, if X is general enough, $\mathscr{H}_1^0(X)$ is in fact a smooth irreducible curve see [IP99, Theorem 4.2.7], and references therein.

Let us remark that a non-hyperelliptic prime Fano threefold X is exotic if and only if it contains infinitely many non-reduced conics (see [BF08]). For $g \geq 9$, the results of [GLN06] and [Pro90] imply that X is non-exotic unless $g = 12$ and X is the Mukai-Umemura threefold, see [MU83]. In fact, the only other known examples of exotic prime Fano threefolds besides Mukai-Umemura's case are those containing a cone. For instance if X is the Fermat quartic threefold in $\mathbb{P}^4$ ($g = 3$), then $\mathscr{H}_1^0(X)$ is a curve with 40 irreducible components, each of multiplicity 2 (see [Ten74]). We do not know if there exist exotic prime Fano threefolds of genus 7. In view of a result of Iliev-Markushevich (restated in Lemma 4.1 further on), this amounts to ask whether there are non-tetragonal smooth curves Γ of genus 7 admitting infinitely many divisors $\mathcal{L}$ of type g^1_5 such that $K_\Gamma - 2\mathcal{L}$ is effective (see Remark 4.2).

Remark that the cohomology groups $\mathrm{H}^{k,k}(X)$ of a prime Fano threefold X of genus g are generated by the divisor class H_X (for $k = 1$), the class L_X of a line contained in X (for $k = 2$), the class P_X of a closed point of X (for $k = 3$). Hence we will denote the Chern classes of a sheaf on X by the integral multiple of the corresponding generator. Recall that $H_X^2 = (2\,g - 2)L_X$.

Applying the theorem of Riemann-Roch to a sheaf F on X, of (generic) rank r and with Chern classes c_1, c_2, c_3, we obtain the following formulas:

$$\chi(F) = r + \frac{11+g}{6}\,c_1 + \frac{g-1}{2}\,c_1^2 - \frac{1}{2}c_2 + \frac{g-1}{3}\,c_1^3 - \frac{1}{2}\,c_1\,c_2 + \frac{1}{2}\,c_3, \tag{2.9}$$

$$\chi(F,F) = r^2 - \frac{1}{2}\Delta(F),$$

and, in case $r = 2$ and $g = 7$, formula (2.9) becomes:

$$\chi(F) = 2 + 3\,c_1 + 3\,c_1^2 - \frac{1}{2}c_2 + 2\,c_1^3 - \frac{1}{2}\,c_1\,c_2 + \frac{1}{2}\,c_3. \tag{2.10}$$

Recall also that a smooth projective surface S is a *K3 surface* if it has trivial canonical bundle and irregularity zero. Note that a general hyperplane section of a smooth non-hyperelliptic prime Fano threefold of genus g is a K3 surface S whose Picard group is generated by the restriction H_S of H_X to S, and whose (sectional) genus equals g. We consider stability with respect to H_S. Given a stable sheaf F of rank r on a K3 surface S with Chern classes c_1, c_2, the dimension at $[F]$ of the moduli space $\mathsf{M}_S(r, c_1, c_2)$ is:

$$\Delta(F) - 2\,(r^2 - 1). \tag{2.11}$$

For this equality we refer for instance to [HL97, Part II, Chapter 6].

Remark 2.1. Assume that X is a smooth prime Fano threefold, and let L be a line contained in X, with $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$. Then we have:

$$\mathrm{ext}^1_X(\mathscr{O}_L, \mathscr{O}_L) = 1, \quad \mathrm{ext}^2_X(\mathscr{O}_L, \mathscr{O}_L) = 0.$$

One can easily check this statement, using (2.2) and (2.6).

Remark 2.2. Let X be a smooth prime Fano threefold, and L a line contained in X. Then by the well-known Hartshorne-Serre correspondence (for instance, by an adaptation of [Har80, Theorem 4.1] to our setup) we can associate to L a rank 2 vector bundle F_L, with $c_1(F_L) = -1$ and $c_2(F_L) = 1$ (see also [Mad02]). Moreover we have the following exact sequence:

$$0 \to \mathscr{O}_X \to F_L \to \mathcal{I}_L(-1) \to 0. \tag{2.12}$$

2.2. **Geometry of Fano threefolds of genus 7.** We recall here the construction of a Fano threefold of genus 7 as a section of the spinor 10-fold, outlined by Mukai in [Muk88], [Muk89]. See also [Muk92], [Muk95a], [IM04a].

Let V be a 10-dimensional $\mathbb{C}$-vector space, equipped with a non-degenerate quadratic form. The algebraic group $\mathsf{Spin}(V)$ corresponds to a Dynkin diagram of type D_5. It admits two 16-dimensional irreducible representations S^+ and S^-, called the half-spin representations, having maximal weight respectively $\lambda_+ = \lambda_4$ and $\lambda_- = \lambda_5$. These representations are naturally dual to each other.

The corresponding roots $\alpha_+ = \alpha_4$ and $\alpha_- = \alpha_5$ give rise to the Hermitian symmetric spaces Σ^+ and Σ^-, defined by $\Sigma^\pm = \mathsf{Spin}(10)/\mathsf{P}(\alpha_\pm)$. These can be seen as the components of the orthogonal Grassmann variety $\mathbb{G}_Q(\mathbb{P}^4, \mathbb{P}(V))$ of 4-dimensional isotropic linear subspaces $\mathbb{P}^4$ contained in the smooth quadric hypersurface Q in $\mathbb{P}^9 = \mathbb{P}(V)$ corresponding to the quadratic form on V. We denote by $\mathcal{U}_\pm$ the restriction of the tautological subbundle on $\mathbb{G}_Q(\mathbb{P}^4, \mathbb{P}(V))$ to $\Sigma^\pm$. We have thus the universal exact sequence:

$$0 \to \mathcal{U}_\pm \to V \otimes \mathscr{O}_{\Sigma^\pm} \to \mathcal{U}_\pm^* \to 0. \tag{2.13}$$

The hyperplane divisors $H_{\Sigma^\pm}$ provide natural equivariant embeddings of $\Sigma^\pm$ into $\mathbb{P}(\mathsf{S}^\pm)$. Given a subvariety $Y \subset \Sigma^\pm$, we denote by H_Y the restriction of $H_{\Sigma^\pm}$ to Y.

Now choose a 9-dimensional vector subspace A of S^+, and consider its (7-dimensional) orthogonal space $B = A^\perp \subset \mathsf{S}^-$ under the duality $(\mathsf{S}^+)^* \cong \mathsf{S}^-$. We define:

$$X = \Sigma^+ \cap \mathbb{P}(A) \subset \mathbb{P}(\mathsf{S}^+), \tag{2.14}$$

$$\Gamma = \Sigma^- \cap \mathbb{P}(B) \subset \mathbb{P}(\mathsf{S}^-). \tag{2.15}$$

If the subspace A is general enough, then X is a smooth prime Fano threefold of genus 7, and any such threefold is obtained in this way. In particular we have $K_X = -H_X$, $H_X^3 = 12$. In turn, the curve Γ is a smooth canonical curve of genus 7, called the *homologically projective dual curve* of X. By [Muk95a, Table 1], we know that the curve Γ is not trigonal nor tetragonal and $W^2_{1,6}$ is empty. Moreover, a general curve of genus 7 is of this kind.

2.2.1. *Semiorthogonal decomposition of the derived category of X.* Here we briefly sketch the construction due to Kuznetsov [Kuz05], of the semiorthogonal decomposition of $\mathbf{D^b}(X)$. We consider the product variety $X \times \Gamma$, together with the two projections $p\colon X \times \Gamma \to X$, $q\colon X \times \Gamma \to \Gamma$.

The symmetric form on V provides the following natural exact sequence on $X \times \Gamma \subset \Sigma^+ \times \Sigma^-$:

$$0 \to \mathscr{E}^* \to \mathcal{U}_- \to \mathcal{U}_+^* \xrightarrow{\alpha} \mathscr{E} \to 0 \tag{2.16}$$

(here $\mathcal{U}_\pm$ denotes also the pull-back of $\mathcal{U}_\pm$ to $X \times \Gamma$). It turns out that $\mathscr{E}$ is a locally free sheaf on $X \times \Gamma$ with the following invariants:

$$c_1(\mathscr{E}) = H_X + H_\Gamma, \tag{2.17}$$

$$c_2(\mathscr{E}) = \frac{7}{12} H_X\, H_\Gamma + 5\, L + \eta, \tag{2.18}$$

where η sits in $\mathrm{H}^3(X, \mathbb{C}) \otimes \mathrm{H}^1(\Gamma, \mathbb{C})$ and satisfies $\eta^2 = 14$.

In view of the results of [Muk01], [Muk95b], [Kuz05] and [IM04a], the vector bundle $\mathscr{E}$ is a universal object for moduli functors on X and Γ in the sense specified as follows.

Theorem 2.3 (Mukai, Iliev-Markushevich, Kuznetsov)**.** *Let $A \subset \mathsf{S}^+$ be chosen so that X defined by* (2.14) *is a smooth threefold of Picard number 1, and define Γ and $\mathscr{E}$ as in* (2.15), (2.16)*. Then:*

i) the curve Γ is isomorphic to $\mathsf{M}_X(2,1,5)$,
ii) the manifold X is isomorphic to the Brill-Noether locus of stable bundles $\mathcal{E}$ on Γ with $\mathrm{rk}(\mathcal{E}) = 2$, $\det(\mathcal{E}) \cong H_\Gamma$, $\mathrm{h}^0(\Gamma, \mathcal{E}) = 5$,
iii) the bundle $\mathscr{E}$ universally represents both moduli problems (i) *and* (ii)*,*
iv) for all $y \in \Gamma$, the sheaf $\mathscr{E}_y$ is a globally generated ACM vector bundle.

Given points $x \in X$, $y \in \Gamma$, and given a vector bundle $\mathscr{F}$ on $X \times \Gamma$, we denote by $\mathscr{F}_y$ (resp. $\mathscr{F}_x$) the bundle over X (resp. over Γ) obtained restricting $\mathscr{F}$ to $X \times \{y\}$ (resp. to $\{x\} \times \Gamma$). We still denote by $\mathcal{U}_+$ (resp. $\mathcal{U}_-$) the restriction of $\mathcal{U}_\pm$ to X (resp. to Γ). The vector bundles $\mathcal{U}_+$ and $\mathcal{U}_-$ have rank 5. We have $c_1(\mathcal{U}_-) = -2H_\Gamma$ and:

$$c_1(\mathcal{U}_+) = -2, \qquad c_2(\mathcal{U}_+) = 24, \qquad c_3(\mathcal{U}_+) = -14.$$

We define the following exact functors:

$$\mathbf{\Phi} : \mathbf{D^b}(\Gamma) \to \mathbf{D^b}(X), \qquad \mathbf{\Phi}(-) = \mathbf{R}p_*(q^*(-) \otimes \mathscr{E}), \tag{2.19}$$

$$\mathbf{\Phi}^! : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^!(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(H_\Gamma))[1], \tag{2.20}$$

$$\mathbf{\Phi}^* : \mathbf{D^b}(X) \to \mathbf{D^b}(\Gamma), \qquad \mathbf{\Phi}^*(-) = \mathbf{R}q_*(p^*(-) \otimes \mathscr{E}^*(-H_X))[3]. \tag{2.21}$$

We recall that $\mathbf{\Phi}$ is fully faithful, $\mathbf{\Phi}^*$ is left adjoint to $\mathbf{\Phi}$, and $\mathbf{\Phi}^!$ is right adjoint to $\mathbf{\Phi}$. The main result of [Kuz05] provides the following semiorthogonal decomposition:

$$\mathbf{D^b}(X) \cong \langle \mathscr{O}_X, \mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{D^b}(\Gamma))\rangle. \tag{2.22}$$

This decomposition will be used to write a canonical resolution of a given sheaf over X. In view of [Gor90], given a sheaf F over X, the decomposition (2.22) provides a functorial exact triangle:

$$\mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F)), \tag{2.23}$$

where $\mathbf{\Psi}$ is the inclusion of the subcategory $\langle \mathscr{O}_X, \mathcal{U}_+^*\rangle$ in $\mathbf{D^b}(X)$ and $\mathbf{\Psi}^*$ is the left adjoint functor to $\mathbf{\Psi}$. The k-th term of the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ can be written as follows:

$$(\mathbf{\Psi}(\mathbf{\Psi}^*(F)))^k \cong \mathrm{Ext}_X^{-k}(F, \mathscr{O}_X)^* \otimes \mathscr{O}_X \oplus \mathrm{Ext}_X^{1-k}(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^*. \tag{2.24}$$

Remark 2.4. Given a sheaf F on X, one can describe more explicitly the map $F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F))$. We do this here, in order to show that the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ is *minimal*, i.e. the only non-zero maps in the complex are from copies of $\mathscr{O}_X$ to copies of $\mathcal{U}_+^*$. In other words, for any k, the differential d^k from the $(k-1)$-st term to the k-th term is strictly upper triangular.

We consider the product $X \times X$ and the projections q_1 and q_2 onto the two factors. One has a natural map $\epsilon : \mathcal{U}_+ \boxtimes \mathcal{U}_+ \to \mathscr{O}_{X\times X}$ with $\mathrm{cok}(\epsilon) \cong \mathscr{O}_\Delta$, obtained restricting the standard resolution of the diagonal on the Grassmannian, see [Kap88]. We denote by $\mathbf{U}$ the complex on $X \times X$ given as $\mathscr{O}_{X\times X}[3] \to \mathcal{U}_+^* \boxtimes \mathcal{U}_+^*[3]$, where the differential is the transpose of ϵ and $\mathscr{O}_{X\times X}$ sits in degree -3. So, there is a natural map $\mathscr{O}_\Delta(1) \to \mathbf{U}$. Then, given a sheaf F on X, the complex $\mathbf{\Psi}(\mathbf{\Psi}^*(F))$ is given by $\mathbf{R}q_{2*}(q_1^*(F(-1)) \otimes \mathbf{U})$, and the map $F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F))$ is induced by $\mathscr{O}_\Delta(1) \to \mathbf{U}$. Finally, we recall the vanishing $\mathrm{Ext}_X^k(\mathcal{U}_+^*, \mathscr{O}_X) = \mathrm{Ext}_X^k(\mathscr{O}_X, \mathcal{U}_+^*) = 0$ for any $k > 0$.

Having this in mind, one can easily prove the minimality statement, indeed [Kap88, Lemma 1.6] applies, and we can use [AO89, Lemma 3.2] to deduce that the differentials between the graded pieces of $\mathbf{R}q_{2*}(q_1^*(F(-1)) \otimes \mathcal{U}_+^* \boxtimes \mathcal{U}_+^*)$ are zero, as well as differentials between the graded pieces of $\mathbf{R}q_{2*}(q_1^*(F(-1)))$.

Remark 2.5. Given an object F of $\mathbf{D^b}(X)$, we have an exact triangle:

$$F \to \mathbf{\Psi}(\mathbf{\Psi}^*(F)) \to \mathbf{\Phi}(\mathbf{\Phi}^!(F))[1],$$

so we may think of $\mathbf{\Phi}(\mathbf{\Phi}^!(F))[1]$ as the *right mutation functor* $R_{\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle}$ with respect to the subcategory $\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$ of $\mathbf{D^b}(X)$ see [Gor90]. Note that is in fact an autoequivalence of ${}^{\perp}\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$.

2.2.2. *Some lemmas on universal bundles.* We close this section with some lemmas regarding the image via the integral functors defined above of some natural sheaves on X and Γ. These results will be needed further on. From the sequence (2.16) we obtain:

$$0 \to \mathscr{E}^* \to \mathcal{U}_- \to \mathscr{G} \to 0, \tag{2.25}$$

$$0 \to \mathscr{G} \to \mathcal{U}_+^* \to \mathscr{E} \to 0, \tag{2.26}$$

where $\mathscr{G}$ is a rank 3 vector bundle with $c_1(\mathscr{G}) = H_X - H_\Gamma$.

Lemma 2.6. *The vector bundles $\mathcal{U}_+$ and $\mathscr{G}_y$, for any $y \in \Gamma$, are stable and ACM. Moreover, we have $\mathrm{H}^0(\Gamma, \mathscr{G}_x) = 0$ for any $x \in X$.*

Proof. Let us prove first that $\mathrm{H}^0(\Gamma, \mathscr{G}_x) = 0$ for any $x \in X$. Notice that $\mathscr{G}_x \cong \wedge^2 \mathscr{G}_x^*(-1)$, because $\mathscr{G}_x$ has rank 3 and $c_1(\mathscr{G}_x) \cong -H_\Gamma$. Let us dualize (2.25) and restrict it to $\{x\} \times \Gamma$. We obtain an inclusion:

$$\wedge^2 \mathscr{G}_x^*(-1) \hookrightarrow \wedge^2 \mathcal{U}_-^*(-1).$$

Then we have $\mathrm{H}^0(\Gamma, \mathscr{G}_x) \subset \mathrm{H}^0(\Gamma, \wedge^2 \mathcal{U}_-^*(-1))$. So it suffices to show that the latter space is 0. To prove this, one can tensor by $\wedge^2 \mathcal{U}_-^*(-1)$ the Koszul complex:

$$0 \to \wedge^9 A \otimes \mathscr{O}_{\Sigma_-}(-9) \to \cdots \to A \otimes \mathscr{O}_{\Sigma_-}(-1) \to \mathscr{O}_{\Sigma_-} \to \mathscr{O}_\Gamma \to 0,$$

and the conclusion follows applying Bott's theorem on Σ_- to the homogeneous vector bundles $\wedge^2 \mathcal{U}_-^*(-t)$, for $t = 1, \ldots, 10$.

Let us now turn to $\mathcal{U}_+$. Consider the Koszul complex:

$$0 \to \wedge^7 B \otimes \mathscr{O}_{\Sigma_+}(-7) \to \cdots \to B \otimes \mathscr{O}_{\Sigma_+}(-1) \to \mathscr{O}_{\Sigma_+} \to \mathscr{O}_X \to 0,$$

and tensor it with $\mathcal{U}_+$. Applying Bott's theorem on Σ_+ we obtain that, for any t, the homogeneous vector bundles $\mathcal{U}_+(t)$ on Σ have natural cohomology and more precisely we get:

$$\mathrm{H}^k(\Sigma, \mathcal{U}_+(-t)) = 0, \qquad \text{for} \qquad \begin{cases} \text{all } k \text{ and } t = 0, \ldots, 7, \\ k \neq 0 \text{ and } t < 0, \\ k \neq 10 \text{ and } t > 7. \end{cases}$$

Then it easily follows that $\mathcal{U}_+$ is an ACM bundle on X and

$$\mathrm{H}^k(X, \mathcal{U}_+) = 0, \quad \text{for all } k.$$

Applying the same argument to $\wedge^2 \mathcal{U}_+$, we obtain the following:

$$\mathrm{H}^k(X, \wedge^2 \mathcal{U}_+) = 0, \quad \text{for } k \neq 1, \text{ and} \quad \mathrm{h}^1(X, \wedge^2 \mathcal{U}_+) = 1.$$

In particular, Serre duality implies:

$$\mathrm{H}^0(X, \wedge^4 \mathcal{U}_+(1)) = 0, \qquad \mathrm{H}^0(X, \wedge^3 \mathcal{U}_+(1)) = 0. \tag{2.27}$$

By Hoppe's criterion, see [AO94, Theorem 1.2] and [Hop84, Lemma 2.6], this proves stability of $\mathcal{U}_+$.

Recall that the dual of an ACM vector bundle is also ACM. Therefore, the dual bundles of $\mathcal{U}_+$ and $\mathscr{E}_y$ are ACM by (iv) of Theorem 2.3. This easily implies, by (2.25) and (2.26), that the bundle $\mathscr{G}_y$ is ACM. To prove that $\mathscr{G}_y$ is stable, by Hoppe's criterion it is enough to show that the groups $\mathrm{H}^0(X, \mathscr{G}_y^*)$, $\mathrm{H}^0(X, \mathscr{G}_y(-1))$ both vanish. We consider the restriction to $X \times \{y\}$ of (2.26). Since $\mathcal{U}_+^*(-1) \cong \wedge^4 \mathcal{U}_+(1)$, we obtain the latter vanishing by (2.27). Dualizing the same exact sequence and using $\mathrm{H}^1(X, \mathscr{E}_y^*) = 0$, we get the former. $\square$

Lemma 2.7. *Given an object $\mathcal{F}$ in $\mathbf{D^b}(\Gamma)$ and an object F in $\mathbf{D^b}(X)$ we have the following functorial isomorphisms:*

$$\mathbf{R}\mathscr{H}om_X(\mathbf{\Phi}(\mathcal{F}), \mathscr{O}_X) \cong \mathbf{\Phi}(\mathbf{R}\mathscr{H}om_\Gamma(\mathcal{F}, \mathscr{O}_\Gamma)) \otimes \mathscr{O}_X(-1)[1], \tag{2.28}$$

$$\mathbf{R}\mathscr{H}om_\Gamma(\mathbf{\Phi}^!(F), \mathscr{O}_\Gamma) \cong \mathbf{\Phi}^!(\mathbf{R}\mathscr{H}om_X(F, \mathscr{O}_X)) \otimes \omega_\Gamma^*[1]. \tag{2.29}$$

Proof. By Grothendieck duality, (see [Har66, Chapter III], or [Con00]), given a complex $\mathscr{K}$ on $X \times \Gamma$, we have:

$$\mathbf{R}\mathscr{H}om_X(\mathbf{R}p_*(\mathscr{K}), \mathscr{O}_X) \cong \mathbf{R}p_*(\omega_\Gamma \otimes \mathbf{R}\mathscr{H}om_{X\times\Gamma}(\mathscr{K}, \mathscr{O}_{X\times\Gamma}))[1], \tag{2.30}$$

$$\mathbf{R}\mathscr{H}om_\Gamma(\mathbf{R}q_*(\mathscr{K}), \mathscr{O}_\Gamma) \cong \mathbf{R}q_*(\omega_X \otimes \mathbf{R}\mathscr{H}om_{X\times\Gamma}(\mathscr{K}, \mathscr{O}_{X\times\Gamma}))[3], \tag{2.31}$$

and the isomorphisms are functorial. Recall that $\omega_X \cong \mathscr{O}_X(-1)$ and $\omega_\Gamma \cong \mathscr{O}_\Gamma(H_\Gamma)$. So by (2.17) we have $\mathscr{E}^* \otimes \omega_\Gamma \cong \mathscr{E} \otimes \mathscr{O}_X(-1)$. Then, setting $\mathscr{K} = q^*(\mathcal{F}) \otimes \mathscr{E}$ in (2.30), we get (2.28). Setting $\mathscr{K} = p^*(F) \otimes \mathscr{E}^* \otimes \omega_\Gamma[1]$ in (2.31), we obtain (2.29). $\square$

Lemma 2.8. *The following relations hold on Γ, for each point $y \in \Gamma$:*

$$\mathbf{\Phi}^*(\mathscr{O}_X) \cong \mathcal{U}_-, \qquad \mathbf{\Phi}^*(\mathcal{U}_+^*) \cong \mathscr{O}_\Gamma, \qquad \mathbf{\Phi}^*(\mathscr{E}_y) \cong \mathscr{O}_y, \tag{2.32}$$

$$\mathbf{\Phi}^!(\mathscr{O}_X) = 0, \qquad \mathbf{\Phi}^!(\mathcal{U}_+^*) = 0, \qquad \mathbf{\Phi}^!(\mathscr{E}_y) \cong \mathscr{O}_y, \tag{2.33}$$

and on X:

$$\mathcal{H}^0(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \cong \mathcal{U}_+^*, \qquad \mathcal{H}^1(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \cong \mathcal{U}_+(1), \tag{2.34}$$
$$\mathcal{H}^k(\mathbf{\Phi}(\mathscr{O}_\Gamma)) = 0, \qquad \textit{for } k \neq 0, 1,$$

$$\mathbf{\Phi}(\mathscr{O}_y) \cong \mathscr{E}_y. \tag{2.35}$$

Proof. The isomorphism (2.35) follows immediately from the definition of $\mathbf{\Phi}$. Since the functor $\mathbf{\Phi}$ is fully faithful we easily obtain also the relations $\mathbf{\Phi}^*(\mathscr{E}_y) \cong \mathbf{\Phi}^!(\mathscr{E}_y) \cong \mathscr{O}_y$. It is clear that $\mathbf{\Phi}^!(\mathscr{O}_X) = \mathbf{\Phi}^!(\mathcal{U}_+^*) = 0$.

The isomorphism $\mathbf{\Phi}^*(\mathcal{U}_+^*) \cong \mathscr{O}_\Gamma$ is proved in [Kuz05, Lemma 5.6]. Twisting (2.25) by $\mathscr{O}_{X\times\Gamma}(-H_X)$ and taking $\mathbf{R}q_*$, we get $\mathbf{\Phi}^*(\mathscr{O}_X) \cong \mathcal{U}_-$. Indeed, we have $\mathrm{H}^k(X, \mathscr{G}_y(-H_X)) = 0$ for any integer k, since the vanishing for $k = 1, 2$ follows from the fact that $\mathscr{G}_y$ is ACM (by Lemma 2.6), and the vanishing for $k = 0, 3$ follows from the fact that $\mathscr{G}_y$ is stable (again by Lemma 2.6).

Given $x \in X$, we restrict (2.26) to $\{x\} \times \Gamma$ and taking global sections we get $(\mathcal{U}_+^*)_x \subset \mathrm{H}^0(\Gamma, \mathscr{E}_x)$, by Lemma 2.6. Notice that $\dim(\mathcal{U}_+^*)_x = 5$ and by Brill-Noether theory we know that the bundle $\mathscr{E}_x$ cannot have more than 5 sections, see [BF98]. Hence $\mathcal{U}_{+,x}^* \cong \mathrm{H}^0(\Gamma, \mathscr{E}_x)$ for all $x \in X$. We obtain the isomorphism $\mathcal{H}^0(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \cong \mathcal{U}_+^*$. By (2.28) we get:

$$\mathcal{H}^1(\mathbf{\Phi}(\mathscr{O}_\Gamma))^* \cong \mathcal{H}^0(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \otimes \mathscr{O}_X(-1).$$

This gives the isomorphism $\mathcal{H}^1(\mathbf{\Phi}(\mathscr{O}_\Gamma)) \cong \mathcal{U}_+(1)$. $\square$

The following corollary of Lemma 2.7 has been pointed out by the referee. We set τ for the functor $\mathcal{F} \mapsto \mathbf{R}\mathscr{H}om_\Gamma(\mathcal{F}, \omega_\Gamma)$ defined on $\mathbf{D^b}(\Gamma)$.

Corollary 2.9. *Set T for the functor $F \mapsto \mathbf{\Phi}(\mathbf{\Phi}^!(\mathbf{R}\mathscr{H}om_X(F, \mathscr{O}_X)))[1]$. Then T is an autoequivalence of ${}^\perp\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$. Moreover, we have $\mathbf{\Phi}^! \circ T = \tau \circ \mathbf{\Phi}^!$.*

Proof. It is clear that the image of T is contained in the image $\mathbf{\Phi}$ of the subcategory ${}^\perp\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$. Moreover, we have $T(F) = 0$, for any object F in $\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle$, since $T(\mathscr{O}_X) = T(\mathcal{U}_+^*) = 0$. Using Lemmas 2.7 and 2.8. it is easy to show that $T(\mathscr{E}_y) \cong \mathscr{E}_y[-1]$ for all $y \in \Gamma$. Note that ${}^\perp\langle \mathscr{O}_X, \mathcal{U}_+^* \rangle = \mathbf{\Phi}(\mathbf{D^b}(\Gamma))$, and $\mathbf{\Phi}(\mathscr{O}_y) \cong \mathscr{E}_y$.

Therefore, the natural isomorphism $T(\mathscr{E}_y) \cong \mathscr{E}_y[-1]$ proves that T is an autoequivalence of $\mathbf{\Phi}(\mathbf{D^b}(\Gamma))$.

To prove $\mathbf{\Phi}^! \circ T = \tau \circ \mathbf{\Phi}^!$, it suffices to use (2.29). $\square$

3. Rank 2 stable sheaves on prime Fano threefolds

In this section we present some results concerning rank 2 stable sheaves F with $c_1(F) = 1$ on a smooth non-hyperelliptic prime Fano threefold Y, with special attention to the case $g \geq 6$. We will first analyze the cases of minimal c_2 (see the next subsection) and then look for bundles with higher c_2.

3.1. **Rank 2 stable sheaves with $c_1 = 1$ and minimal c_2.** We provide a lower bound on $c_2(F)$ for the existence of F, namely $\mathsf{M}_Y(2, 1, c_2)$ is non-empty if and only if $c_2(F) \geq m_g = \lceil \frac{g+2}{2} \rceil$. Then we describe some properties of F in the cases $c_2 = m_g$ (see Proposition 3.5) and $c_2 = m_g + 1$ (see Proposition 3.7). This description is deeply inspired on the analysis of the case $g = 8$ pursued by Iliev and Manivel in [IM07].

Lemma 3.1. *Let Y be a smooth non-hyperelliptic Fano threefold of genus g, and let F be a rank 2 stable sheaf on Y with $c_1(F) = H_Y$. Then we have:*

$$c_2(F) \geq \frac{g+2}{2}. \tag{3.1}$$

Proof. Let $S \subset Y$ be a general hyperplane section surface. Since Y is non-hyperelliptic, by Moishezon's theorem [Moĭ67], we have $\mathrm{Pic}(S) \cong \mathbb{Z} = \langle H_S \rangle$. Consider the restriction $F_S = F \otimes \mathscr{O}_S$ and notice that the sheaf F_S is still torsion-free. Moreover it is semistable by Maruyama's theorem ([Mar81]), hence stable since $c_1(F_S) = H_S$ and $\mathrm{Pic}(S) = \langle H_S \rangle$. Since S is a K3 surface, the dimension of the moduli space $\mathsf{M}_S(2, 1, c_2(F_S))$ equals $4\, c_2(F_S) - 2\, g - 4$. So this number has to be non-negative, and we obtain (3.1). $\square$

In view of the previous lemma we define:

$$m_g = \left\lceil \frac{g+2}{2} \right\rceil. \tag{3.2}$$

Lemma 3.2. *Let C be a curve in $\mathscr{H}^0_d(Y)$, with $d < m_g$ Then C is Cohen-Macaulay and we have $\mathrm{H}^k(Y, \mathcal{I}_C) = 0$ for all k.*

Proof. First observe that the curve C has no isolated or embedded points. Indeed, the purely 1-dimensional piece $\tilde{C}$ of C is a curve of degree d and arithmetic genus ℓ, where ℓ is the length of the zero-dimensional piece of C. In order to see that, for $\ell > 0$, this leads to a contradiction, one notes that since $\mathrm{H}^0(Y, \mathcal{I}_{\tilde{C}}) = 0$, we have $\mathrm{h}^2(Y, \mathcal{I}_{\tilde{C}}) \geq \chi(\mathcal{I}_{\tilde{C}}) = \ell$. Thus we would have a non-zero element of $\mathrm{Ext}^1_Y(\mathcal{I}_C(1), \mathscr{O}_Y)$, corresponding to a rank 2 sheaf F with $c_1(F) = 1$, $c_2(F) = d$. It is easy to see that the sheaf F would be stable. Indeed, assuming that there exists a destabilizing torsion-free subsheaf K, then it is easy to check that $\mathrm{rk}(K) = 1$ and

$c_1(K) = 1$ and we have the following commutative diagram:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\ 0 \arrow[r] & \mathscr{O}_Y \arrow[r] \arrow[d, equal] & F \arrow[r] \arrow[d] & \mathcal{I}_C(1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{O}_Y \arrow[r] & S \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 0 \\ & & 0 & 0 & \end{tikzcd}$$

where T has rank 0 and $c_1(T) = 0$. This implies that T is supported at a subvariety $Z \subset Y$ of dimension less than or equal to 1. It follows that $\mathrm{Ext}^1_Y(T, \mathscr{O}_Y) \cong \mathrm{H}^2(Z, T(-1))^* = 0$ and so $S \cong \mathscr{O}_Y \oplus T$. Since the map $\mathscr{O}_Y \to S$ is the composition of injective maps and T is a torsion sheaf, it easily follows that F must contain $\mathscr{O}_Y$ as direct summand. This means that the exact sequence in the middle of the above diagram splits, a contradiction.

Hence the sheaf F is stable, thus contradicting Lemma 3.1. The above argument implies that the group $\mathrm{H}^2(Y, \mathcal{I}_C)$ vanishes. Note that this implies the statement by Riemann-Roch. $\square$

Definition 3.3. A zero-dimensional subscheme Z of a smooth K3 surface S of Picard number 1 and sectional genus g is of type $\mathscr{Z}^k_\ell$ if it is locally a complete intersection, and if $\mathrm{len}(Z) = \ell, \mathrm{h}^1(S, \mathcal{I}_Z(1)) = k$ and $\mathrm{h}^0(S, \mathcal{I}_Z(1)) = (g+1) - \ell - k$.

Lemma 3.4. *Let S be a K3 surface of Picard number 1 and sectional genus g, and let $m = m_g$ be defined by (3.2). Then S contains no subscheme of type $\mathscr{Z}^k_\ell$ for any $k \geq 1$ and for any $\ell \leq m + k - 2$. Moreover if $g \geq 6$, then S contains no subscheme of type $\mathscr{Z}^k_\ell$ for any $k \geq 2$ and any $\ell \leq m + k - 1$.*

Proof. We split the induction argument in two steps.

Step 1. There are no subschemes of S of type $\mathscr{Z}^1_\ell$ for any $\ell \leq m - 1$.

It is easy to see that a point must be of type $\mathscr{Z}^0_1$, and that there exists no scheme of type $\mathscr{Z}^k_1$ for $k \geq 1$. Now consider a subscheme $Z \subset S$ of type $\mathscr{Z}^1_\ell$ for $\ell \geq 2$, and the sheaf F on S associated to Z by the non-trivial extension

$$0 \to \mathscr{O}_S \to F \to \mathcal{I}_Z(1) \to 0. \tag{3.3}$$

Notice that $c_1(F) = 1$, $c_2(F) = \ell$. If F is locally free, then it is stable since $F(-1)$ has no non-zero global sections. Hence we get $\ell \geq m$, since the quantity (2.11) must be non-negative. If F is not locally free, then the pair $(\mathscr{O}(1), Z)$ does not satisfy the Cayley-Bacharach property (see [HL97]) and so there exists a subscheme $Z' \subset Z$ of type $\mathscr{Z}^1_{\ell-1}$. Iterating this argument on Z' we obtain either that $\ell \geq m$ or that there is a scheme of type $\mathscr{Z}^1_1$ which is impossible.

By induction on $k \geq 1$ one easily proves that there are no subschemes of S of type $\mathscr{Z}^k_\ell$ with $\ell \leq m + k - 2$.

Step 2. We assume now that $g \geq 6$ and we prove that there are no subschemes of S of type $\mathscr{Z}^2_{m+1}$. Suppose that Z is a subscheme of type $\mathscr{Z}^2_{m+1}$. Let F be the rank 3 sheaf associated to Z by the extension:

$$0 \to \mathscr{O}_S \otimes \mathrm{H}^1(S, \mathcal{I}_Z(1)) \to F \to \mathcal{I}_Z(1) \to 0. \tag{3.4}$$

Since by the previous step there exists no subscheme of type $\mathscr{Z}^2_m$, it follows that F is locally free. By (3.4), it follows that $\mathrm{H}^1(S, F) = \mathrm{H}^2(S, F) = 0$. It is easy to see that the groups $\mathrm{H}^0(S, F^*)$ and $\mathrm{H}^0(S, F(-1))$ vanish, hence F is a rank 3 stable bundle on S with $c_1(F) = 1$ and $c_2(F) = m + 1$. By (2.11), the dimension of the moduli space $\mathsf{M}_S(3, 1, c_2(F))$ equals $6\,c_2(F) - 4\,g - 12$ and for $g \geq 6$ this dimension is negative, a contradiction.

Finally by induction on $k \geq 2$ one easily proves that there are no subschemes of S of type $\mathcal{Z}^k_\ell$ with $\ell \leq m + k - 1$.

$\square$

The following proposition is essentially due to Iliev and Manivel, see [IM07].

Proposition 3.5 (Iliev-Manivel). *Let Y be a smooth non-hyperelliptic Fano threefold of genus g and set $m = m_g$. Let F be a rank 2 stable sheaf on Y, with $c_1(F) = 1$, $c_2(F) = m$, $c_3(F) \geq 0$. Then F is a locally free sheaf and satisfies:*

$$\mathrm{H}^k(Y, F(-1)) = 0, \qquad \mathrm{H}^j(Y, F) = 0, \qquad \text{for all } k \in \mathbb{Z} \text{ and for all } j \geq 1.$$

Moreover if $g \geq 6$, then F is also globally generated and ACM.

Proof. First of all one proves $\mathrm{H}^2(Y, F) = 0$. Indeed any non-trivial element of $\mathrm{H}^2(Y, F)^* \cong \mathrm{Ext}^1_Y(F, \mathscr{O}_Y(-1))$ provides an extension of the form:

$$0 \to \mathscr{O}_Y(-1) \to \tilde{F} \to F \to 0,$$

where $\tilde{F}$ is a rank 3 sheaf which is easily seen to be semistable. This sheaf satisfies $c_1(\tilde{F}) = 0$ and $c_2(\tilde{F}) = m - (2g - 2) < 0$, which contradicts Bogomolov's inequality (2.1). Since $\chi(F) = g + 3 - m \geq 0$, it follows that there exists a non-zero global section of F.

Fix now a general hyperplane section S of Y. We set $E = F^{**}$ and $E_S = E \otimes \mathscr{O}_S$. We can assume that E_S is locally free on S. It is easy to see that E is a rank 2 stable sheaf with $c_1(E) = 1$, and this implies $\mathrm{H}^0(Y, E(-1)) = 0$ by stability and $\mathrm{H}^3(Y, E(-1)) = \mathrm{H}^3(Y, E) = 0$ by Serre duality and stability. Since $\mathrm{h}^0(Y, F) \neq 0$, one sees that $\mathrm{h}^0(Y, E) \neq 0$. Let Z be the zero locus of a general section of E_S. Note that Z has dimension zero and $\mathrm{len}(Z) = m$, and recall the exact sequence:

$$0 \to \mathscr{O}_S \to E_S \to \mathcal{I}_Z(1) \to 0. \tag{3.5}$$

It is easy to prove that $\mathrm{H}^0(S, E_S(-1)) = 0$, hence E_S is stable by Hoppe's criterion. In particular we get $\mathrm{H}^2(S, E_S)^* \cong \mathrm{H}^0(S, E_S^*) = 0$, so the induced map $\mathrm{H}^1(S, \mathcal{I}_Z(1)) \to \mathrm{H}^2(S, \mathscr{O}_S)$ is surjective.

We can now prove that F is locally free. Indeed, assume the contrary, and write down the double dual exact sequence:

$$0 \to F \to E \to T \to 0, \tag{3.6}$$

where T is a torsion sheaf with $c_1(T) = 0$ and $c_2(T) \leq 0$. By the minimality of m, we must have $c_2(E) \geq c_2(F)$ and so we get $c_2(T) = c_2(E) - c_2(F) \geq 0$. Hence $c_2(T)$ vanishes. Thus the sheaf T is supported on a subscheme of codimension 3 and $c_3(T) = c_3(E) - c_3(F) \geq 0$. By Lemma 3.4 the subscheme Z must be of type $\mathcal{Z}^1_m$, hence from (3.5), using $\mathrm{H}^1(S, \mathscr{O}_S) = 0$ and $\mathrm{H}^2(S, E_S) = 0$, we get $\mathrm{H}^1(S, E_S) = 0$. Then we consider the exact sequence:

$$0 \to E(-1) \to E \to E_S \to 0. \tag{3.7}$$

One can show $\mathrm{H}^2(Y, E) = 0$, by the same argument that we used for F. Then, taking global sections, we obtain $\mathrm{H}^2(Y, E(-1)) = 0$, so:

$$\chi(E(-1)) = -\,\mathrm{h}^1(Y, E(-1)) \leq 0. \tag{3.8}$$

Since E is reflexive, by a straightforward generalization of [Har80, Proposition 2.6], we also have $\chi(E(-1)) = c_3(E)/2 \geq 0$. We deduce $c_3(E) = 0$, so E is locally free. Moreover, from (3.8) we also obtain $\mathrm{H}^1(Y, E(-1)) = 0$. Note incidentally that, using (3.7), this implies also $\mathrm{H}^1(Y, E) = 0$. Let us now observe that, since $c_3(E) = 0$, the assumption $c_3(F) \geq 0$ forces $c_3(T) = 0$. We have thus proved that $T = 0$, so the sheaf F is isomorphic to E, hence it is locally free. Moreover, since $F \cong E$, we have proved the desired vanishing results too.

Now let us assume that $g \geq 6$ and show that F is globally generated. Following [IM07, Proposition 5.4] one reduces to show that for any point $x \in Y$ and for a general surface S' through x, the vector bundle $F_{S'} = F \otimes \mathscr{O}_{S'}$ is globally generated. Clearly it is enough to prove that $\mathcal{I}_Z(1)$ is globally generated, where we denote again by Z the zero locus of a general global section of $F_{S'}$. This amounts to show that Z is cut out scheme-theoretically by its linear span, in other word that Z cannot be contained in a subscheme $Z' \subset S'$ of type $\mathscr{Z}^2_{m+1}$. But no such subscheme exists by Lemma 3.4, as soon as $g \geq 6$.

It remains to show that F is ACM, and this follows by Griffith's theorem, [SS85, Theorem 5.52], since F is globally generated. □

Remark 3.6. Note that if F is an ACM bundle on a smooth prime Fano threefold Y and S is a hyperplane section surface, then the restriction F_S is ACM too. In particular if Y is a prime Fano threefold of genus 7, the bundle $\mathscr{E}_y$, introduced in the previous section, is ACM for any $y \in \mathsf{M}_Y(2, 1, 5)$ and its restriction to S is ACM too.

Proposition 3.7 (Iliev-Manivel)**.** *Let Y be a smooth non-hyperelliptic Fano threefold of genus $g \geq 6$ and set $m = m_g$. Let F be a rank 2 stable sheaf on Y, with $c_1(F) = 1$, $c_2(F) = m+1$, $c_3(F) \geq 0$. Then either F is a locally free sheaf or there exists an exact sequence:*

$$0 \to F \to E \to \mathscr{O}_L \to 0, \tag{3.9}$$

where E is as in Proposition 3.5 and L is a line contained in Y. Moreover, we have:

$$\mathrm{H}^k(Y, F(-1)) = \mathrm{H}^k(Y, F) = 0, \qquad \text{for } k = 1, 2. \tag{3.10}$$

Proof. We work as in the previous proposition. First of all we prove that $\mathrm{H}^2(Y, F) = 0$, (since $m + 1 < 2g - 2$ as soon as $g > 3$). If F is not locally free, then we consider the sheaves $E = F^{**}$ and $T = E/F$ and the exact sequence (3.6). Recall that no subscheme of type $\mathscr{Z}^k_{m+1}$ for $k \geq 2$ exists by Lemma 3.4 since $g \geq 6$. So the same argument we used in the previous proof to establish the property of being locally free this time proves that $c_2(E) = m$ and $c_2(T) = -1$. Since E is reflexive (and thus $c_3(E) \geq 0$), Proposition 3.5 implies that E is locally free and globally generated.

Therefore the support of T is a line $L \subset Y$, and we may take a general hyperplane section S such that $L \cap S = x$, a point of Y. A general global section of F_S (respectively, of E_S) vanishes on a subscheme $Z' \subset S$ (respectively, $Z \subset S$), the scheme Z is of type $\mathscr{Z}^1_m$, and we have:

$$0 \to \mathcal{I}_{Z'}(1) \to \mathcal{I}_Z(1) \to \mathscr{O}_x \to 0. \tag{3.11}$$

Since the sheaf E is globally generated, $\mathcal{I}_Z(1)$ is too, hence Z is cut sheaf-theoretically by hyperplanes. Then the map $\mathrm{H}^0(S, \mathcal{I}_{Z'}(1)) \to \mathrm{H}^0(S, \mathcal{I}_Z(1))$ induced by (3.11) is not an isomorphism. We obtain $\mathrm{h}^1(S, \mathcal{I}_{Z'}(1)) = 1$, which easily implies $\mathrm{H}^1(S, F_S) = 0$. At this point, following the argument of the previous proposition, one can easily prove the following chain of implications:

$$\mathrm{H}^2(Y, F) = \mathrm{H}^1(S, F_S) = 0 \Rightarrow \mathrm{H}^2(Y, F(-1)) = 0 \Rightarrow c_3(F) = 0 \Rightarrow$$
$$\Rightarrow \mathrm{H}^1(Y, F(-1)) = 0 \Rightarrow \mathrm{H}^1(Y, F) = 0.$$

This proves the vanishing (3.10).

Now we mimic a technique of Druel, see [Dru00]. Namely, since $\mathrm{H}^1(Y, F(-1)) = 0$, we have $\mathrm{H}^0(Y, T(-1)) = 0$. It follows that T is a Cohen-Macaulay curve and by a Hilbert polynomial computation we obtain $T \cong \mathscr{O}_L$ for a certain line $L \subset Y$. $\square$

3.2. **A good component of the moduli space $\mathsf{M}_Y(2,1,d)$.** Throughout this section we will assume that Y is an ordinary, non-hyperelliptic smooth prime Fano threefold. In particular we assume that the Hilbert scheme $\mathscr{H}_1^0(Y)$ has a generically smooth component.

We first restate a result concerning the moduli space $\mathsf{M}_Y(2,1,m_g)$, see [BF08, Proposition 3.5]. Recall that the non-emptyness of this space is derived from a case by case analysis, going back to [Mad00] for $g = 3$, [Mad02] for $g = 4,5$, [Gus82] for $g = 6$, [IM04a], [IM07c], [Kuz05], for $g = 7$, [Gus83], [Gus92], [Muk89] for $g = 8$, [IR05] for $g = 9$, [Muk89] for $g = 10$, [Kuz96] (see also [Sch01], [Fae07]) for $g = 12$.

Theorem 3.8. *Let Y be a smooth ordinary non-hyperelliptic prime Fano threefold of genus g and let F be a sheaf in $\mathsf{M}_Y(2,1,m_g)$. Then F is locally free and ACM. Furthermore, the moduli space $\mathsf{M}_Y(2,1,m_g)$ contains a sheaf F satisfying the conditions:*

$$\mathrm{Ext}^2_Y(F,F) = 0, \tag{3.12}$$

$$F \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(1), \qquad \text{for some line } L \text{ with } N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1). \tag{3.13}$$

The dimension of $\mathsf{M}_Y(2,1,m_g)$ equals 0 if g is even and 1 if g is odd.

Now, we construct inductively a component of $\mathsf{M}_Y(2,1,d)$, for all $d \geq m_g$. This component is generically smooth of the expected dimension and its general element F is locally free and satisfies $\mathrm{H}^1(Y, F(-1)) = 0$.

Lemma 3.9. *Let Y be a smooth ordinary prime Fano threefold of genus g, let $L \subset Y$ be a line belonging to a generically smooth irreducible component H of $\mathscr{H}_1^0(Y)$ and choose an irreducible component M of $\mathsf{M}_Y(2,1,d)$, with $\dim(\mathsf{M}) = n$. Assume that, for a given locally free sheaf E lying in M, with $\hom_Y(E, \mathscr{O}_L) = 1$, we have an exact sequence of the form:*

$$0 \to F \to E \to \mathscr{O}_L \to 0. \tag{3.14}$$

Then the set of sheaves F fitting in (3.14) with $[E] \in \mathsf{M}$ and $[L] \in \mathsf{H}$ is an irreducible $(n+1)$-dimensional subvariety of $\mathsf{M}_Y(2,1,d+1)$.

Proof. It is easy to see that F is a rank 2 stable (non-reflexive) sheaf with $c_1(F) = 1$, $c_2(F) = d+1$, $c_3(F) = 0$. Note that a surjective map $E \to \mathscr{O}_L$ exists if and only if $E \otimes \mathscr{O}_L$ is isomorphic to $\mathscr{O}_L \oplus \mathscr{O}_L(1)$, which happens if and only if $\hom_Y(E, \mathscr{O}_L) = 1$. This is equivalent to the condition $\mathrm{Ext}^1_Y(E, \mathscr{O}_L) = 0$. This vanishing takes place for general $[L] \in \mathsf{H}$ and for general $[E] \in \mathsf{M}$.

Note that the condition $\hom_Y(E, \mathscr{O}_L) = 1$, implies that the map $\sigma : E \to \mathscr{O}_L$ is unique up to a non-zero scalar, so the kernel of σ is determined (up to isomorphism) by E and L. Therefore we have a rational map $\mathsf{M} \times \mathsf{H} \to \mathsf{M}_Y(2,1,d+1)$ which associates to the general member of $\mathsf{M} \times \mathsf{H}$ the sheaf $F = \ker(\sigma)$, where σ generates $\mathrm{Hom}_Y(E, \mathscr{O}_L)$. This map is generically injective, since E is recovered as F^{**} and $\mathscr{O}_L$ as the quotient F/F^{**}. Note that both M and H are irreducible, respectively of dimension n, and 1. Thus the image of the rational map above is irreducible of dimension $n+1$. $\square$

Remark 3.10. Let Y be a smooth ordinary prime Fano threefold and d be an integer. If a sheaf F in $\mathsf{M}_Y(2,1,d)$ satisfies $\mathrm{H}^1(Y, F(-1)) = 0$, then we have $\mathrm{H}^k(Y, F(-1)) = 0$ for any k. Indeed the cases $k = 0,3$ follow by stability and Serre duality. By (2.9) it is easy to compute that $\chi(F(-1)) = 0$, and this implies the vanishing for $k = 2$.

Remark 3.11. Let Y be a smooth prime Fano threefold and d be an integer. If a sheaf F in $\mathsf{M}_Y(2,1,d)$ satisfies $\mathrm{H}^1(Y, F(-1)) = 0$, then we have $\mathrm{H}^1(Y, F(-t)) = 0$, for any $t \geq 1$. Indeed taking a general hyperplane section S of Y we have the restriction exact sequence, for any integer t,

$$0 \to F(-1-t) \to F(-t) \to F_S(-t) \to 0. \tag{3.15}$$

Note that the sheaf F_S is semistable, by Maruyama's theorem. This implies $\mathrm{H}^0(Y, F_S(-t)) = 0$ for any $t \geq 1$. Then, taking cohomology of (3.15), we deduce that $\mathrm{H}^1(Y, F(-t)) = 0$, for any $t \geq 1$.

Theorem 3.12. *Let Y be a smooth ordinary prime Fano threefold of genus g, set $m = m_g$, $L \subset Y$ be a line in Y with $N_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$, and x be a point of L. Then, for any integer $d \geq m$, there exists a rank 2 stable locally free sheaf F_d with $c_1(F_d) = 1$, $c_2(F_d) = d$, and satisfying:*

$$\mathrm{Ext}^2_Y(F_d, F_d) = 0, \tag{3.16}$$

$$\mathrm{H}^1(Y, F_d(-1)) = 0, \tag{3.17}$$

$$\mathrm{H}^0(L, F_d(-2\,x)) = 0. \tag{3.18}$$

The sheaf F_d thus belongs to a generically smooth component of $\mathsf{M}_X(2,1,d)$ of dimension $2d - g - 2$.

Proof. We work by induction on $d \geq m$. For $d = m$, the sheaf F provided by Theorem 3.8 is satisfies all our requirements. Indeed, the only property that we need to check in this case is (3.18), but this is clear by (3.13).

Let us now construct a sheaf F_d in $\mathsf{M}_X(2,1,d)$. By induction we can choose a rank 2 locally free sheaf F_{d-1} with $c_1(F_{d-1}) = 1$ (so that $F^*_{d-1} \cong F_{d-1}(-1)$), $c_2(F_{d-1}) = d-1$, satisfying $\mathrm{Ext}^2_Y(F_{d-1}, F_{d-1}) = 0$, $\mathrm{H}^1(Y, F_{d-1}(-1)) = 0$, and $\mathrm{H}^0(L, F_{d-1}(-2\,x)) = 0$. From the last vanishing it easily follows that $F_{d-1} \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(x)$. Therefore there exists a (unique up to a non-zero scalar) surjective morphism $F_{d-1} \otimes \mathscr{O}_L \to \mathscr{O}_L$. Then we get a projection σ as the composition of surjective morphisms: $F_{d-1} \to F_{d-1} \otimes \mathscr{O}_L \to \mathscr{O}_L$. We denote by F_d the kernel of σ and we have the exact sequence:

$$0 \to F_d \to F_{d-1} \xrightarrow{\sigma} \mathscr{O}_L \to 0. \tag{3.19}$$

We claim that the sheaf F_d lies in $\mathsf{M}_X(2,1,d)$ and satisfies (3.16), (3.17) and (3.18). In view of Lemma 3.9, F_d is a stable torsion-free sheaf with $c_1(F_d) = 1$ and $c_2(F_d) = d$. We have $\mathrm{H}^1(Y, F_d(-1)) = 0$ since $\mathrm{H}^1(Y, F_{d-1}(-1)) = 0$ by induction and $\mathrm{H}^0(Y, \mathscr{O}_L(-1)) = 0$. So (3.17) holds.

In order to prove (3.16), let us apply the functor $\mathrm{Hom}_Y(F_d, -)$ to (3.19). This gives the exact sequence:

$$\mathrm{Ext}^1_Y(F_d, \mathscr{O}_L) \to \mathrm{Ext}^2_Y(F_d, F_d) \to \mathrm{Ext}^2_Y(F_d, F_{d-1}).$$

We will prove that both the first and the last terms of the above sequence vanish. To prove the vanishing of the latter, apply $\mathrm{Hom}_Y(-, F_{d-1})$ to the exact sequence (3.19). We get the exact sequence:

$$\mathrm{Ext}^2_Y(F_{d-1}, F_{d-1}) \to \mathrm{Ext}^2_Y(F_d, F_{d-1}) \to \mathrm{Ext}^3_Y(\mathscr{O}_L, F_{d-1}).$$

By induction, we have $\mathrm{Ext}^2_Y(F_{d-1}, F_{d-1}) = 0$. Serre duality yields:

$$\mathrm{Ext}^3_Y(\mathscr{O}_L, F_{d-1})^* \cong \mathrm{H}^0(Y, \mathscr{O}_L \otimes F^*_{d-1}(-1)) \cong \mathrm{H}^0(L, F_{d-1}(-2x)) = 0.$$

Therefore we obtain $\mathrm{Ext}^2_Y(F_d, F_{d-1}) = 0$. To show the vanishing of the group $\mathrm{Ext}^1_Y(F_d, \mathscr{O}_L)$, we apply the functor $\mathrm{Hom}_Y(-, \mathscr{O}_L)$ (3.19). We are left with the exact sequence:

$$\mathrm{Ext}^1_Y(F_{d-1}, \mathscr{O}_L) \to \mathrm{Ext}^1_Y(F_d, \mathscr{O}_L) \to \mathrm{Ext}^2_Y(\mathscr{O}_L, \mathscr{O}_L).$$

The rightmost term vanishes by Remark 2.1. By Serre duality on L we get $\mathrm{Ext}^1_Y(F_{d-1}, \mathscr{O}_L) \cong \mathrm{H}^1(L, F^*_{d-1}) \cong \mathrm{H}^0(L, F_{d-1}(-2x))^*$. But this group vanishes by induction. We have thus established (3.16). Note that, since clearly $\mathrm{Hom}_Y(F_d, F_d) = \mathrm{Ext}^3_Y(F_d, F_d) = 0$ by stability, Riemann-Roch gives:

$$\mathrm{ext}^1_Y(F_d, F_d) = 2\,d - g - 2, \tag{3.20}$$

Now let us prove property (3.18). Tensoring (3.19) by $\mathscr{O}_L$ we get the following exact sequence of sheaves on L

$$0 \to \mathscr{T}or_1^Y(\mathscr{O}_L, \mathscr{O}_L) \to F_d \otimes \mathscr{O}_L \to F_{d-1} \otimes \mathscr{O}_L \to \mathscr{O}_L \to 0. \tag{3.21}$$

By (2.3) we know that $\mathscr{T}or_1^Y(\mathscr{O}_L, \mathscr{O}_L) \cong N_L^* \cong \mathscr{O}_L \oplus \mathscr{O}_L(x)$, since L is general in $\mathscr{H}_1^0(Y)$ and Y is ordinary. Now we twist (3.21) by $\mathscr{O}_L(-2x)$ and take global sections. By induction $\mathrm{H}^0(L, F_{d-1}(-2\,x)) = 0$, so our claim (3.18) follows easily.

Finally, we would like to flatly deform the sheaf F_d to a stable vector bundle F, and we claim that (3.17), (3.16), (3.18) will hold for F too. Indeed, (3.17) and (3.16) will still hold on F by semicontinuity (see [Har77, Theorem 12.8] and [BPS80, Satz 3 (i)]). For (3.18), we notice that, by stability of F_d and by Remark 3.11, we have $\mathrm{H}^0(Y, F_d \otimes \mathscr{O}_L(-2)) \cong \mathrm{H}^1(Y, F_d \otimes \mathcal{I}_L(-2))$. Now let F_L be the vector bundle associated to L as in Remark 2.2. Tensoring by $F_d(-1)$ the sequence (2.12) and taking cohomology, since $\mathrm{H}^1(Y, F_d(-1)) = \mathrm{H}^2(Y, F_d(-1)) = 0$ by Remark 3.10, we get $\mathrm{H}^1(Y, F_d \otimes \mathcal{I}_L(-2)) \cong \mathrm{H}^1(Y, F_d \otimes F_L(-1))$. Hence we conclude that a deformation of F_d will satisfy property (3.18) by semicontinuity of the dimension of the cohomology group $\mathrm{H}^1(Y, F_d \otimes F_L(-1))$. We have thus proved (3.17), (3.16), and (3.18) for F. We note further that the equality $\mathrm{ext}^1_Y(F, F) = 2\,d - g - 2$ implies here that $[F]$ lies in a generically smooth component of $\mathsf{M}_X(2, 1, d)$ of dimension $2d - g - 2$.

Thus it only remains to check that a general deformation F of F_d in $\mathsf{M}_Y(2, 1, d)$ is a locally free sheaf. In order to show this, we consider the double dual exact sequence:

$$0 \to F \to F^{**} \to T \to 0, \tag{3.22}$$

where T is a torsion sheaf whose support W has dimension at most 1. We would like to prove $T = 0$. Is is easy to see that F^{**} is stable. This implies $\mathrm{H}^0(Y, F^{**}(-1)) = 0$ which in turn gives $\mathrm{H}^0(Y, T(-1)) = 0$, so W contains no isolated or embedded points, i.e. it is a Cohen-Macaulay curve.

We will now argue that the exact sequence (3.22) is not of the form (3.19). Indeed, recall that the variety $\mathsf{M}_Y(2, 1, d-1)$ is smooth at the point corresponding to the locally free sheaf F_{d-1}, since $\mathrm{Ext}^2_Y(F_{d-1}, F_{d-1}) = 0$ by induction hypothesis. Let M be the component of $\mathsf{M}_Y(2, 1, d-1)$ containing $[F_{d-1}]$ and H the component of $\mathscr{H}_1^0(Y)$ containing $[L]$. The dimension of M equals $\mathrm{ext}^1_Y(F_{d-1}, F_{d-1}) = 2\,d - g - 4$. So by Lemma 3.9, the set of sheaves fitting as kernel of (3.19), with the middle term lying in M, has dimension $2\,d - g - 3$. Now, if for a general element $[F]$ in an irreducible open neighborhood of F_d, the sequence (3.22) was of the form (3.19), then the sheaf $[F^{**}]$ would lie in M (for F^{**} specializes to F_{d-1}) and the quotient $[T]$ would lie in H (for T specializes to $\mathscr{O}_L$). But we have proved that the set of such sheaves F has dimension $2\,d - g - 3$, while $[F]$ lies in a component of dimension one greater (see (3.20)).

Finally, let us show that, assuming $T \neq 0$, we are lead to a contradiction: this will finish the proof. To do this, we show that the support W of T must have degree 1. In fact we prove that $c_2(T) = -1$, and we only need to show $c_2(T) \geq -1$. Note first the equality $\chi(T(t)) = -\,\mathrm{h}^1(Y, T(t))$ for any negative integer t. Recall that, by [Har80, Remark 2.5.1], we have $\mathrm{H}^1(Y, F^{**}(t)) = 0$ for all $t \ll 0$. Thus, tensoring

(3.19) by $\mathscr{O}_Y(t)$ and taking cohomology, we obtain $\mathrm{h}^1(Y, T(t)) \leq \mathrm{h}^2(Y, F(t))$ for all $t \ll 0$. Further, for any integer t, we can compute:

$$c_1(T(t)) = 0, \quad c_2(T(t)) = c_2(T) = d - c_2(F^{**}), \quad c_3(T(t)) = c_3(F^{**}) - (2t+1)c_2(T),$$

hence by Riemann-Roch:

$$\chi(T(t)) = -c_2(T)(t+1) + \frac{c_3(F^{**})}{2}.$$

Since F is a deformation of F_d, semicontinuity gives $\mathrm{h}^2(Y, F(t)) \leq \mathrm{h}^2(Y, F_d(t))$. Now, (3.19) provides $\mathrm{h}^2(Y, F_d(t)) = \mathrm{h}^1(Y, \mathscr{O}_L(t)) = -\chi(\mathscr{O}_L(t)) = -t-1$. Summing up we have, for $t \ll 0$:

$$-(t+1)c_2(T) + \frac{c_3(F^{**})}{2} \geq t+1,$$

which implies $c_2(T) \geq -1$. We have thus proved that T is of the form $\mathscr{O}_L(a)$, for some $[L] \in \mathscr{H}_1^0(Y)$, and for some integer a. Then $c_3(T) = c_3(F^{**}) = 1 + 2\,a$, so $a \geq 0$, see [Har80, Proposition 2.6]. On the other hand we have seen $\mathrm{H}^0(Y, T(-1)) = 0$, so $a \leq 0$. But then T should be of the form $\mathscr{O}_L$, a contradiction. $\square$

By Theorem 3.12 and Lemma 3.9 we can pose the following.

Definition 3.13. Choose a component $\mathsf{M}(m_g)$ of the moduli space $\mathsf{M}_Y(2, 1, m_g)$ containing a sheaf F satisfying the properties listed in Theorem 3.8. Then, for each $d \geq m_g + 1$, we recursively define $\mathsf{N}(d)$ as the set of non-reflexive sheaves fitting as kernel in an exact sequence of the form (3.19), with $F_{d-1} \in \mathsf{M}(d-1)$, and $\mathsf{M}(d)$ as the component of the moduli scheme $\mathsf{M}_Y(2, 1, d)$ containing $\mathsf{N}(d)$. In view of Theorem 3.12 the component $\mathsf{M}(d)$ is generically smooth of dimension $2d - g - 2$ and contains $\mathsf{N}(d)$ as an irreducible divisor.

Remark 3.14. Making use of Theorem 3.12, it is possible to classify all ACM bundles of rank 2 and $c_1 = 1$ on smooth non-hyperelliptic ordinary prime Fano threefolds. We refer to [BF08] for a complete investigation.

4. Rational cubics on Fano threefolds of genus 7

Let X be a smooth prime Fano threefold of genus 7, and let Γ be its homologically projectively dual curve. For $1 \leq d \leq 4$, the subset of $\mathscr{H}_d^0(X)$ containing rational normal curves is described by the results of [IM07c]. It is known to have dimension d, and to be isomorphic to $W^1_{1,5}$ for $d = 1$, isomorphic to $\Gamma^{(2)}$ for $d = 2$, and birational to $\Gamma^{(3)}$ for $d = 3$. The isomorphism $\mathscr{H}_2^0(X) \cong \Gamma^{(2)}$ was also proved by Kuznetsov, making use of the semiorthogonal decomposition of $\mathbf{D^b}(X)$.

Here we make more precise the result on cubics, showing that the Hilbert scheme $\mathscr{H}_3^0(X)$ is in fact isomorphic to the symmetric cube $\Gamma^{(3)}$.

The following result is due to Iliev-Markushevich, [IM07c]. This reformulation will be used further on.

Lemma 4.1 (Iliev-Markushevich)**.** *Let X be a smooth prime Fano threefold of genus 7. Then we have the following two bijective morphisms:*

$$\mathscr{H}_1^0(X) \to W^1_{1,5}, \qquad L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L)[-1], \tag{4.1}$$

$$\mathscr{H}_1^0(X) \to W^2_{1,7}, \qquad L \mapsto \mathbf{\Phi}^!(\mathscr{O}_L(-1)). \tag{4.2}$$

Proof. For any $y \in \Gamma$, we have $\mathscr{E}_y^* \otimes \mathscr{O}_L \cong \mathscr{O}_L \oplus \mathscr{O}_L(-1)$, so $\mathrm{h}^0(X, \mathscr{E}_y^* \otimes \mathscr{O}_L) = 1$. Indeed $\mathscr{E}_y$ is globally generated for each y and has degree 1 on L. So $\mathcal{H}^k(\mathbf{\Phi}^!(\mathscr{O}_L))_y = 0$ for all $y \in \Gamma$ and all $k \neq -1$. Hence $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle on Γ, and has degree 5 by Grothendieck-Riemann-Roch. Now observe that, using (2.34) and the spectral sequence (2.4), we get:

$$\begin{aligned} \mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(\mathscr{O}_L)[-1]) &\cong \mathrm{Hom}_X(\mathbf{\Phi}(\mathscr{O}_\Gamma), \mathscr{O}_L[-1]) \cong \\ &\cong \mathrm{Hom}_X(\mathcal{H}^1(\mathbf{\Phi}(\mathscr{O}_\Gamma)), \mathscr{O}_L) \cong \\ &\cong \mathrm{H}^0(L, \mathcal{U}_+^*(-1)). \end{aligned}$$

Since $\mathcal{U}_+^*$ is globally generated and $c_1(\mathcal{U}_+^*) = 2$, one sees easily that this space must have dimension 2. So $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ lies in $W^1_{1,5}$. Denote $\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(\mathscr{O}_L)[-1])$ by A_L.

To show that our map is injective, first observe that $\mathrm{Ext}^k_X(\mathscr{O}_L, \mathscr{O}_X) = 0$ for all k, while $\mathrm{Ext}^k_X(\mathscr{O}_L, \mathcal{U}_+) = 0$ for $k \neq 3$, and $\mathrm{Ext}^3_X(\mathscr{O}_L, \mathcal{U}_+) = A_L^*$. Making use of the exact triangle (2.23), this gives the isomorphisms:

$$\mathcal{H}^k(\mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_L))) \cong \begin{cases} \mathscr{O}_L & \text{for } k = 0, \\ A_L \otimes \mathcal{U}_+^* & \text{for } k = -1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Then $\mathcal{H}^0(\mathbf{\Phi}(\mathbf{\Phi}^!(\mathscr{O}_L))) \cong \mathscr{O}_L$, and we have injectivity of (4.1). Surjectivity follows, since $\mathscr{H}^0_1(X)$ and $W^1_{1,5}$ are isomorphic by [IM07c, Proposition 2.1].

Set now $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_L)[-1] \in W^1_{1,5}$. Applying (2.29), since $\mathbf{R}\mathscr{H}om_X(\mathscr{O}_L, \mathscr{O}_X)[2] \cong \mathscr{O}_L(-1)$, we obtain the functorial isomorphism:

$$\mathbf{\Phi}^!(\mathscr{O}_L(-1)) \cong \mathcal{L}^* \otimes \omega_\Gamma.$$

Clearly the degree of $\mathcal{L}^* \otimes \omega_\Gamma$ is 7 and by Serre duality we get $\mathrm{h}^0(\Gamma, \mathcal{L}^* \otimes \omega_\Gamma) = \mathrm{h}^1(\Gamma, \mathcal{L}) = 3$, hence the sheaf $\mathcal{L}^* \otimes \omega_\Gamma$ lies in $W^2_{1,7}$. This says that the map appearing in (4.2) is everywhere defined. Hence is it bijective since the map defined by (4.1) is. $\square$

Remark 4.2. In view of the isomorphism $\mathscr{H}^0_1(X) \cong W^1_{1,5}$, we note that the threefold X is exotic if and only if $W^1_{1,5}$ has a component which is non-reduced at any point. It is well-known (see e.g. [ACGH85, Proposition 4.2]) that $[\mathcal{L}]$ is a singular point of $W^1_{1,5}$ if and only if the Petri map:

$$\pi_{\mathcal{L}} : \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathrm{H}^0(\Gamma, \mathcal{L}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma, \omega_\Gamma).$$

is not injective. Note that, since the curve Γ is not tetragonal, any line bundle $\mathcal{L}$ in $W^1_{1,5}$ is globally generated. Therefore $\ker(\pi_{\mathcal{L}})$ is isomorphic to $\mathrm{H}^0(\Gamma, \mathcal{L}^* \otimes \mathcal{L}^* \otimes \omega_\Gamma)$.

This proves that the threefold X is exotic if and only if Γ admits infinitely many line bundles $\mathcal{L}$ in $W^1_{1,5}$ such that $\mathcal{L}^* \otimes \mathcal{L}^* \otimes \omega_\Gamma$ is effective.

Our next goal is to investigate the Hilbert scheme $\mathscr{H}^0_3(X)$. We will need the following lemma.

Lemma 4.3. *Let C be any Cohen-Macaulay curve of degree $d \geq 3$ and arithmetic genus p_a contained in X. Then $\mathbf{\Phi}^!(\mathscr{O}_C)$ is a vector bundle on Γ of rank $d - 2 + 2p_a$.*

Proof. The following argument is inspired on the proof of [Kuz05, Lemma 5.1]. We have to prove that, for each $y \in \Gamma$, the group $\mathrm{H}^0(X, \mathscr{E}_y^* \otimes \mathscr{O}_C)$ vanishes. By (2.26), it is enough to prove:

$$\mathrm{H}^0(C, \mathcal{U}_+) = 0. \tag{4.4}$$

Assume the contrary, and consider a non-zero global section u in $\mathrm{H}^0(C, \mathcal{U}_+)$. Let U be the 1-dimensional subspace spanned by u. By (2.13), we have $U \subset \mathrm{H}^0(C, \mathcal{U}_+) \subset \mathrm{H}^0(C, \mathscr{O}_X \otimes V) \cong V$. Set $V' = U^\perp / U$. Then the orthogonal Grassmann variety $\mathbb{G}_Q(\mathbb{P}^3, \mathbb{P}(V')) \subset \Sigma_+$ is a quadric and clearly the curve C is contained in $X' := \mathbb{G}_Q(\mathbb{P}^3, \mathbb{P}(V')) \cap X$. Recall that X is a linear section of Σ^+, then X must contain either a 2-dimensional quadric or a plane. But this is impossible by Lefschetz theorem.

This proves that $\mathbf{\Phi}^!(\mathscr{O}_C)$ is a vector bundle on Γ. By Riemann-Roch formula we conclude that its rank equals $-\chi(\mathscr{E}_y^* \otimes \mathscr{O}_C) = d - 2 + 2p_a$. □

We are now in position to prove the following result.

Theorem 4.4. *Let X be a smooth prime Fano threefold of genus 7, and Γ be its homologically projectively dual curve. The map $\psi : \mathscr{O}_C \mapsto (\mathbf{\Phi}^!(\mathscr{O}_C))^* \otimes \omega_\Gamma$ gives an isomorphism between $\mathscr{H}_3^0(X)$ and $\Gamma^{(3)}$. In particular $\mathscr{H}_3^0(X)$ is a smooth irreducible threefold.*

Proof. We have seen in Lemma 4.3 that, for any element C of $\mathscr{H}_3^0(X)$, the sheaf $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_C)$ is a line bundle on Γ, and $\mathcal{L}$ has degree 9 by Grothendieck-Riemann-Roch. Therefore by stability of $\mathscr{E}_x$ for any $x \in X$, we have $\mathrm{H}^1(\Gamma, \mathcal{L} \otimes \mathscr{E}_x) = 0$, so $\mathbf{\Phi}(\mathcal{L})$ is a locally free sheaf of rank 18 on X. Moreover by Grothendieck-Riemann-Roch one sees that $\deg(\mathbf{\Phi}(\mathcal{L})) = 8$.

Let us show that $\mathbf{\Phi}(\mathcal{L})$ lies in $W^3_{1,9}$. Note first that, since $\deg(\mathbf{\Phi}(\mathcal{L}) = 8$, we have $\mathcal{H}^1(\mathbf{\Phi}(\mathcal{L})) = 0$, so the cohomology of the complex $(\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_C)))$ can appear only in degree -1 or 0, by the decomposition (2.23) applied to the sheaf $\mathscr{O}_C$. By formula (2.24), we have $(\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_C)))^k = 0$ for $k \neq -1, -2, -3$ and

$$\begin{aligned}
(\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_C)))^{-3} &= \mathscr{O}_X^a, \\
(\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_C)))^{-2} &= \mathscr{O}_X^{a+2} \oplus \mathcal{U}^{*b}, \\
(\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_C)))^{-1} &= \mathcal{U}^{*b+4},
\end{aligned}$$

where $a = \mathrm{h}^0(C, \mathscr{O}_X(-1))$ and $b = \mathrm{h}^0(C, \mathcal{U}_+^*(-1))$, and clearly, by Riemann-Roch formula, $a + 2 = \mathrm{h}^1(C, \mathscr{O}_X(-1))$ and $b + 4 = \mathrm{h}^1(C, \mathcal{U}_+^*(-1))$. It follows that the cohomology of the complex $(\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_C)))$ is concentrated in degree -1. Moreover since $\mathcal{H}^k(\mathbf{\Psi}(\mathbf{\Psi}^*(\mathscr{O}_C))) = 0$ for $k = -3$ and $k = -2$, then we have that the differential $d^{-2} : \mathscr{O}_X^a \to \mathscr{O}_X^{a+2} \oplus \mathcal{U}^{*b}$ is injective and that $\ker(d^{-1}) = \mathrm{Im}(d^{-2})$. By the minimality of the complex (see Remark 2.4), one can now easily prove $a = b = 0$, that is

$$\mathrm{h}^1(C, \mathscr{O}_X(-1)) = 2, \qquad \mathrm{H}^0(C, \mathscr{O}_X(-1)) = 0. \tag{4.5}$$

$$\mathrm{h}^1(C, \mathcal{U}_+^*(-1)) = 4, \qquad \mathrm{H}^0(C, \mathcal{U}_+^*(-1)) = 0, \tag{4.6}$$

and we get the following exact sequence:

$$0 \to \mathscr{O}_X^2 \to (\mathcal{U}_+^*)^4 \xrightarrow{\zeta_C} \mathbf{\Phi}(\mathcal{L}) \to \mathscr{O}_C \to 0. \tag{4.7}$$

Then we have:

$$\mathrm{h}^0(\Gamma, \mathcal{L}) = \hom_\Gamma(\mathscr{O}_\Gamma, \mathcal{L}) = \hom_\Gamma(\mathbf{\Phi}^*(\mathcal{U}_+^*), \mathcal{L}) = \hom_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathcal{L})) = 4,$$

where the last equality is obtained applying the functor $\mathrm{Hom}_X(\mathcal{U}_+^*, -)$ to the exact sequence (4.7) and using the fact that the group $\mathrm{H}^0(C, \mathcal{U}_+)$ vanishes by (4.4). So $\mathcal{L}$ is an element of $W^3_{1,9}$.

This implies the map ζ_C corresponds to the natural evaluation map, and it is thus uniquely determined. This proves that the mapping $\varphi : \mathscr{H}_3^0(X) \to W^3_{1,9}$ defined by $\varphi(\mathscr{O}_C) = \mathbf{\Phi}^!(\mathscr{O}_C)$ is injective.

Observe that the correspondence $\tau : \mathcal{L} \mapsto \mathcal{L}^* \otimes \omega_\Gamma$ provides an isomorphism between $W^3_{1,9}$ and $W^0_{1,3}$. On the other hand note that $W^0_{1,3} \cong \Gamma^{(3)}$, indeed the curve Γ is not trigonal (see [Muk95a, Table 1]). Since $\mathscr{H}_3^0(X)$ is projective of dimension at least 3 and $\Gamma^{(3)}$ is irreducible and of dimension 3, it follows that the map $\varphi : \mathscr{H}_3^0(X) \to W^3_{1,9}$ is surjective. We have thus that $\psi = \tau \circ \varphi$ is a bijection.

We will show now that the tangent space $\mathrm{Ext}^1_X(\mathcal{I}_C,\mathcal{I}_C)$ to $\mathscr{H}^0_3(X)$ at $[C]$ is identified with the tangent space to $W^3_{1,9}$ at $[\mathcal{L}]$. By (4.5), we easily get the vanishing $\mathrm{H}^k(X,\mathcal{I}_C(-1)) = 0$ for $k \neq 2,3$. By (4.6) and since $\mathrm{H}^k(X,\mathcal{U}_+^*(-1)) = 0$ by Serre duality and semiorthogonality of $\mathscr{O}_X$ and $\mathcal{U}_+^*$, we get:

$$\mathrm{H}^k(X,\mathcal{U}_+^*(-1)\otimes\mathcal{I}_C) = 0, \qquad \text{for } k \neq 2. \tag{4.8}$$

Moreover by (4.4) and since $\mathrm{H}^k(X,\mathcal{U}_+) = 0$, we obtain $\mathrm{H}^k(X,\mathcal{U}_+\otimes\mathcal{I}_C) = 0$, for $k \neq 2$. Then by (2.13) and Lemma 3.2, we obtain:

$$\mathrm{H}^k(X,\mathcal{U}_+^*\otimes\mathcal{I}_C) = 0, \qquad \text{for } k \neq 1. \tag{4.9}$$

Set $A_C = \mathrm{H}^2(X,\mathcal{U}_+^*(-1)\otimes\mathcal{I}_C)$ and $B_C = \mathrm{H}^2(X,\mathcal{I}_C(-1))$. Note that the dimension of A_C is 4, the dimension of B_C is 2. One also computes $\mathrm{h}^3(X,\mathcal{I}_C(-1)) = 1$, and $\mathrm{h}^2(X,\mathcal{U}_+\otimes\mathcal{I}_C) = \mathrm{h}^1(X,\mathcal{U}_+^*\otimes\mathcal{I}_C) = 1$. Note that $\mathbf{\Phi}^!(\mathcal{I}_C) \cong \mathbf{\Phi}^!(\mathscr{O}_C)[-1] = \mathcal{L}[-1]$. Therefore we have the following exact sequences:

$$0 \to B_C\otimes\mathscr{O}_X \to F_C \to \mathcal{I}_C \to 0, \tag{4.10}$$

$$0 \to F_C \to \mathscr{O}_X \oplus \left(A_C\otimes\mathcal{U}_+^*\right) \xrightarrow{(\varsigma_C,\zeta_C)} \mathbf{\Phi}(\mathcal{L}) \to 0, \tag{4.11}$$

where (4.10) is the universal extension corresponding to $\mathrm{Ext}^1_X(\mathcal{I}_C,\mathscr{O}_X)$, and (4.11) follows from (2.23) applied to F_C, since $\mathbf{\Phi}^!(F_C) \cong \mathbf{\Phi}^!(\mathcal{I}_C)$. Here the map ς_C is obtained lifting to $\mathbf{\Phi}(\mathcal{L})$ the projection $\mathscr{O}_X \to \mathscr{O}_C$ and ζ_C is given by (4.7). By Lemma 3.2, we have $\mathrm{Ext}^k_X(\mathscr{O}_X,\mathcal{I}_C) = 0$ for all k. Therefore applying the functor $\mathrm{Hom}_X(-,\mathcal{I}_C)$ to (4.10) we obtain, for each k, an isomorphism:

$$\mathrm{Ext}^k_X(F_C,\mathcal{I}_C) \cong \mathrm{Ext}^k_X(\mathcal{I}_C,\mathcal{I}_C).$$

Moreover, applying the functor $\mathrm{Hom}_X(-,\mathcal{I}_C)$ this time to (4.11), we get an exact sequence:

$$\mathrm{Ext}^1_X(\mathcal{I}_C,\mathcal{I}_C) \to \mathrm{Ext}^2_X(\mathbf{\Phi}(\mathcal{L}),\mathcal{I}_C) \xrightarrow{\eta_C} \mathrm{Ext}^2_X(\mathcal{U}_+^*,\mathcal{I}_C)\otimes A_C^* \to \mathrm{Ext}^2_X(\mathcal{I}_C,\mathcal{I}_C), \tag{4.12}$$

where $\eta_C = \mathrm{Ext}^2_X(\zeta_C,\mathcal{I}_C)$. Consider now the curve Γ and denote by e the natural evaluation map:

$$e_{\mathscr{O}_\Gamma,\mathcal{L}} : \mathrm{Hom}_\Gamma(\mathscr{O}_\Gamma,\mathcal{L})\otimes\mathscr{O}_\Gamma \to \mathcal{L}.$$

Observe that the transpose of the Petri map $\pi_{\mathcal{L}}^\top$ (see 2.8) is obtained applying the functor $\mathrm{Ext}^1_\Gamma(-,\mathcal{L})$ to the map e. We have the natural isomorphisms:

$$\begin{aligned}
&\mathrm{Ext}^2_X(\mathbf{\Phi}(\mathcal{L}),\mathcal{I}_C) \cong \mathrm{Ext}^1_\Gamma(\mathcal{L},\mathcal{L}), \\
&\mathrm{Ext}^2_X(\mathcal{U}_+^*,\mathcal{I}_C) \cong \mathrm{Ext}^1_\Gamma(\mathscr{O}_\Gamma,\mathcal{L}), && \text{by (4.8)}, \\
&A_C = \mathrm{Ext}^2_X(\mathcal{U}_+(1),\mathcal{I}_C) \cong \mathrm{Hom}_\Gamma(\mathscr{O}_\Gamma,\mathcal{L}), && \text{by (4.9)}.
\end{aligned}$$

Remark now that, since ζ_C is uniquely determined up to a non-zero scalar, we may assume that it coincides with $\mathcal{H}^0(\mathbf{\Phi}(e))$. Moreover, by (4.8) we must have $\eta_C = \mathrm{Ext}^2_X(\mathbf{\Phi}(e),\mathcal{I}_C)$. We conclude that:

$$\pi_{\mathcal{L}}^\top = \mathrm{Ext}^1_\Gamma(e,\mathbf{\Phi}^!(\mathcal{I}_C)[1]) = \mathrm{Ext}^2_X(\mathbf{\Phi}(e),\mathcal{I}_C) = \eta_C.$$

From this discussion we conclude that, using (4.12), one can identify $\mathrm{Ext}^2_X(\mathcal{I}_C,\mathcal{I}_C)$ with the space of obstructions of $W^3_{1,9}$ at $\mathcal{L}$, and $\mathrm{Ext}^1_X(\mathcal{I}_C,\mathcal{I}_C)$ with the tangent space of $W^3_{1,9}$ at $\mathcal{L}$. Hence since $W^3_{1,9}$ is smooth (being isomorphic to $\Gamma^{(3)}$), then $\mathscr{H}^0_3(X)$ is smooth too, and the map $\varphi : \mathscr{H}^0_3(X) \to W^3_{1,9}$ is a local isomorphism between smooth varieties. By [Har92, Theorem 14.9], it follows that φ is an isomorphism, and ψ is an isomorphism too. $\square$

5. Vector bundles on Fano threefolds of genus 7

In this section, we assume that X is a smooth prime Fano threefold of genus 7, and we let Γ be its homologically projectively dual curve. We set up a birational correspondence between the component $\mathsf{M}(d)$ of Definition 3.13 and a component of the Brill-Noether variety $W^{2\,d-11}_{d-5,5\,d-24}$. This correspondence will turn out to be an isomorphism for $d = 6$.

5.1. **Vanishing results.** In order to setup the correspondence mentioned above, we will have to prove that various cohomology groups are zero. This is the purpose of the next series of lemmas.

Lemma 5.1. *Let $d \geq 6$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ such that:*

$$\mathrm{H}^1(X, F(-1)) = 0, \tag{5.1}$$

then we have:

$$\mathrm{Ext}^k_X(\mathscr{E}_y, F) = 0, \qquad \text{for all } y \in \Gamma \text{ and for all } k \neq 1.$$

Proof. Notice that for $k < 0$ and for $k > 3$ the claim is obvious since X has dimension 3. For $k = 2$, the claim amounts to $\mathrm{Hom}_X(F, \mathscr{E}_y(-1)) = \mathrm{Hom}_X(F, \mathscr{E}_y^*) = 0$, which follows from stability of F and $\mathscr{E}_y$.

For $k = -1$, we have to show that $\mathrm{Hom}_X(\mathscr{E}_y, F) = 0$. Assume the contrary, and remark that any non-trivial map $f : \mathscr{E}_y \to F$ provides an isomorphism $\mathscr{E}_y \to F^{**}$. Composing f with the natural injection $F \hookrightarrow F^{**} \cong \mathscr{E}_y$, we would get an isomorphism since $\mathscr{E}_y$ is a stable sheaf. But $c_2(\mathscr{E}_y) = 5$, while $c_2(F) = d \geq 6$, a contradiction.

For $k = 1$, let us show that $\mathrm{Ext}^1_X(F, \mathscr{E}_y^*) = 0$. Applying the functor $\mathrm{Hom}_X(F, -)$ to the restriction of (2.25) to $X \times \{y\}$, we get:

$$\mathrm{Hom}_X(F, \mathscr{G}_y) \to \mathrm{Ext}^1_X(F, \mathscr{E}_y^*) \to \mathrm{Ext}^1_X(F, \mathscr{O}_X) \otimes (\mathcal{U}_-)_y. \tag{5.2}$$

It is easy to see that the term on the left hand side vanishes by virtue of stability of $\mathscr{G}_y$ and F (see Lemma 2.6). On the other hand, by the assumption $\mathrm{H}^1(X, F(-1)) = 0$ we have $\mathrm{Ext}^1_X(F, \mathscr{O}_X) \cong \mathrm{H}^2(X, F(-1)) = 0$, since $\chi(F(-1)) = 0$. $\square$

Lemma 5.2. *Let $d \geq 6$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ satisfying* (5.1)*. Then the following equalities hold*

$$\mathrm{Ext}^k_X(F, \mathscr{O}_X) = 0, \qquad \text{for any } k \in \mathbb{Z}, \tag{5.3}$$

$$\mathrm{Ext}^k_X(F, \mathcal{U}_+) = 0, \qquad \text{for any } k \neq 2, \tag{5.4}$$

$$\mathrm{ext}^2_X(F, \mathcal{U}_+) = 2d - 10. \tag{5.5}$$

Proof. By stability of F, the assumption (5.1) and by the Riemann-Roch formula (2.10), we get immediately (5.3).

On the other hand if $k = 0, 3$, then we have $\mathrm{Ext}^k_X(F, \mathcal{U}_+) = 0$ by stability of the sheaves $\mathcal{U}_+$ and F. Since $\mathrm{Ext}^1_X(F, \mathscr{O}_X) = 0$, by applying the functor $\mathrm{Hom}_X(F, -)$ to the sequence (2.13) we get $\mathrm{Ext}^1_X(F, \mathcal{U}_+) \cong \mathrm{Hom}_X(F, \mathcal{U}_+^*)$, which vanishes by stability of F and $\mathcal{U}_+$. This proves (5.4). Finally, by the Riemann-Roch formula (2.10) we obtain the last equality. $\square$

Lemma 5.3. *Let $d \geq 7$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ satisfying* (5.1)*. Then we have*

$$\mathrm{Hom}_X(\mathcal{U}_+^*, F) = 0.$$

Proof. Fix a point y in Γ. Applying the functor $\mathrm{Hom}_X(-,F)$ to the exact sequence (2.26) we obtain an exact sequence:

$$0 \to \mathrm{Hom}_X(\mathscr{E}_y, F) \to \mathrm{Hom}_X(\mathcal{U}_+^*, F) \to \mathrm{Hom}_X(\mathscr{G}_y, F).$$

By Lemma 5.1 we know that the leftmost term vanishes. Assume that the rightmost does not, and consider a non-zero map $f : \mathscr{G}_y \to F$. Set $F' = \mathrm{Im}(f)$ and note that, by stability of the sheaves F and $\mathscr{G}_y$, we must have $\mathrm{rk}(F') = 2$ and $c_1(F') = 1$. We have thus an exact sequence:

$$0 \to F' \to F \to T \to 0, \tag{5.6}$$

where T is a torsion sheaf, with $\dim(\mathrm{supp}(T)) \leq 1$. Note that F' is stable, since any destabilizing subsheaf would destabilize also F. By [Har80, Propositions 1.1 and 1.9], the sheaf $\ker(f)$ must be a line bundle of degree zero. This means that $\ker(f) \cong \mathscr{O}_X$, and we have an exact sequence:

$$0 \to \mathscr{O}_X \to \mathscr{G}_y \to F' \to 0.$$

So F' satisfies $c_2(F') = 7$, $c_3(F') = 2$. Hence by (5.6) it follows that $d = c_2(F) = 7 + c_2(T) \leq 7$, since $c_2(T)$ is non-negative. Hence we have $d = 7$ and in this case we have $c_2(T) = 0$, and $c_3(T) = -2$. But this is a contradiction since $c_3(T)$ must be non-negative. We have thus proved our claim. □

5.2. **Canonical resolution of a bundle in $\mathsf{M}_X(2,1,d)$.** We will show here that a sheaf F in $\mathsf{M}_X(2,1,d)$ which satisfies $\mathrm{H}^1(X, F(-1)) = 0$ admits a canonical resolution having two terms, see the formula (5.8) below. Recall that, if the threefold X is ordinary, such a sheaf exists for all $d \geq 6$ by Theorem 3.12.

Proposition 5.4. *Let $d \geq 6$ and let F be a sheaf in $\mathsf{M}_X(2,1,d)$ such that $\mathrm{H}^1(X, F(-1)) = 0$. Then $\mathbf{\Phi}^!(F)$ is a simple vector bundle on Γ, with:*

$$\mathrm{rk}(\mathbf{\Phi}^!(F)) = d - 5, \qquad \deg(\mathbf{\Phi}^!(F)) = 5\,d - 24. \tag{5.7}$$

Moreover, F admits the following canonical resolution:

$$0 \to \mathrm{Ext}^2_X(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^* \xrightarrow{\zeta_F} \mathbf{\Phi}(\mathbf{\Phi}^!(F)) \to F \to 0, \tag{5.8}$$

where $\mathbf{\Phi}(\mathbf{\Phi}^!(F))$ is a simple vector bundle.

Proof. Consider the stalk over a point $y \in \Gamma$ of the sheaf $\mathcal{H}^k(\mathbf{\Phi}^!(F))$. We have:

$$\mathcal{H}^k(\mathbf{\Phi}^!(F))_y \cong \mathrm{Ext}^{k+1}_X(\mathscr{E}_y, F) \otimes \omega_{\Gamma,y}. \tag{5.9}$$

Hence by Lemma 5.1 it follows that this group vanishes for all $y \in \Gamma$ and for all $k \neq 0$. This implies that $\mathbf{\Phi}^!(F)$ is a locally free sheaf. By Riemann-Roch we have $-\chi(\mathscr{E}_y, F) = \chi(F, \mathscr{E}_y^*) = d - 5$, so the rank of $\mathbf{\Phi}^!(F)$ is $d - 5$. It follows easily by Grothendieck-Riemann-Roch that $\deg(\mathbf{\Phi}^!(F)) = 5\,d - 24$.

Let us now prove that $\mathbf{\Phi}^!(F)$ is a simple bundle. If $d = 6$, then $\mathbf{\Phi}^!(F)$ is a line bundle, hence it is obviously simple. For $d \geq 7$ we want to prove that the group $\mathrm{Hom}_\Gamma(\mathbf{\Phi}^!(F), \mathbf{\Phi}^!(F)) \cong \mathrm{Hom}_X(\mathbf{\Phi}(\mathbf{\Phi}^!(F)), F)$ is 1-dimensional. Applying the functor $\mathrm{Hom}_X(-,F)$ to the sequence (5.8) we obtain

$$\mathrm{Hom}_X(\mathbf{\Phi}(\mathbf{\Phi}^!(F)), F) \cong \mathrm{Hom}_X(F, F),$$

since the term $\mathrm{Hom}_X(\mathcal{U}_+^*, F)$ vanishes by Lemma 5.3. Hence $\mathbf{\Phi}^!(F)$ is simple, for F is. Since the functor $\mathbf{\Phi}$ is fully faithful, it follows that also the vector bundle $\mathbf{\Phi}(\mathbf{\Phi}^!(F))$ is simple.

It remains to exhibit the resolution (5.8). Note that, by formula (2.24) and Lemma 5.2 we get that the complex $(\mathbf{\Psi}(\mathbf{\Psi}^*(F)))$ is concentrated in degree -1 and isomorphic to $\mathrm{Ext}^2_X(F, \mathcal{U}_+)^* \otimes \mathcal{U}_+^*$. Hence the exact triangle (2.23) provides the resolution (5.8) for F. □

Lemma 5.5. *Let $d \geq 7$ and let F be as in the previous proposition, and set $A_F = \mathrm{Ext}^2_X(F,\mathcal{U}_+)^*$. Then we have the natural isomorphism:*

$$A_F \cong \mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(F)) \cong \mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))). \tag{5.10}$$

In particular $\mathrm{h}^0(\Gamma, \mathbf{\Phi}^!(F)) = 2d - 10$.

Proof. By Lemma 5.3 we know that $\mathrm{Hom}_X(\mathcal{U}_+^*, F) = 0$. Therefore, applying the functor $\mathrm{Hom}_X(\mathcal{U}_+^*, -)$ to the resolution (5.8) we obtain:

$$A_F = \mathrm{Ext}^2_X(F,\mathcal{U}_+)^* \cong \mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))).$$

Now we can use the spectral sequence (2.4) to show the isomorphisms:

$$\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(F)) \cong \mathrm{Hom}_\Gamma(\mathscr{O}_\Gamma, \mathbf{\Phi}^!(F)) \cong \mathrm{Ext}^1_X(\mathcal{U}_+(1), F) \cong A_F.$$

Indeed, by Lemma 5.3 we have $\mathrm{Hom}_X(\mathcal{U}_+^*, F) = 0$, and clearly $\mathrm{Ext}^{-1}_X(\mathcal{U}_+^*, F) = 0$. So the last statement follows from (5.5). □

In order to set up our correspondence between $\mathsf{M}(d)$ and $W^{2\,d-11}_{d-5,5\,d-24}$, we have to prove that a general vector bundle $\mathbf{\Phi}^!(F)$ is stable. This is done in the next lemma.

Lemma 5.6. *For each integer $d \geq 6$, there exists a Zariski dense open subset $\Omega(d) \subset \mathsf{M}(d)$, such that each point F_d of $\Omega(d)$ satisfies $\mathrm{H}^1(X, F_d(-1)) = 0$, and $\mathbf{\Phi}^!(F_d)$ is a stable sheaf.*

Proof. Let us prove the statement by induction on $d \geq 6$. If $d = 6$, $\mathbf{\Phi}^!(F_6)$ is stable since it is a line bundle. Suppose now $d > 6$ and assume that $\mathbf{\Phi}^!(F_{d-1})$ is stable. Recall that $\mathcal{L} = \mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle of degree 5 by Lemma 4.1. Applying the functor $\mathbf{\Phi}^!$ to the sequence (3.19), we get

$$0 \to \mathcal{L} \to \mathbf{\Phi}^!(F_d) \to \mathbf{\Phi}^!(F_{d-1}) \to 0. \tag{5.11}$$

Notice that the extension (5.11) is non-trivial because $\mathbf{\Phi}^!(F_d)$ is indecomposable since it is simple (see Proposition 5.4).

Since, by formulas (5.7), we know that $\mu(\mathbf{\Phi}^!(F_d)) = \frac{5d-24}{d-5} = 5 + \frac{1}{d-5}$, it is enough to prove that $\mathbf{\Phi}^!(F_d)$ is semistable. Assume by contradiction that there exists a subsheaf $\mathcal{K}$ destabilizing $\mathbf{\Phi}^!(F_d)$ of rank $r < d - 5$ and degree c. Since $\mathbf{\Phi}^!(F_{d-1})$ is stable, we must have

$$5 + \frac{1}{d-5} < \frac{c}{r} \leq 5 + \frac{1}{d-6},$$

from which we get

$$0 < \frac{c}{r} - \frac{5d-24}{d-5} \leq \frac{1}{(d-5)(d-6)}.$$

It is easy to check that the only possibility is $r = d - 6$ and $c = 5d - 29$, and so we would have $\mathcal{K} \cong \mathbf{\Phi}^!(F_{d-1})$ and (5.11) would split, a contradiction. Hence $\mathbf{\Phi}^!(F_d)$ is stable. Therefore the same holds for a general point of $\mathsf{M}(d)$ by Maruyama's result [Mar76]. □

Lemma 5.7. *Let $d \geq 6$ and let F be a locally free sheaf in $\Omega(d)$. Then $\mathbf{\Phi}^!(F)$ is globally generated and we have the exact sequence*

$$0 \to (\mathbf{\Phi}^!(F))^* \to \mathrm{Ext}^2_X(F,\mathcal{U}_+)^* \otimes \mathscr{O}_\Gamma \to \mathbf{\Phi}^!(F) \to 0. \tag{5.12}$$

Proof. Consider the complex $\mathbf{\Phi}^*(F)$. Let us compute the stalk over the point $y \in \Gamma$ of the sheaf $\mathcal{H}^k(\mathbf{\Phi}^*(F))$. We obtain:

$$\mathcal{H}^{-k}(\mathbf{\Phi}^*(F))_y \cong \mathrm{Ext}^k_X(F, \mathscr{E}_y)^*.$$

For $k = 3$, this group vanishes by stability of F and $\mathscr{E}_y$. For $k = 2$, this group vanishes as well. Indeed, applying the functor $\mathrm{Hom}_X(F, -)$ to (2.26), and by stability of $\mathscr{G}_y$, we are reduced to show that $\mathrm{Ext}^2_X(F, \mathcal{U}_+^*) = 0$. But this follows easily applying $\mathrm{Hom}_X(F, -)$ to (2.13).

Now observe that $\mathrm{Hom}_X(F, \mathscr{E}_y) = 0$ as soon as F is locally free. So $\mathbf{\Phi}^*(F)[-1]$ is a locally free sheaf. Moreover, applying (2.29) and using the definition of $\mathbf{\Phi}^*$ and $\mathbf{\Phi}^!$, we get:

$$\mathbf{\Phi}^*(F)[-1] \cong (\mathbf{\Phi}^!(F))^*.$$

Remark that, for any sheaf $\mathcal{F}$ on the curve Γ, since the functor $\mathbf{\Phi}$ is fully faithful, we have:

$$\mathbf{\Phi}^*(\mathbf{\Phi}(\mathcal{F})) \cong \mathcal{F}.$$

Thus, applying the functor $\mathbf{\Phi}^*$ to (5.8), we obtain, in view of (2.32), the exact sequence (5.12). Hence, the sheaf $\mathbf{\Phi}^!(F)$ is globally generated. $\square$

In the next section we will study the space $\mathsf{M}_X(2, 1, d)$, focusing first on the case $d \geq 7$, where we give a birational description. The case $d = 6$ will be treated in greater detail afterwards.

5.3. **The moduli spaces $\mathsf{M}_X(2, 1, d)$, with $d \geq 7$.** Here we show that the component $\mathsf{M}(d)$ of the variety $\mathsf{M}_X(2, 1, d)$ containing the sheaves arising from the construction of Theorem 3.12 is birational to a component $\mathsf{W}(d)$ of $W^{2\,d-11}_{d-5,5\,d-24}$. Recall that in Lemma 5.6 we have introduced the open set $\Omega(d) \subset \mathsf{M}(d)$. Every sheaf $F \in \Omega(d)$ satisfies the following two conditions:

i) the group $\mathrm{H}^1(X, F(-1))$ vanishes,
ii) the vector bundle $\mathcal{F} = \mathbf{\Phi}^!(F)$ is stable.

Theorem 5.8. *Let X be a smooth ordinary prime Fano threefold of genus 7, and let F be a sheaf in $\Omega(d)$. Then the map*

$$\varphi : \Omega(d) \to W^{2\,d-11}_{d-5,5\,d-24} \quad \varphi : F \mapsto \mathcal{F} = \mathbf{\Phi}^!(F)$$

is injective. Moreover, denoting by $\mathsf{W}(d)$ the irreducible component of $W^{2\,d-11}_{d-5,5\,d-24}$ containing the image of φ, we have that the tangent space (resp. the space of obstruction) to $\mathsf{W}(d)$ at the point $[\mathbf{\Phi}^!(F)]$ is naturally identified with $\mathrm{Ext}^1_X(F, F)$ (resp. with $\mathrm{Ext}^2_X(F, F)$). Therefore, $\mathsf{M}(d)$ and $\mathsf{W}(d)$ are birational varieties of dimension $2d - 9$.

Proof. Note that the map φ is well defined by Proposition 5.4 and Lemmas 5.5, 5.6. Set again $A_F = \mathrm{Ext}^2_X(F, \mathcal{U}_+)^*$ and $\mathcal{F} = \mathbf{\Phi}^!(F)$. Keep in mind the natural isomorphism (5.10), and the natural evaluation map:

$$e = e_{\mathscr{O}, \mathcal{F}} : A_F \otimes \mathscr{O}_\Gamma \to \mathcal{F}.$$

Recall that the transpose of the Petri map $\pi_{\mathcal{F}}^{\top}$ equals the map $\mathrm{Ext}^1_\Gamma(e, \mathcal{F})$, obtained applying $\mathrm{Hom}_\Gamma(-, \mathcal{F})$ to e.

In order to show that φ is injective, we consider the resolution (5.8) provided by Proposition 5.4, and we notice that the map ζ_F agrees, up to a non-zero scalar, with the map $\mathcal{H}^0(\mathbf{\Phi}(e))$, obtained taking the cohomology in degree zero of the map $\mathbf{\Phi}(e)$. Indeed, by Lemma 5.5 we get $\mathrm{Hom}_X(\mathcal{U}_+^*, \mathbf{\Phi}(\mathbf{\Phi}^!(F))) \cong A_F$, in particular ζ_F is uniquely determined up to a non-zero scalar. This proves that φ is injective, for $\mathrm{cok}(\mathcal{H}^0(\mathbf{\Phi}(e))) \cong F$ is the unique preimage of $\mathcal{F} = \mathbf{\Phi}^!(F)$.

Let us now setup the natural identifications of tangent and obstruction spaces. We have the natural isomorphisms:

$$\mathrm{Ext}^1_X(\mathbf{\Phi}(\mathcal{F}), F) \cong \mathrm{Ext}^1_\Gamma(\mathcal{F}, \mathcal{F}),$$

$$\mathrm{Ext}^1_X(\mathcal{U}_+^*, F) \cong \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathscr{O}_\Gamma), F) \cong \mathrm{Ext}^1_\Gamma(\mathscr{O}_\Gamma, \mathcal{F}). \tag{5.13}$$

Here, to prove (5.13), by (2.4) and (2.34) it suffices to show $\mathrm{Ext}^2_X(\mathcal{U}_+(1), F) = 0$ and $\mathrm{Ext}^3_X(\mathcal{U}_+(1), F) = 0$. By Serre duality we have $\mathrm{Ext}^k_X(\mathcal{U}_+(1), F)^* \cong \mathrm{Ext}^{3-k}_X(F, \mathcal{U}_+)$, which vanishes, for $k = 2, 3$, by Lemma 5.2. Now, applying the functor $\mathrm{Hom}_X(-, F)$ to (5.8) we obtain an exact sequence:

$$\mathrm{Ext}^1_X(F, F) \to \mathrm{Ext}^1_X(\mathbf{\Phi}(\mathcal{F}), F) \xrightarrow{\eta_F} \mathrm{Ext}^1_X(\mathcal{U}_+^*, F) \otimes A_F^* \to \mathrm{Ext}^2_X(F, F),$$

where the map η_F is defined as $\mathrm{Ext}^1_X(\zeta_F, F)$. But since $\zeta_F = \mathbf{\Phi}(e)$, we have:

$$\eta_F = \mathrm{Ext}^1_X(\mathbf{\Phi}(e), F) = \mathrm{Ext}^1_\Gamma(e, \mathbf{\Phi}^!(F)) = \mathrm{Ext}^1_\Gamma(e, \mathcal{F}) = \pi^\top_{\mathcal{F}}.$$

We have thus constructed the required identification of the tangent space to $W^{2\,d-11}_{d-5,5\,d-24}$ at the point $[\mathcal{F}]$ (i.e. of $\ker(\pi^\top_{\mathcal{F}})$) with $\mathrm{Ext}^1_X(F, F)$. The same argument identifies the obstruction space with $\mathrm{Ext}^2_X(F, F)$.

To finish the proof, recall by Theorem 3.12 that the component $\mathsf{M}(d)$ of $\mathsf{M}_X(2, 1, d)$ is generically smooth of dimension $2d - 9$. Therefore the same holds for the component of $W^{2\,d-11}_{d-5,5\,d-24}$ which contains the image of φ. We denote by $\mathsf{W}(d)$ this component. We have thus proved that φ is a birational map from $\mathsf{M}(d)$ to $\mathsf{W}(d)$. $\square$

5.4. **Universal sheaves.** For any $d \geq 6$, we let $\mathsf{P}(d)$ be the moduli space of stable vector bundles on Γ of rank $d - 5$ and degree $5\,d - 24$. Thus $\mathsf{W}(d)$ is a subvariety of $\mathsf{P}(d)$. Since the rank and the degree are coprime, it is well known that this moduli space is fine. So we denote by $\mathscr{P}$ the universal bundle over $\Gamma \times \mathsf{P}(d)$, and by abuse of notation to the product $\Gamma \times \mathsf{W}(d)$.

Theorem 5.9. *For $d \geq 7$, the moduli space $\Omega(d) \subset \mathsf{M}(d)$ is fine.*

Proof. We would like to exhibit a universal sheaf $\mathscr{F}$ over $X \times \Omega(d)$ such that, for a given closed point z of $\Omega(d)$ representing a stable sheaf F, the restriction of $\mathscr{F}$ to $X \times \{z\}$ is isomorphic to F.

Recall by Theorem 5.8 that φ maps $\Omega(d) \subset \mathsf{M}(d)$ to an open subset of $\mathsf{W}(d)$. Consider the projections:

$$\begin{tikzcd} & X \times \Gamma \times \Omega(d) \arrow[dl, "p\times 1"'] \arrow[dr, "q\times\varphi"] & \\ X \times \Omega(d) & & \Gamma \times \varphi(\Omega(d)) \end{tikzcd}$$

We consider the pull-back to $X \times \Gamma \times \Omega(d)$ of the map $\alpha : \mathcal{U}_+^* \to \mathscr{E}$ of (2.16). We tensor this map with $(q \times \varphi)^*(\mathscr{P})$. We have thus a morphism:

$$\mathcal{U}_+^* \boxtimes (q \times \varphi)^*(\mathscr{P}) \xrightarrow{\alpha \boxtimes 1} \mathscr{E} \boxtimes (q \times \varphi)^*(\mathscr{P}).$$

We define the universal sheaf $\mathscr{F}$ as the cokernel of the map $(p \times 1)_*(\alpha \boxtimes 1)$. Let us verify that $\mathscr{F}$ has the desired properties. So choose a closed point $z \in \Omega(d) \subset \mathsf{M}(d)$, and consider the corresponding sheaf F_z on X and the vector bundle $\mathscr{P}_{\varphi(z)} \cong \mathbf{\Phi}^!(F_z)$ on Γ. Notice that the sheaf $(q \times \varphi)^*(\mathscr{P}_{\varphi(z)})$ is just $q^*(\mathbf{\Phi}^!(F_z))$. Then, evaluating at the point z the map $(p \times 1)_*(\alpha \boxtimes 1)$ we obtain the map:

$$\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(F_z)) \otimes \mathcal{U}_+^* \to \mathbf{\Phi}(\mathbf{\Phi}^!(F_z)).$$

Recall the natural isomorphism $\mathrm{H}^0(\Gamma, \mathbf{\Phi}^!(F_z)) \cong \mathrm{Ext}^2_X(F_z, \mathcal{U}_+)^*$, and note that, by functoriality, this map must agree with the map ζ_{F_z} given by the resolution (5.8). Thus its cokernel is isomorphic to F_z. $\square$

5.5. **The moduli space $\mathsf{M}_X(2, 1, 6)$.** Here we focus on the moduli space $\mathsf{M}_X(2, 1, 6)$, and we prove that it is isomorphic to the Brill-Noether locus $W^1_{1,6}$ on the homologically projectively dual curve Γ. This makes more precise a result of Iliev-Markushevich, [IM07c]. Then we investigate the subvariety of $\mathsf{M}_X(2, 1, 6)$ consisting of vector bundles which are not globally generated. We will see that these bundles are in one-to-one correspondence with non-reflexive sheaves in $\mathsf{M}_X(2, 1, 6)$.

Finally we will see that these two subsets are interchanged by a natural involution of $\mathsf{M}_X(2,1,6)$. Here is our result.

Theorem 5.10. *Let X be a smooth prime Fano threefold of genus 7.*

A) The map $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ gives an isomorphism of the moduli space $\mathsf{M}_X(2,1,6)$ onto the Brill-Noether variety $W^1_{1,6}$. In particular, $\mathsf{M}_X(2,1,6)$ is a connected threefold. Moreover it is a fine moduli space.

B) If X is not exotic, then $\mathsf{M}_X(2,1,6)$ has at most finitely many singular points. If X is general, then $\mathsf{M}_X(2,1,6)$ is smooth and irreducible.

We prove now the first part of the theorem, while we postpone the second part to the end of the subsection.

Proof of Theorem 5.10, part A. First of all the map $\varphi : F \mapsto \mathbf{\Phi}^!(F)$ is well-defined. Indeed, let F be any sheaf in $\mathsf{M}_X(2,1,6)$. By Proposition 3.7, we know that F satisfies the hypothesis (5.1). Then by Proposition 5.4, $\mathbf{\Phi}^!(F)$ is a line bundle of degree 6 on Γ. Set $\mathcal{L} = \mathbf{\Phi}^!(F)$. We have to prove that $\mathcal{L}$ admits at least two independent global sections. If F is locally free, by Lemma 5.7 we have that $\mathcal{L}$ is globally generated. Moreover the exact sequence (5.12), implies $\mathrm{h}^0(\Gamma,\mathcal{L}) \geq 2$, since $\mathrm{h}^0(\Gamma,\mathcal{L}^*) = 0$. It follows that $\mathbf{\Phi}^!(F)$ lies in $W^1_{1,6}$. If F is not locally free, then it fits in the exact sequence (3.9). Recall that by Lemma 4.1 we know that $\mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ is a line bundle $\mathcal{M}$ contained in $W^1_{1,5}$. Hence, applying $\mathbf{\Phi}^!$ to the exact sequence (3.9), we obtain:

$$0 \to \mathcal{M} \to \mathcal{L} \to \mathscr{O}_y \to 0, \tag{5.14}$$

where y is a point of Γ. Therefore we have again $\mathrm{h}^0(\Gamma,\mathcal{L}) \geq \mathrm{h}^0(\Gamma,\mathcal{M}) \geq 2$, and $\mathbf{\Phi}^!(F)$ lies in $W^1_{1,6}$. Note that in this case the open subset $\Omega(6)$ coincides in fact with all of $\mathsf{M}_X(2,1,6)$. Then, with the same proof of Theorem 5.9 one can show that the moduli space $\mathsf{M}_X(2,1,6)$ is fine.

We prove now that the map φ is injective. Note that the equality $\mathrm{h}^0(\Gamma,\mathcal{L}) = 2$ must be attained for all $\mathcal{L}$, since $W^2_{1,6}$ is empty in view of Mukai's result (see [Muk95a, Table 1]). Then by the spectral sequence:

$$\mathrm{Hom}_X(\mathcal{U}_+^*, F) \oplus \mathrm{Ext}^1_X(\mathcal{U}_+(1), F) \Rightarrow \mathrm{Hom}_\Gamma(\mathscr{O}_\Gamma, \mathcal{L})$$

we obtain $\mathrm{Hom}_X(\mathcal{U}_+^*, F) = 0$. The same conclusion of Lemma 5.5 follows for $d = 6$. Therefore, the map ζ_F appearing in the resolution (5.8) provided by Proposition 5.4 is uniquely determined for any element $\mathcal{L}$ lying in the image of φ. In particular $\mathrm{cok}(\zeta_F) \cong F$ is the unique preimage of $\mathcal{L}$.

Since $\chi(F,F) = -2$, and clearly $\mathrm{Ext}^3_X(F,F) = 0$, by [HL97, Chapter 4.5], the space $\mathsf{M}_X(2,1,6)$ is a proper scheme of dimension at least 3. The dimension of $\mathsf{M}_X(2,1,6)$ follows, together with connectedness, for the variety $\mathsf{M}_X(2,1,6)$ embeds in $W^1_{1,6}$ which is a connected threefold (see for instance [ACGH85, IV, Theorem 5.1 and V, Theorem 1.4]).

We will now provide an inverse map of φ. Take a line bundle $\mathcal{L}$ in $W^1_{1,6}$, and denote by $e_{\mathcal{L}} = e_{\mathscr{O}_\Gamma,\mathcal{L}} : \mathrm{H}^0(\Gamma,\mathcal{L}) \otimes \mathscr{O}_\Gamma \to \mathcal{L}$ the natural evaluation map. We distinguish two cases according to whether $\mathcal{L}$ is globally generated or not. In the former, we have an exact sequence:

$$0 \to \mathcal{L}^* \to \mathrm{H}^0(\Gamma,\mathcal{L}) \otimes \mathscr{O}_\Gamma \xrightarrow{e_{\mathcal{L}}} \mathcal{L} \to 0. \tag{5.15}$$

Since, for any $x \in X$ the vector bundle $\mathscr{E}_x$ is stable, we have $\mathrm{H}^0(\Gamma, \mathscr{E}_x \otimes \mathcal{L}^*) = \mathrm{H}^1(\Gamma, \mathscr{E}_x \otimes \mathcal{L}) = 0$. Therefore, applying the functor $\mathbf{\Phi}$ to (5.15), and using the duality formula (2.28), we obtain an exact sequence:

$$0 \to \mathrm{H}^0(\Gamma,\mathcal{L}) \otimes \mathcal{U}_+^* \to \mathbf{\Phi}(\mathcal{L}) \to \mathbf{\Phi}(\mathcal{L})^*(1) \to \mathrm{H}^0(\Gamma,\mathcal{L})^* \otimes \mathcal{U}_+(1) \to 0.$$

It is easy to see that the image of the middle map in the above sequence is a reflexive sheaf F (by [Har80, Proposition 1.1]) of rank 2 with $c_1(F) = 1$, $c_2(F) = 6$, $c_3(F) = 0$, sitting in the following exact sequence:

$$0 \to \mathrm{H}^0(\Gamma, \mathcal{L}) \otimes \mathcal{U}_+^* \xrightarrow{\mathrm{H}^0(\mathbf{\Phi}(e_{\mathcal{L}}))} \mathbf{\Phi}(\mathcal{L}) \to F \to 0. \tag{5.16}$$

Note that F is locally free, so it is also stable, once we prove $\mathrm{Hom}_X(\mathscr{O}_X(1), F) = 0$. Recall that $\mathrm{Ext}^k_X(\mathscr{O}_X(1), \mathcal{U}_+^*) = 0$ for any integer k. Further, it is easy to see that $\mathbf{\Phi}^*(\mathscr{O}_X(1)) = 0$. Then, applying the functor $\mathrm{Hom}_X(\mathscr{O}_X(1), -)$ to (5.16), we have:

$$\mathrm{Hom}_X(\mathscr{O}_X(1), F) \cong \mathrm{Hom}_X(\mathscr{O}_X(1), \mathbf{\Phi}(\mathcal{L})) \cong \mathrm{Hom}_\Gamma(\mathbf{\Phi}^*(\mathscr{O}_X(1)), \mathcal{L}) = 0,$$

and so $F \in \mathsf{M}_X(2, 1, 6)$. Thus we can define $\omega(\mathcal{L}) = \mathrm{cok}(\mathrm{H}^0(\mathbf{\Phi}(e_{\mathcal{L}})))$. Applying the functor $\mathbf{\Phi}^!$ to (5.16), we find $\mathbf{\Phi}^!(F) \cong \mathcal{L}$, so $\varphi(\omega(\mathcal{L})) = \mathcal{L}$.

It remains to find an inverse image via φ of a non-globally generated sheaf $\mathcal{L}$. In this case, the image $\mathcal{M} \subset \mathcal{L}$ of $e_{\mathcal{L}}$ must be a line bundle, with $\mathrm{h}^0(\Gamma, \mathcal{M}) = \mathrm{h}^0(\Gamma, \mathcal{L}) = 2$. Then $\mathcal{M}$ must lie in $W^1_{1,5}$, since Γ has no g^1_4 by [Muk95a]. We have an exact sequence of the form (5.14), for some $y \in \Gamma$. Applying the functor $\mathbf{\Phi}$ to this sequence, by Lemma 4.1 and formula (4.3), we obtain:

$$0 \to A_L \otimes \mathcal{U}_+^* \xrightarrow{\mathrm{H}^0(\mathbf{\Phi}(e_{\mathcal{L}}))} \mathbf{\Phi}(\mathcal{L}) \to \mathscr{E}_y \to \mathscr{O}_L \to 0,$$

where L is the line contained in X such that $\mathcal{M} \cong \mathbf{\Phi}^!(\mathscr{O}_L)[-1]$ and $A_L = \mathrm{H}^0(\Gamma, \mathcal{M}) \cong \mathrm{H}^0(\Gamma, \mathcal{L})$. It is easy to see that the image of the middle map in the exact sequence above is a sheaf $F \in \mathsf{M}_X(2, 1, 6)$. We define again $\omega(\mathcal{L}) = \mathrm{cok}(\mathrm{H}^0(\mathbf{\Phi}(e_{\mathcal{L}})))$ and since $\mathbf{\Phi}^!(F) \cong \mathcal{L}$, it follows $\varphi(\omega(\mathcal{L})) = \mathcal{L}$.

By the proof of Theorem 5.8, we also have that $\omega \circ \varphi$ is the identity on $\mathsf{M}_X(2, 1, 6)$ and this completes the proof of part (A). $\square$

Now we will analyze the space $\mathsf{M}_X(2, 1, 6)$ in greater detail.

Lemma 5.11. *Let F be a sheaf in $\mathsf{M}_X(2, 1, 6)$. Then either F is globally generated, or there exists an exact sequence:*

$$0 \to I \to F \to \mathscr{O}_L(-1) \to 0, \tag{5.17}$$

where L is a line contained in X and I is a sheaf fitting into:

$$0 \to \mathscr{E}_y^* \to \mathrm{H}^0(X, F) \otimes \mathscr{O}_X \to I \to 0. \tag{5.18}$$

Proof. If the sheaf F fits into the exact sequence (5.17), it cannot be globally generated, since $\mathscr{O}_L(-1)$ has no global sections. So let us prove the converse implication.

Assume thus that F is not globally generated, let I (respectively, T and K) be the image (respectively, the cokernel and the kernel) of the natural evaluation map $e_{\mathscr{O},F} : \mathrm{H}^0(X, F) \otimes \mathscr{O}_X \to F$. By Proposition 3.7, we have $\mathrm{H}^k(X, F) = 0$, for each $k \neq 0$, and $\mathrm{h}^0(X, F) = 4$.

Note that, by definition of $e_{\mathscr{O},F}$ the sheaf I admits 4 independent global sections. Then, by stability of $\mathscr{O}_X$ and F, I must be a torsion-free sheaf of rank 2, with $c_1(I) = 1$, $c_2(I) \geq 6$, and I is stable as well. It is easy to see that K is thus a stable reflexive (by [Har80, Proposition 1.1]) sheaf of rank 2 with $c_1(K) = -1$, $c_2(K) = 12 - c_2(I)$. Then we have $c_3(K) \geq 0$ and by Lemma 3.1 it follows $c_2(K) \geq 5$.

Assume first that $c_2(I) = 7$. Then we can apply Proposition 3.5 to the sheaf $K(1)$ to prove that K is locally free. It follows that K is of the form $\mathscr{E}_y^*$ for some y by virtue of Theorem 2.3. So, using $\mathrm{H}^0(X, T) = 0$, we obtain that T is isomorphic to $\mathscr{O}_L(-1)$ by a Hilbert polynomial computation. This concludes the proof if $c_2(I) = 7$.

Let us assume now that $c_2(I) = 6$, which implies $c_2(K) = 6$ and $c_3(K) = -c_3(I) \geq 0$ for the sheaf K is reflexive. In this case Proposition 3.7 implies that

K is locally free so $c_3(K) = 0$. This implies $c_3(I) = 0$, so $T = 0$ and F is globally generated. $\square$

We can consider the closed subvarieties of the Brill-Noether variety $W^1_{1,6}$ described by the following conditions:

$$\mathsf{G} = \{\mathcal{L} \in W^1_{1,6} \,|\, \mathcal{L} \text{ is not globally generated}\} \tag{5.19}$$

$$\mathsf{C} = \{\mathcal{L} \in W^1_{1,6} \,|\, \mathcal{L} \text{ is contained in a line bundle lying in } W^2_{1,7}\}. \tag{5.20}$$

Moreover, we have the following involution:

$$\tau : W^1_{1,6} \to W^1_{1,6}, \qquad \mathcal{L} \mapsto \mathcal{L}^* \otimes \omega_\Gamma.$$

Proposition 5.12. *The sets* C *and* G *are interchanged by the involution* τ*, and are both isomorphic to the product* $\Gamma \times W^1_{1,5}$*. The intersection* $\mathsf{C} \cap \mathsf{G} \subset W^1_{1,6}$ *is an* 8*-tuple cover of the curve* $W^1_{1,5}$*.*

Proof. Given a line bundle $\mathcal{L}$ in G, we consider as in the proof of Theorem 5.10, part (A). the image $\mathcal{M} \subset \mathcal{L}$ of the natural evaluation map $e_{\mathscr{O}_\Gamma,\mathcal{L}}$. We have $\mathcal{M} \in W^1_{1,5}$ and an exact sequence of the form (5.14), for some $y \in \Gamma$. This defines a map $\mathsf{G} \to \Gamma \times W^1_{1,5}$, which is injective since $\mathrm{Ext}^1_\Gamma(\mathscr{O}_y, \mathcal{M}) \cong \mathbb{C}$. This map is surjective too, indeed the unique extension from $\mathscr{O}_y$ to $\mathcal{M}$ must lie in $W^1_{1,6}$, since $W^2_{1,6}$ is empty. Setting $\mathcal{N} = \mathcal{M}^* \otimes \omega_\Gamma$, we have:

$$\mathrm{h}^0(\Gamma, \mathcal{M}) = \mathrm{h}^1(\Gamma, \mathcal{N}) = 2, \qquad \mathrm{h}^1(\Gamma, \mathcal{M}) = \mathrm{h}^0(\Gamma, \mathcal{N}) = 3.$$

It follows that $\mathcal{N}$ lies in $W^2_{1,7}$. Dualizing the sequence (5.14) and tensoring by ω_Γ, we obtain the exact sequence:

$$0 \to \tau(\mathcal{L}) \to \mathcal{N} \to \mathscr{O}_y \to 0. \tag{5.21}$$

So the line bundle $\tau(\mathcal{L})$ lies in C. Since this procedure is reversible, we have proved that the involution τ interchanges the subsets G and C. Note that the map $\tau : \mathcal{M} \mapsto \mathcal{M}^* \otimes \omega_\Gamma$ gives an isomorphism from $W^1_{1,5}$ to $W^2_{1,7}$.

Let us now describe the intersection $\mathsf{C} \cap \mathsf{G} \subset W^1_{1,6}$. Recall that the map $\varphi_{|\mathcal{N}|}$ associated to a given $\mathcal{N} \in W^2_{1,7}$ embeds Γ into $\mathbb{P}^2$ as a septic. This septic is smooth away from 8 double points $y_1, \ldots, y_8$, see [IM07c, Lemma 2.6]. For each y_i we have a unique $\mathcal{M}_i \in W^1_{1,5}$ given by the projection from the double point y_i. On the other hand any subbundle $\mathcal{M} \in W^1_{1,5}$ of $\mathcal{N}$ must correspond to the projection from a double point.

Now fix a line bundle $\mathcal{N}$ in $W^2_{1,7} \cong W^1_{1,5}$. A subbundle $\mathcal{L} \in \mathsf{C}$ of $\mathcal{N}$ corresponds to the projection from a smooth point y as soon as $\mathcal{L}$ is globally generated. Therefore, $\mathcal{L}$ lies in $\mathsf{C} \cap \mathsf{G}$ if and only if we have:

$$\mathcal{M}_i \subset \mathcal{L} \subset \mathcal{N}, \qquad \text{for some } i = 1, \ldots, 8.$$

For each $\mathcal{M}_i$ there is a unique $\mathcal{L}_i$ satisfying the above condition. Thus we have realized $\mathsf{C} \cap \mathsf{G}$ as an 8-tuple cover of $W^1_{1,5}$. $\square$

We consider now the pull-back θ of τ to $\mathsf{M}_X(2,1,6)$, i.e. we set:

$$\theta : \mathsf{M}_X(2,1,6) \to \mathsf{M}_X(2,1,6), \qquad \theta = \varphi^{-1} \circ \tau \circ \varphi.$$

We will next show that θ can be seen on $\mathsf{M}_X(2,1,6)$ in terms of the functor T of Corollary 2.9.

Proposition 5.13. *Let* F *be an element of* $\mathsf{M}_X(2,1,6)$*. Then we have:*

i) the sheaf F *is not locally free if and only if* $\mathbf{\Phi}^!(F)$ *lies in* G*.*
ii) the sheaf F *is not globally generated if and only if* $\mathbf{\Phi}^!(F)$ *lies in* C*.*

Moreover the function θ is an involution which interchanges the two subsets of sheaves which are not locally free, and not globally generated.

Finally, for each F in $\mathsf{M}_X(2,1,6)$ we have:

$$\theta(F) = \varphi^{-1}(T(F)) = \varphi^{-1}\mathbf{\Phi}^!(\mathbf{R}\mathscr{H}om_X(F,\mathscr{O}_X))[1]. \tag{5.22}$$

Proof. We have already proved the implication "$\Leftarrow$" of (i) in Lemma 5.7. To prove the converse, we consider a sheaf F which is not locally free. Then F fits into an exact sequence of the form (3.9). Applying the functor $\mathbf{\Phi}^!$ to this sequence and setting $\mathcal{L} = \mathbf{\Phi}^!(F)$ we obtain an exact sequence of the form (5.14) for some $\mathcal{M}$ in $W^1_{1,5}$ (see Lemma 4.1). Since $\mathrm{H}^0(\Gamma,\mathcal{M}) \cong \mathrm{H}^0(\Gamma,\mathcal{L})$, the evaluation map $\mathrm{H}^0(\Gamma,\mathcal{L}) \otimes \mathscr{O}_\Gamma \to \mathcal{L}$ cannot be surjective, so $\mathcal{L}$ lies in G.

To prove (ii), in view of Lemma 5.11, we have to show that the line bundle $\mathbf{\Phi}^!(F)$ lies in C if and only if the sheaf F fits into (5.17), for some I fitting in (5.18). To show "$\Rightarrow$" of (ii), we let F be any such sheaf, and recall by Lemma 4.1 that $\mathscr{N} = \mathbf{\Phi}^!(\mathscr{O}_L(-1))$ lies in $W^2_{1,7}$. Since $\mathbf{\Phi}^!(\mathscr{O}_X) = 0$, by the exact sequence (5.18) we have:

$$\mathbf{\Phi}^!(I)[1] \cong \mathbf{\Phi}^!(\mathscr{E}_y) \cong \mathscr{O}_y.$$

Thus applying the functor $\mathbf{\Phi}^!$ to (5.17) we obtain an exact sequence:

$$0 \to \mathbf{\Phi}^!(F) \to \mathscr{N} \to \mathscr{O}_y \to 0,$$

and $\mathbf{\Phi}^!(F)$ lies in C.

To prove the converse implication, we consider a globally generated sheaf F and the exact sequence:

$$0 \to K \to \mathrm{H}^0(X,F) \otimes \mathscr{O}_X \to F \to 0. \tag{5.23}$$

Remark that K is a locally free sheaf and K^* lies in $\mathsf{M}_X(2,1,6)$ as well. We note that, applying (2.29), we get the natural isomorphism:

$$\mathbf{\Phi}^!(K) \cong \mathbf{\Phi}^!(K^*)^* \otimes \omega_\Gamma[-1].$$

On the other hand, by (5.23) we get $\mathbf{\Phi}^!(K) \cong \mathbf{\Phi}^!(F)[-1]$. Then we have:

$$\mathbf{\Phi}^!(F) \cong \tau(\mathbf{\Phi}^!(K^*)).$$

But $\mathbf{\Phi}^!(K^*)$ is globally generated by Lemma 5.7, hence we are done since τ interchanges C and G. We have thus established (i) and (ii).

It follows that θ interchanges the sheaves which are not locally free, and the sheaves which are not globally generated, and clearly θ is an involution.

To show the expression (5.22) of θ, recall that $\mathbf{\Phi}^! \circ T = \tau \circ \mathbf{\Phi}^!$ by Corollary 2.9. Therefore, for any F in $\mathsf{M}_X(2,1,6)$ we have $\theta(F) = \varphi^{-1}(\mathbf{\Phi}^!(T(F)))$. Since for any object a in $\mathbf{D^b}(X)$ we have $\mathbf{\Phi}^!(\mathbf{\Phi}(\mathbf{\Phi}^!(a))) \cong \mathbf{\Phi}^!(a)$, it follows that $\theta(F) = \varphi^{-1}\mathbf{\Phi}^!(\mathbf{R}\mathscr{H}om_X(F,\mathscr{O}_X))[1]$. □

The following lemma follows a suggestion of Dimitri Markushevich.

Lemma 5.14. *The set of singular points $[\mathcal{L}]$ of $W^1_{1,6}$, such that $\mathcal{L}$ is globally generated, is in bijection with the set of even effective theta-characteristics on Γ. In particular, this set if finite. Moreover, it is empty if Γ is outside a divisor in the moduli space of curves of genus 7, and of cardinality 1 if Γ is general in that divisor.*

Proof. According to Mukai's classification in [Muk95a]), the smooth curve section Γ of the spinor 10-fold satisfies $W^2_{1,6} = \emptyset$, and a general curve of genus 7 is of this form. Now recall that a line bundle $\mathcal{L}$ lies in the singular locus of $W^1_{1,6}$ if and only if the Petri map:

$$\pi_\mathcal{L} : \mathrm{H}^0(\Gamma,\mathcal{L}) \otimes \mathrm{H}^0(\Gamma,\mathcal{L}^* \otimes \omega_\Gamma) \to \mathrm{H}^0(\Gamma,\omega_\Gamma).$$

is not injective. Note that the kernel of this map is isomorphic to $\mathrm{H}^0(\Gamma, \mathcal{L}^* \otimes \mathcal{L}^* \otimes \omega_\Gamma)$. Therefore, the above map is injective unless $\mathcal{L} \otimes \mathcal{L} \cong \omega_\Gamma$, which means that $\mathcal{L}$ is an even effective theta-characteristic.

By [TiB87, Theorem 2.16] the set of curves of genus 7 admitting an even effective theta-characteristic form a divisor in the moduli space of curves of genus 7, and the general curve in this divisor has precisely one even effective theta-characteristic. This concludes the proof. □

Proof of Theorem 5.10, part B. Let us assume X to be non-exotic, and prove that $\mathsf{M}_X(2,1,6)$ has at most finitely many singular points. In view of Theorem 5.10, part A, the space $\mathsf{M}_X(2,1,6)$ is isomorphic to $W^1_{1,6}$. The number of singular points of $W^1_{1,6}$ which correspond to globally generated line bundles is finite by Lemma 5.14.

We consider thus a line bundle $\mathcal{L} \in \mathsf{G}$ which is a singular point of $W^1_{1,6}$. Since τ is an isomorphism, we can also assume that $\tau(\mathcal{L})$ is not globally generated, i.e. $\mathcal{L} \in \mathsf{C} \cap \mathsf{G}$. In particular we must have an exact sequence of the form (5.14), and there exists $\mathcal{N} \in W^2_{1,7}$ such that $\mathcal{M} \subset \mathcal{L} \subset \mathcal{N}$. Applying τ, we also get $\tau(\mathcal{N}) \subset \tau(\mathcal{L}) \subset \tau(\mathcal{M})$. Now, as in the proof of Lemma 5.14, we note that $\tau(\mathcal{L})$ lies in the singular locus of $W^1_{1,6}$ if and only if the Petri map $\pi_{\mathcal{L}}$ is not injective. In this case the kernel is isomorphic to $\mathrm{H}^0(\Gamma, \tau(\mathcal{N})^* \otimes \mathcal{L})$. Since we assume that this space is non-zero, we have an inclusion $\tau(\mathcal{N}) \subset \mathcal{L}$. It follows that $\mathcal{M} \cong \tau(\mathcal{N})$, since $\mathcal{L}$ contains a unique line bundle lying in $W^1_{1,5}$. We have thus an inclusion $\mathcal{M} \subset \tau(\mathcal{M})$, which means $\mathrm{H}^0(\Gamma, \mathcal{M}^* \otimes \mathcal{M}^* \otimes \omega_\Gamma) \neq 0$, so that $\mathcal{M}$ is a singular point of $W^1_{1,5}$ (see Remark 4.2). Since $W^1_{1,5} \cong \mathscr{H}^0_1(X)$ by Lemma 4.1, and since X is not exotic, the number of singular points of this form in $W^1_{1,6}$ is finite, and we are done.

Finally, note that if X is general, then the curve Γ is general. Then it is well-known that $W^1_{1,6}$ is smooth and irreducible, see for instance [ACGH85, V, Theorem 1.6]. It follows that $\mathsf{M}_X(2,1,6)$ is a smooth irreducible threefold. □

5.6. **The space $\mathsf{M}_X(2,1,6)$ as a subspace of $\mathsf{M}_S(2,1,6)$.** In this section we let X be a *general* prime Fano threefold of genus 7. Let S be a general hyperplane section of X. Assume in particular that S is a K3 surface of Picard number 1 and sectional genus 7. In this paragraph we will show that $\mathsf{M}_X(2,1,6)$ is isomorphic to a Lagrangian submanifold of $\mathsf{M}_S(2,1,6)$. This provides an instance of a general remark of Tyurin, [Tyu04].

We want to prove now that given *any* sheaf $F \in \mathsf{M}_X(2,1,6)$, its restriction $F_S = F \otimes \mathscr{O}_S$ is stable. Assume first that F_S is locally free. By Remark 3.11, we know that $\mathrm{H}^1(X, F(-2)) = 0$. Hence, by the exact sequence (3.15) with $t = 0$, and by (3.10), it follows that $\mathrm{H}^0(S, F_S(-1)) = 0$. This implies that F_S is stable, by Hoppe's criterion. Assume now that F_S is not locally free. Then by Proposition 3.7 it fits in the exact sequence (3.9) for some stable vector bundle $E \in \mathsf{M}_X(2,1,5)$ and a line $L \subset X$. Since E is stable and ACM by Proposition 3.5, it easily follows by the restriction sequence that $\mathrm{H}^0(S, E_S(-1)) = 0$, hence E_S is stable. Then if there exists a destabilizing subsheaf of F_S, it would destabilize also E_S, a contradiction.

Hence we define a restriction map:

$$\rho_S : \mathsf{M}_X(2,1,6) \to \mathsf{M}_S(2,1,6), \qquad [F] \mapsto [F_S].$$

Lemma 5.15 (Tyurin)**.** *Let S be a general hyperplane section of X. Then the map ρ_S is a closed immersion.*

Proof. We prove that the differential $d(\rho_S)_{[F]}$ is injective at any point $[F]$ of $\mathsf{M}_X(2,1,6)$. Applying the functor $\mathrm{Hom}_X(F,-)$ to (3.15) (with $t = 0$) we get:

$$\mathrm{Ext}^1_X(F, F(-1)) \to \mathrm{Ext}^1_X(F, F) \xrightarrow{\delta} \mathrm{Ext}^1_X(F, F_S).$$

Note that the leftmost term in the above sequence vanishes since $\mathrm{Ext}^2_X(F,F)=0$, indeed $\mathsf{M}_X(2,1,6)$ is a non-singular threefold by Theorem 5.10, part B. So δ is injective. Recall that $\mathrm{Ext}^1_X(F,F)$ is naturally isomorphic to $T_{[F]}\mathsf{M}_X(2,1,6)$, hence we are done if we prove that $\mathrm{Ext}^1_X(F,F_S)$ is naturally isomorphic to $T_{[F_S]}\mathsf{M}_S(2,1,6)$.

Let us prove that $\mathrm{Ext}^1_X(F,F_S)\cong\mathrm{Ext}^1_S(F_S,F_S)$. Indeed, denoting by ι the inclusion of the surface S in X, we have

$$\mathrm{Ext}^1_X(F,F_S)\cong\mathrm{Ext}^1_X(F,\iota_*\iota^*F)\cong\mathrm{Ext}^1_S(\iota^*F,\iota^*F)\cong\mathrm{Ext}^1_S(F_S,F_S),$$

where the second isomorphism above holds if $\mathbf{L}_k\iota^*(F)=0$ for any $k>0$. But this is true by Proposition 3.7, if S does not contain L and we can assume by generality that S does not contain lines. This concludes the proof. □

Lemma 5.16. *Let F be a sheaf in $\mathsf{M}_X(2,1,6)$ and fix a general hyperplane section S. Then there is at most one sheaf $F'\not\cong F$ in $\mathsf{M}_X(2,1,6)$ such that $F_S\cong F'_S$.*

Moreover, the set of sheaves F admitting a sheaf F' satisfying the above condition is finite.

Proof. Assume that the sheaf F_S is isomorphic to F'_S. It is easy to see that this isomorphism lifts to an isomorphisms $F\cong F'$ if $\mathrm{Ext}^1(F,F'(-1))=0$. We assume thus that this group is non-trivial.

We assume first that F is globally generated. We have an exact sequence of the form:

$$0\to K\to \mathrm{H}^0(X,F)\otimes\mathscr{O}_X\to F\to 0, \tag{5.24}$$

and K is a reflexive sheaf by [Har80, Proposition 1.1], hence $K(1)$ is a locally free sheaf in $\mathsf{M}_X(2,1,6)$. Applying $\mathrm{Hom}_X(-,F'(-1))$ to (5.24), one obtains $\mathrm{Hom}_X(K,F'(-1))\neq 0$, so $F'\cong K(1)$. This implies the first statement.

Let us now turn to the second one. Note that F' is locally free. We consider the symmetric square of (5.24):

$$0\to\mathrm{Sym}^2(F')^*\to\mathrm{H}^0(X,F)\otimes(F')^*\to\wedge^2\mathrm{H}^0(X,F)\otimes\mathscr{O}_X\to\mathscr{O}_X(1)\to 0,$$

and we take global sections. Since $\mathrm{H}^0(X,(F')^*)=0$, $\wedge^2(F')^*\cong\mathscr{O}_X(-1)$ and $\mathrm{H}^1(X,\mathrm{Sym}^2(F')^*)\cong\mathrm{H}^1(X,(F')^*\otimes(F')^*)\cong\mathrm{Ext}^2_X(F',F')^*=0$, we obtain an injection $\iota_F:\wedge^2\mathrm{H}^0(X,F)\hookrightarrow\mathrm{H}^0(X,\mathscr{O}_X(1))$. Note that $\dim(\mathrm{cok}(\iota_F))=3$, hence setting $\Lambda_F=\mathbb{P}(\mathrm{cok}(\iota_F))\subset\mathbb{P}^8=\mathbb{P}(\mathrm{H}^0(X,\mathscr{O}_X(1)))$, we define a correspondence:

$$\Lambda:\mathsf{M}_X(2,1,6)\to\mathbb{G}(2,8),\quad \Lambda:F\mapsto\Lambda_F.$$

Clearly we have $\dim(\mathrm{Im}(\Lambda))\leq 3$.

Now we fix a general hyperplane section S. Taking global sections of the restriction of the symmetric square of (5.24), we obtain an exact commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \wedge^2\mathrm{H}^0(X,F) \arrow[r, "\iota_F"] \arrow[d, equal] & \mathrm{H}^0(X,\mathscr{O}_X(1)) \arrow[d] \\
0 \arrow[r] & \mathrm{H}^1(S,\mathrm{Sym}^2F_S^*) \arrow[r] & \wedge^2\mathrm{H}^0(S,F_S) \arrow[r] & \mathrm{H}^0(S,\mathscr{O}_S(1)).
\end{tikzcd} \tag{5.25}$$

Note that $\mathrm{H}^1(S,\mathrm{Sym}^2F_S^*)\neq 0$. Indeed since $K_S\cong F_S^*$, then the exact sequence (5.24) (restricted to S) provides a non-trivial element in $\mathrm{Ext}^1_S(F_S,F_S^*)\cong\mathrm{H}^1(S,F_S^*\otimes F_S^*)\cong\mathrm{H}^1(S,\mathrm{Sym}^2F_S^*)$, where we use $\wedge^2F_S^*\cong\mathscr{O}_S(-1)$.

Then the diagram (5.25) induces a projection $\mathrm{H}^0(S,\mathscr{O}_S(1))\to\mathrm{cok}(\iota_F)$ and so the hyperplane defining the surface S must contain Λ_F. We denote by $\mathbb{G}_S$ the set of planes of $\mathbb{G}(2,8)$ contained in $\mathbb{P}(\mathrm{H}^0(S,\mathscr{O}_S(1)))=\mathbb{P}^7$. We have proved that if ρ_S is not injective at $[F]$, then $\Lambda_F\in\mathbb{G}_S$. Clearly $\mathbb{G}_S\subset\mathbb{G}(2,8)$ is a subvariety of codimension 3 and corresponds to the choice of a general global section of the rank

3 universal bundle on $\mathbb{G}(2,8)$, which is globally generated. Hence, the set of planes contained in $\mathrm{Im}(\Lambda) \cap \mathbb{G}_S$ must be finite. This proves the second statement.

It remains to prove the claim when F is not globally generated (say over a line $L' \subset X$, see Lemma 5.11). The sheaf F_S thus fails to be globally generated over the point $x = L' \cap S$. In turn, the sheaf F' is not globally generated, say over a line $L \subset X$, and we must have either $L = L'$, or $L \cap L' = x$.

In the first case, we will prove that $F \cong F'$. Indeed by Lemma 5.11 we know that F and F' fit in (5.17), (5.18), and

$$0 \to I' \to F' \to \mathscr{O}_L(-1) \to 0, \tag{5.26}$$

$$0 \to \mathscr{E}_{y'}^* \to \mathrm{H}^0(X, F') \otimes \mathscr{O}_X \to I' \to 0. \tag{5.27}$$

Restricting the sequences (5.17) and (5.26) to S, we get $I_S \cong I'_S$. Note that the bundle $\mathscr{E}_{y'}^*$ is ACM by Remark 3.6, hence $\mathrm{H}^1(S, \mathscr{E}_{y'}^*) = 0$. Then from the sequences (5.18) and (5.27), we deduce that $(\mathscr{E}_y^*)_S \cong (\mathscr{E}_{y'}^*)_S$. But this implies the isomorphism $\mathscr{E}_y \cong \mathscr{E}_{y'}$, and thus the isomorphism $F \cong F'$. Indeed the restriction map from $\mathsf{M}_S(2,1,5) \to \mathsf{M}_X(2,1,5)$ is known to be injective, because it corresponds to the embedding of Γ as linear section of $\mathsf{M}_X(2,1,5)$.

Then we may assume $L \cap L' = x$. Using (5.17) and (5.18), it is easy to see that $\mathrm{Ext}^1_X(F, F'(-1)) = 0$ unless F is not locally free over the line L. Set $E = F^{**}$, and recall that E belongs to $\mathsf{M}(2,1,5)$. Then we have $(F'_S)^{**} \cong E_S$, hence $(F')^{**} \cong E$. Hence F' is not locally free either, and in turn its non-locally free locus must be L'. We have:

$$0 \to F' \to E \to \mathscr{O}_{L'} \to 0. \tag{5.28}$$

This proves the first statement. The set of pairs (F, F'), with $F_S \cong F'_S$, and F not globally generated, is in natural bijection with the set of pairs of lines (L, L') which meet at a point of S. Since S is general, the curve spanned by the intersection points of lines in $\mathscr{H}_1^0(X)$ meets S at a finite number of points. $\square$

Lemma 5.17. *Let F be a sheaf fitting into* (3.9) *and F' fitting in* (5.28). *Fix a general hyperplane section S and assume that $L \cap S = L' \cap S = x$. Then $F_S \cong F'_S$.*

Proof. Note that the sequences (3.9) and (5.28) induce two inclusions of $\mathrm{H}^0(X, F)$ and $\mathrm{H}^0(X, F')$ as subspaces of $\mathrm{H}^0(X, E)$. The intersection of these two subspaces is not empty. Let s be a non-zero global section of E which belongs to such intersection. Clearly s vanishes on a curve C which meets both L and L'. The zero locus of s as an element of $\mathrm{H}^0(X, F)$ (respectively, of $\mathrm{H}^0(X, F')$) is the curve $D = C \cup L$ (respectively, $D' = C \cup L'$), where C is the zero locus of s as an element of $\mathrm{H}^0(X, E)$. Thus $Z = D \cap S$ (resp. $Z' = D' \cap S$) is the vanishing locus of s as a global section of F_S (resp. of F'_S) and we have the two exact sequences

$$0 \to \mathscr{O}_S \to F_S \to \mathcal{I}_Z(1) \to 0, \qquad 0 \to \mathscr{O}_S \to F'_S \to \mathcal{I}'_Z(1) \to 0. \tag{5.29}$$

But since $L \cap L' = x \in S$, we have $Z = Z'$. Moreover by Lemma 3.4, we have $\mathrm{h}^1(S, \mathcal{I}_Z(1)) \leq 1$. So since $\mathrm{H}^1(S, \mathcal{I}_Z(1)) \cong \mathrm{Ext}^1_S(\mathcal{I}_Z(1), \mathscr{O}_S)^*$ by Serre duality, the two extensions in (5.29) are equivalent, and we obtain $F_S \cong F'_S$. $\square$

Recall that $\mathsf{M}_S(2,1,6)$ is a holomorphic symplectic manifold with respect to the Mukai form, see [Muk84].

Theorem 5.18. *Let X be a general prime Fano threefold of genus* 7, *S be a general hyperplane section of X, and let ρ_S be the restriction map from $\mathsf{M}_X(2,1,6)$ to $\mathsf{M}_S(2,1,6)$.*

The image $\rho_S(\mathsf{M}_X(2,1,6))$ is a Lagrangian subvariety of $\mathsf{M}_S(2,1,6)$. The singular locus of $\rho_S(\mathsf{M}_X(2,1,6))$ is non-empty and consists of finitely many double points.

Proof. Recall that since X is general, then $\mathsf{M}_X(2,1,6)$ is smooth, by Theorem 5.10, part B. We have seen in Lemmas 5.15 and 5.16 that ρ_S is a closed embedding outside a finite subset R of $\mathsf{M}_X(2,1,6)$. Lemma 5.17 implies that this set is non-empty.

By the proof of Lemmas 5.16 and 5.17, we have that the preimage of a singular point of $\rho_S(\mathsf{M}_X(2,1,6))$ consists of precisely two points of $\mathsf{M}_X(2,1,6)$, hence the singular locus consists of double points.

The image $\rho_S(\mathsf{M}_X(2,1,6)\setminus R)$ is a Lagrangian submanifold by a remark of Tyurin, see [Tyu04, Proposition 2.2]. □

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