POSTULATION OF GENERAL QUARTUPLE FAT POINT SCHEMES IN $\mathbf{P}^3$

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Abstract. We study the postulation of a general union Y of double, triple, and quartuple points of $\mathbf{P}^3$. We prove that Y has the expected postulation in degree $d \geq 41$, using the Horace differential lemma. We also discuss the cases of low degree with the aid of computer algebra.

1. Introduction

In this paper we study the postulation of general fat point schemes of $\mathbf{P}^3$. A *fat point* mP is a zero-dimensional subscheme of $\mathbf{P}^3$ supported at a point P and with $(\mathcal{I}_{P,\mathbf{P}^3})^m$ as its ideal sheaf. A *general fat point scheme* $Y = m_1P_1 + \ldots + m_kP_k$, with $m_1 \geq \ldots \geq m_k \geq 1$, is a general zero-dimensional scheme such that Y_{red} is a union of k points and for each i the connected component of Y supported at P_i is the fat point m_iP_i. We call multiplicity of Y the maximal multiplicity, m_1, of its components. We recall that $\mathrm{length}(mP) = \binom{m+2}{3}$, for any $m \geq 1$.

Studying the postulation of Y means to compute the dimension of the space of hypersurfaces of any degree containing the scheme Y. In other words this is equivalent to compute the dimension of the space of homogeneous polynomials of any degree vanishing at the point P_i and with all their derivatives, up to multiplicity $m_i - 1$, vanishing at P_i. We say that Y has *good postulation* if such a dimension is the expected one.

This problem has been investigated by many authors in the case of $\mathbf{P}^2$. In particular we recall the important Harbourne-Hirschowitz conjecture (see the survey [7] and references therein). This conjecture characterizes all the general fat point schemes not having good postulation, and has been proved in some special cases. We mention also an analogous conjecture in the case of $\mathbf{P}^3$, due to Laface and Ugaglia (see [11]). In the case of general unions of double points, that is when $m_i = 2$ for any i, the famous Alexander-Hirschowitz theorem gives a complete answer in the case of $\mathbf{P}^n$, for any $n \geq 2$, (see [1, 2], for a survey see [5]). For arbitrary multiplicities and arbitrary projective varieties there is a beautiful asymptotic theorem by Alexander and Hirschowitz [3].

Here we will study the case of general fat point scheme $Y \subset \mathbf{P}^3$ of multiplicity 4. The case of multiplicity 3 was considered by the first author in [4], where he proved that a general union $Y \subset \mathbf{P}^3$ of triple and double points has good postulation in degree $d \geq 7$.

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Our main result is the following:

Theorem 1. *Assume char*$(\mathbf{K}) \neq 2, 3$*. Fix non-negative integers* d, x, y, z *such that* $d \geq 41$*. Let* $Y \subset \mathbf{P}^3$ *be a general union of* x *4-points,* y *3-points and* z *2-points. Then* Y *has good postulation, i.e.*

- *if* $20x + 10y + 4z \leq \binom{d+3}{3}$*, then* $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_Y(d)) = 0$*,*
- *if* $20x + 10y + 4z \geq \binom{d+3}{3}$*, then* $\mathrm{h}^0(\mathbf{P}^3, \mathcal{I}_Y(d)) = 0$*.*

The proof is based on the well known *Horace differential lemma.* We point out that this asymptotic result is not proved by induction on the degree, hence it does not depend on the cases of low degree.

The cases where $d \leq 40$ can be analyzed with the help of computer algebra. We have checked that if $d \leq 8$ there exist some cases where a general fat point scheme Y of multiplicity 4 does not have good postulation in degree d. This happens in particular if the number of quartuple points contained in Y is high. On the other hand, we found that if $9 \leq d \leq 13$ any general fat point scheme Y of multiplicity 4 has good postulation in degree d. We expect that the same is true also for $14 \leq d \leq 40$, even if we did not perform the computations.

With the same kind of computation one may start to investigate the cases of fat point schemes of multiplicity higher than 4 for low degree. In Section 4.1 we have collected some partial results in this direction.

These numerical experiments lead us to pose the following question, which we believe is interesting even for low multiplicities cases:

Question 2. Let $Y \subset \mathbf{P}^n$ be a general fat point scheme of multiplicity $m \geq 2$. Let $d(n, m)$ be a function such that for any $d \geq d(n, m)$ the scheme Y has good postulation in degree d. For fixed n is it possible to take as $d(n, m)$ a function polynomial (or even linear) in m? Is it possible to take $d(3, m) = 3m$?

Note that by [11, Example 7.7] we know that $d(3, m) > 2m$. We also know that $d(3, m) > 2m + 1$. In fact, the referee suggested us the following example: 9 general 9-points of $\mathbf{P}^3$ have not good postulation in degree 19.

Notice that our question concerns an upper estimate which is not sharp. It seems difficult to find a sharp estimate, and of course it would be very interesting. For other results related to this subject see also [12] and [8].

Here is the plan of the paper. In Section 2 we give some preliminary lemmas. Section 3 is devoted to the proof of the main result of the paper (Theorem 1), while in Section 4 we give some details on the cases of low degree.

2. Preliminaries

Throughout the paper we will work on the n-dimensional projective space $\mathbf{P}^n$ over an algebraically closed field $\mathbf{K}$.

For any smooth n-dimensional connected projective variety A, any $P \in A$ and any integer $m > 0$, an *m-fat point of A* (or just m-point) $\{mP, A\}$ is defined to be the $(m-1)$-th infinitesimal neighborhood of P in A, i.e. the closed subscheme of A with $(\mathcal{I}_{P,A})^m$ as its ideal sheaf. Thus $\{mP, A\}_{red} = \{P\}$ and length$(\{mP, A\}) = \binom{n+m-1}{n}$. We will write mP instead of $\{mP, A\}$ when the space A is clear from the context, and mostly we will consider $A = \mathbf{P}^n$ for $n = 2, 3$. We call *general fat point scheme of* A a union $Y = m_1P_1 + \ldots + m_kP_k$, with $m_1 \geq \ldots \geq m_k \geq 1$, and $P1, \ldots, P_k$ general points of $\mathbf{P}^n$. We denote $\deg(Y) = \sum \mathrm{length}(m_iP_i)$.

Given a positive integer d, we will say that a zero-dimensional scheme Y of $\mathbf{P}^n$ has *good postulation in degree* d if the following conditions hold:

(a) if $\deg(Y) \leq \binom{n+d}{n}$, then $\mathrm{h}^1(\mathbf{P}^n, \mathcal{I}_Y(d)) = 0$,
(b) if $\deg(Y) \geq \binom{n+d}{n}$, then $\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_Y(d)) = 0$.

Given a general fat point scheme Y of $\mathbf{P}^n$ and a hyperplane $H \subset \mathbf{P}^n$ we will call *trace* of Y the subscheme $Y \cap H \subset H$ and *residual* of Y the scheme $\mathrm{Res}_H(Y) \subset \mathbf{P}^n$ with ideal sheaf $\mathcal{I}_Y : \mathcal{O}_{\mathbf{P}^n}(-H)$. Notice that if X is a m-point supported on H, then its trace $X \cap H$ is a m-point of H and its residual $\mathrm{Res}_H(X)$ is a $(m-1)$-point of $\mathbf{P}^n$.

The trace and the residual of a fat point scheme Y of $\mathbf{P}^n$ fit in the following well known *Castelnuovo exact sequence*

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Y)}(d-1) \to \mathcal{I}_Y(d) \to \mathcal{I}_{Y \cap H}(d) \to 0.$$

A straightforward consequence of the Castelnuovo exact sequence is the following form of the so called *Horace lemma*, which we will often use in the sequel. For more details see e.g. [5, Section 4].

Lemma 3. *Let $H \subset \mathbf{P}^n$ be a hyperplane and $Y \subset \mathbf{P}^n$ a fat point scheme of $\mathbf{P}^n$. Then we have*

$$\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_Y(d)) \leq \mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) + \mathrm{h}^0(H, \mathcal{I}_{Y \cap H}(d))$$

$$\mathrm{h}^1(\mathbf{P}^n, \mathcal{I}_Y(d)) \leq \mathrm{h}^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) + \mathrm{h}^1(H, \mathcal{I}_{Y \cap H}(d))$$

The basic tool we will need is the so called *Horace differential lemma*, introduced by Alexander and Hirschowitz. This technique allows us to take a *differential trace* and a *differential residual*, instead of the classical ones. For an explanation of the geometric intuition of the Horace differential lemma see [3, Section 2.1]. Here we give only an idea of how the lemma works.

Let Y be an m-point of $\mathbf{P}^n$ supported on a hyperplane $H \subset \mathbf{P}^n$. Following the language of Alexander and Hirschowitz we can describe Y as formed by infinitesimally piling up some subschemes of H, called *layers*. For example the layers of a 3-point $\{3P, \mathbf{P}^n\}$ are $\{3P, H\}$, $\{2P, H\}$, and $\{P, H\}$. Then the differential trace can be any of these layers and the differential residual is a *virtual* zero-dimensional scheme formed by the remaining layers. We will denote these virtual schemes by writing the subsequent layers from which they are formed. These layers are obtained intersecting with the hyperplane H and taking the residual many times. In particular the notation e.g. $X = (\{3P, H\}, \{2P, H\})$ means that $X \cap H = \{3P, H\}$ and $\mathrm{Res}_H(X) \cap H = \{2P, H\}$, and, finally, $\mathrm{Res}_H(\mathrm{Res}_H(X)) \cap H = \emptyset$.

In this paper we will apply several times the following result which is a particular case of the Horace differential lemma (see [3, Lemma 2.3]).

Lemma 4 (Alexander-Hirschowitz)**.** *Fix an integer $m \geq 2$ and assume that char$(\mathbf{K}) = 0$ or char$(\mathbf{K}) > m$. Let X be an m-point of $\mathbf{P}^n$ supported at P and $H \subset \mathbf{P}^n$ a hyperplane. Then for $i = 0, 1$ we have*

$$\mathrm{h}^i(\mathbf{P}^n, \mathcal{I}_X(d)) \leq \mathrm{h}^i(\mathbf{P}^n, \mathcal{I}_R(d-1)) + \mathrm{h}^i(H, \mathcal{I}_T(d))$$

where the differential residual R *and the* differential trace T *are virtual scheme of the following type:*

(i)	$m=2$:	$T=\{P,H\}$;	$R=\{2P,H\}$	$(1,3)$
(ii)	$m=3$:	$T=\{P,H\}$;	$R=(\{3P,H\},\{2P,H\})$	$(1,6,3)$
(iii)	$m=3$:	$T=\{2P,H\}$;	$R=(\{3P,H\},\{P,H\})$	$(3,6,1)$
(iv)	$m=4$:	$T=\{P,H\}$;	$R=(\{4P,H\},\{3P,H\},\{2P,H\})$	$(1,10,6,3)$
(v)	$m=4$:	$T=\{2P,H\}$;	$R=(\{4P,H\},\{3P,H\},\{P,H\})$	$(3,10,6,1)$
(vi)	$m=4$:	$T=\{3P,H\}$;	$R=(\{4P,H\},\{2P,H\},\{P,H\})$	$(6,10,3,1)$

In the previous lemma, for each case in the statement we write in the last column the list of the lengths of the fat points of H that we will obtain intersecting many times with H. Throughout the paper, when we will apply Lemma 4, we will specify which case we are considering by recalling this sequence of the lengths. For example if we apply Lemma 4, case (i), we will say the we apply the lemma with respect to the sequence $(1,3)$.

Remark 5. Let $X \subseteq Y \subset \mathbf{P}^n$ zero-dimensional schemes. Then it is immediate to see that $\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_Y(d)) \le \mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_X(d))$.

We recall here a particular case of a result of Mignon (see [10, Theorem 1]).

Lemma 6 (Mignon)**.** *Let $X \subset \mathbf{P}^2$ be a general fat point scheme of multiplicity* 4 *(that is a general collection of multiple points of multiplicity at most* 4*) and $d \ge 12$. Then X has good postulation, i.e. we have*

(a) *if* $\deg(X) \le \binom{d+2}{2}$, *then* $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_X(d)) = 0$,

(b) *if* $\deg(X) \ge \binom{d+2}{2}$, *then* $\mathrm{h}^0(\mathbf{P}^2, \mathcal{I}_X(d)) = 0$.

If $X \subset \mathbf{P}^2$ is a general fat point scheme of multiplicity 3*, and $d \ge 9$, then X has good postulation.*

The following lemma is equivalent to [4, Remark 2]. We give here a complete proof for the reader's convenience.

Lemma 7. *Fix integers $d > 0$, $z > 0$, $\gamma \ge 0$, a hyperplane $H \subset \mathbf{P}^n$ and a zero-dimensional scheme $Y \subset \mathbf{P}^n$. Let X be the union of Y and z general simple points supported on H. If the following conditions*

$$\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_Y(d)) \le \gamma + z, \quad \text{and} \quad \mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) \le \gamma, \tag{1}$$

take place, then it follows that

$$\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_X(d)) \le \gamma.$$

Equivalently if the following conditions

$$\mathrm{h}^1(\mathbf{P}^n, \mathcal{I}_Y(d)) \le \max(0, \gamma + \deg(X) - \binom{d+n}{n}) =: \beta,$$

and

$$\mathrm{h}^1(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) \le \max(0, \gamma + \deg(\mathrm{Res}_H(Y)) - \binom{d+n-1}{n}),$$

take place, then it follows that

$$\mathrm{h}^1(\mathbf{P}^n, \mathcal{I}_X(d)) \le \beta.$$

Proof. Notice that, since for any scheme Z and any integer d we have

$$\mathrm{h}^1(\mathbf{P}^n, \mathcal{I}_Z(d)) = \mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_Z(d)) - \binom{d+n}{n} + \deg(Z),$$

then the two formulations of the lemma are equivalent, since clearly we have $\deg(X) = \deg(Y) + z$.

Let us assume that the two conditions in (1) hold and, for any positive integer p, let us denote by Y_p the union of Y and p general simple points of H. Let r be the maximal integer p such that $\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_{Y_p}(d)) = \mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_Y(d)) - p$. Obviously $0 \leq r \leq \mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_Y(d)) \leq \gamma + z \leq z$. Since $Y_{r+1} \setminus Y_r$ is a general point of H, it follows that H is contained in the base locus of the linear system $|\mathcal{I}_{Y_r}(d)|$. This implies that $\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_{Y_r}(d)) = \mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_{\mathrm{Res}_H(Y)}(d-1)) \leq \gamma$. Then, since Y_r can be identified with a subscheme of X, by Remark 5, we conclude that $\mathrm{h}^0(\mathbf{P}^n, \mathcal{I}_X(d)) \leq \gamma$. □

The following numerical lemma will be used in the sequel.

Lemma 8. *Fix non-negative integers t, a, b, c, e, f, g such that $t \geq 14$,*

$$10a + 6b + 3c + u + 6e + 3f + g \leq \binom{t+2}{2} \tag{2}$$

and (e, f, g) is one of the following triples: $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,1,0)$, $(0,1,1)$, $(0,1,2)$, $(1,0,0)$, $(1,0,1)$, $(1,0,2)$, $(1,1,0)$. Then we get the following inequality

$$6a + 3b + c + 10e + 10f + 10g \leq \binom{t+1}{2}. \tag{3}$$

Moreover, if $e + f + g \leq 2$, then (3) holds for any $t \geq 12$. If $e = f = g = 0$, then (3) holds for any $t \geq 3$.

Proof. In order to prove (3), it is sufficient to check the inequality

$$4a + 3b + 2c + u - 4e - 7f - 9g \geq t + 1.$$

From (2) it follows that

$$10a \geq \binom{t+2}{2} - 6b - 3c - u - 9,$$

hence the inequality above comes from

$$\frac{2}{5}\left(\binom{t+2}{2} - 9\right) - 25 \geq t + 1,$$

which is true for all $t \geq 14$. If $e + f + g \leq 2$, then we have to check

$$\frac{2}{5}\left(\binom{t+2}{2} - 9\right) - 18 \geq t + 1,$$

which holds for any $t \geq 12$. Finally, the last statement follows easily. □

3. Proof of the main theorem

This section is devoted to the proof of Theorem 1. Throughout the section we assume that the characteristic of the base field $\mathbf{K}$ is different from 2 and 3, and we fix an hyperplane $H \subset \mathbf{P}^3$.

In the different steps of the proof we will work with zero-dimensional schemes a little more general than a union of fat points. In particular, we will say that a zero-dimensional scheme Y is *of type* $(\star)$ if its irreducible components are of the following type:

- m-points, with $2 \leq m \leq 4$ supported at general points of $\mathbf{P}^3$,

- m-points, with $1 \leq m \leq 4$, or virtual schemes arising as residual in the list of Lemma 4, supported at general points of H.

In the following lemma we describe a basic step that we will apply several times in the sequel.

Lemma 9. *Let Y be a zero-dimensional scheme of type $(\star)$. For $2 \leq i \leq 4$, let c_i be the number of i-points of Y not supported in H. If the following condition holds*

$$\beta := \binom{t+2}{2} - \deg(Y \cap H) \geq 0, \tag{4}$$

then it is possible to degenerate Y to a scheme X such that one of the following possibilities is verified:

(I) $\deg(X \cap H) = \binom{t+2}{2}$,

(II) $\deg(X \cap H) < \binom{t+2}{2}$, *and all the irreducible components of X are supported on H. This is possible only if $c_2 + c_3 + c_4 \leq 2$ and $c_2 + c_3 + c_4 < \beta$.*

In both cases we also have

$$\deg(\mathrm{Res}_H(X) \cap H) \leq \binom{t+1}{2} \tag{5}$$

Proof. First of all we can assume that $\beta \geq 0$ is minimal. Indeed we can change the scheme Y by specializing on H some other component which are not supported on H. Let us denote now by Y' the union of the connected components of Y intersecting H.

By minimality of β it follows that if $c_2 > 0$ then $\beta < 3$, if $c_2 = 0$ and $c_3 > 0$ then $\beta < 6$, if $c_2 = c_3 = 0$ and $c_4 > 0$ then $\beta < 10$. If $c_2 = c_3 = c_4 = 0$ and $\beta > 0$, we are obviously in case (II).

We degenerate now Y to a scheme X described as follows. The scheme X contains all the connected components of Y'. Write

$$\beta = 6e + 3f + g$$

for a unique triple of non-negative integers (e, f, g) in the following list: $(0,0,0)$, $(0,0,1)$, $(0,0,2)$, $(0,1,0)$, $(0,1,1)$, $(0,1,2)$, $(1,0,0)$, $(1,0,1)$, $(1,0,2)$, $(1,1,0)$ (i.e. in the list of Lemma 8). Notice that if $c_2 > 0$ then $e = f = 0$ and $g \leq 2$, if $c_2 = 0$ and $c_3 > 0$ then $e = 0$ and $f + g \leq 3$, if $c_2 = c_3 = 0$ and $c_4 > 0$ then $e + f + g \leq 3$.

Consider first the case $c_2 > 0$ and recall that in this case $e = f = 0$ and $g \leq 2$. Assume now $c_2 \geq g$. Take as X a general union of Y', c_4 4-points, c_3 3-points, $(c_2 - g)$ 2-points, g virtual schemes obtained applying Lemma 4 at g general points of H with respect to the sequence $(1, 3)$. Clearly we have $\deg(X \cap H) = \binom{t+2}{2}$ and we are in case (I).

Let us see now how to specialize Y to X in the remaining cases. If $c_2 = 1 < g$ and $c_3 + c_4 \geq 1$, then in the previous step we apply Lemma 4 using the unique 2-point and one 3-point (or 4-point respectively) with respect to the sequence $(1, 6, 3)$ (or $(1, 10, 6, 3)$ respectively) and we conclude in the same way, obtaining (I). If $c_2 = 1 < g$ and $c_3 = c_4 = 0$, then we apply Lemma 4 to the unique double point with respect to the sequence $(1, 3)$, and we are in case (II).

Assume now $c_2 = 0$ and $c_3 > 0$. If $c_3 \geq f + g$ we take as X a general union of Y', c_4 4-points, $c_3 - f - g$ 3-points, f virtual schemes obtained applying Lemma 4 at f general points of H with respect to the sequence $(3, 6, 1)$ and g virtual schemes obtained applying Lemma 4 at g general points of H with respect to the sequence

$(1,6,3)$. If $0 < c_3 < f+g$ and $c_4 \geq f+g-c_3$, then in the previous step we apply Lemma 4 using c_3 3-point, and $(f+g-c_3)$ 4-points, with respect to the sequences $(3,10,6,1)$ or $(1,10,6,3)$. In all these cases we clearly have $\deg(X \cap H) = \binom{t+2}{2}$, so we are in case (I). If $c_2 = 0$, $0 < c_3 < f+g$ and $c_4 < f+g-c_3$, then we have either $c_3 \leq 1$ and $c_4 \leq 1$, or $c_3 = 2$ and $c_4 = 0$, and in both cases $\beta > c_3 + c_4$. In this cases we can specialize all the components on H, possibly applying Lemma 4 and we are in case (II).

Now, assume that $c_2 = c_3 = 0$ and $c_4 > 0$. If $c_4 \geq e+f+g$, then we take as X a general union of Y', $(c_4-e-f-g)$ 4-points, e virtual schemes obtained applying Lemma 4 at e general points of H with respect to the sequence $(6,10,3,1)$, f virtual schemes obtained applying Lemma 4 at f general points of H with respect to the sequence $(3,10,6,1)$ and g virtual schemes obtained applying Lemma 4 at g general points of H with respect to the sequence $(1,10,6,3)$. Thus we have again $\deg(X \cap H) = \binom{t+2}{2}$, that is we are in case (I). If $c_2 = c_3 = 0$ and $0 < c_4 < e+f+g$, then we are in case (II), because we can specialize all the quartuple points on H (possibly applying Lemma 4), since $c_4 \leq e+f+g+1 \leq 2$ and $\beta > c_4$.

Finally, we note that the property (5) follows immediately by the construction above and by Lemma 8. □

Given a scheme Y of type $(\star)$ satisfying (4), we will say that Y is of type (I) if, when we apply Lemma 9 to Y, we are in case (I). Otherwise we say that Y is of type (II).

We fix now (and we will use throughout this section) the following notation, for any integer t: given a scheme Y_t of type $(\star)$ and satisfying (4), we will denote by X_t the specialization described in Lemma 9. We write the residual $\mathrm{Res}_H(X_t) = Y_{t-1} \cup Z_{t-1}$, where Y_{t-1} is the union of all unreduced components of $\mathrm{Res}_H(X_t)$ and $Z_{t-1} = \mathrm{Res}_H(X_t) \setminus Y_{t-1}$. Clearly Z_{t-1} is the union of finitely many simple points of H. Thus at each step $t \mapsto t-1$, we will have

$$Y_t \mapsto X_t \mapsto \mathrm{Res}_H(X_t) = Y_{t-1} \cup Z_{t-1}.$$

For any integer t, we set $z_t := \sharp(Z_t)$, $\alpha_t := \deg(Y_t) = \deg(X_t)$, and

$$\delta_t := \max\left(0, \binom{t+2}{3} - \deg(Y_{t-1} \cup Z_{t-1})\right).$$

We fix the following statements:

- $\mathsf{A}(t) = \{Y_t \text{ has good postulation in degree } t\}$,
- $\mathsf{B}(t) = \{\mathrm{Res}_H(X_t) \text{ has good postulation in degree } t-1\}$,
- $\mathsf{C}(t) = \{\mathrm{h}^0(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(Y_{t-1})}(t-2)) \leq \delta_t\}$.

Claim 10. Fix $t \geq 13$. If Y_t is a zero-dimensional scheme of type (II), then it has good postulation, i.e. $\mathsf{A}(t)$ is true. Moreover also $\mathsf{B}(t)$ is true.

Proof. Since Y_t is of type (II), when we apply Lemma 9 to Y_t, we obtain a specialization X_t whose all irreducible components are supported on H and such that $\deg(X_t \cap H) \leq \binom{t+1}{2}$.

We prove now the vanishing $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_Y(t)) = 0$. By semicontinuity, it is enough to prove the vanishing $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_X(t)) = 0$. Notice that by taking the residual of X_t with respect to H for at most four times we get at the end the empty set.

Since $\deg(X_t \cap H) \leq \binom{t+2}{2}$, and $t \geq 12$, by Lemma 6 it follows the vanishing $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{X \cap H}(t)) = 0$. Let us denote by R_{t-1} the residual $\mathrm{Res}_H(X)$ and recall that any component of R_{t-1} is supported on H. We need to check now that $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{R_{t-1}}(t)) = 0$.

In order to do this we take again the trace and the residual with respect to H. By (5) we know that $\deg(\mathrm{Res}_H(X_t) \cap H) \le \binom{t+1}{2}$ then again by Lemma 6, since $t-1 \ge 12$, we have $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{R_{t-1}\cap H}(t-1)) = 0$.

We repeat now this step taking $R_{t-2} := \mathrm{Res}_H(R_{t-1})$ and noting that the trace $R_{t-2} \cap H$ has degree less or equal than $\binom{t}{2}$, by Lemma 8. Moreover this time the scheme $R_{t-2} \cap H$ cannot contain quartuple points, in fact it is a general union of triple, double and simple points. Hence by Lemma 6, since $t-2 \ge 9$ we have $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{R_{t-2}\cap H}(t-2)) = 0$.

We repeat once again the same step and we obtain $R_{t-3} := \mathrm{Res}_H(R_{t-2})$. Now the trace $R_{t-3} \cap H$ contains only double or simple points and so we have again the vanishing $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{R_{t-3}\cap H}(t-3)) = 0$, by the Alexander-Hirschowitz Theorem, since $t-3 \ge 5$. Notice that this time the residual $\mathrm{Res}_H(R_{t-3})$ must be empty and so, since $\mathcal{I}_{\mathrm{Res}_H(R_{t-3})} = \mathcal{O}_{\mathbf{P}^3}$, we obviously have $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(R_{t-3})}(t-4)) = 0$. Hence thanks to Lemma 3 we obtain $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{Y_t}(t)) = 0$.

We also know that

$$\deg(Y_t) = \deg(X_t) \le \binom{t+2}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t-1}{2} \le \binom{t+3}{3} \tag{6}$$

where the second inequality is equivalent to $\binom{t-1}{3} \ge 0$, which is true for any $t \ge 4$. Hence it follows that Y_t has good postulation, that is $\mathsf{A}(t)$ is true.

It is easy to see that also the scheme $\mathrm{Res}(X_t)$ must be of type (II) with respect to degree $t-1$. Hence $\mathsf{B}(t)$ follows from the first part of the proof. □

Claim 11. Fix $t \ge 12$. If Y_t is a zero-dimensional scheme of type (I), then $\mathsf{A}(t)$ is true if $\mathsf{B}(t)$ is true.

Proof. Since Y_t is of type (I), we can apply Lemma 9 and we obtain a specialization X_t such that $\deg(X_t \cap H) = \binom{t+1}{2}$. Thus, by Lemma 6 it follows

$$\mathrm{h}^0(H, \mathcal{I}_{X_t\cap H}(t)) = \mathrm{h}^1(H, \mathcal{I}_{X_t\cap H}(t)) = 0.$$

Then, thanks to Lemma 3, it follows, for $i = 0, 1$,

$$\mathrm{h}^i(\mathbf{P}^3, \mathcal{I}_{X_t}(t)) = \mathrm{h}^i(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(X_t)}(t-1)).$$

Thus in order to prove that the scheme X_t has good postulation in degree t, it is sufficient to check the good postulation of $\mathrm{Res}_H(X_t)$ in degree $t-1$. □

Claim 12. If $\mathsf{A}(t-1)$ and $\mathsf{C}(t)$ are true, then $\mathsf{B}(t)$ is true.

Proof. Recall that we write $\mathrm{Res}_H(X_t) = Y_{t-1} \cup Z_{t-1}$, where Z_{t-1} is a union of simple points supported on H.

By Lemma 7, to check that the scheme $\mathrm{Res}_H(X_t)$ has good postulation in degree $t-1$ (i.e. $\mathsf{B}(t)$), it is sufficient to check the good postulation of Y_{t-1} in degree $t-1$ (i.e. $\mathsf{A}(t-1)$) and to prove that $\mathsf{C}(t)$ is true. □

Claim 13. If Y_t is of type (I), then $\mathsf{B}(t-1)$ implies $\mathsf{C}(t)$.

Proof. The statement $\mathsf{C}(t)$ is true if $\mathrm{h}^0(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(Y_{t-1})}(t-2)) \le \delta_t$.

Note that since $\deg(X_t \cap H) = \binom{t+2}{2}$, we have

$$\deg(\mathrm{Res}_H(X_t)) = \deg(Y_{t-1} \cup Z_{t-1}) = \alpha_{t-1} + z_{t-1} = \alpha_t - \binom{t+2}{2},$$

and thus it follows

$$\delta_t := \max\left(0, \binom{t+2}{3} - \alpha_{t-1} - z_{t-1}\right) = \max\left(0, \binom{t+3}{3} - \alpha_t\right).$$

Notice that, by (5) we have $\deg(\mathrm{Res}_H(X_t) \cap H) \leq \binom{t+1}{2}$. Hence, it follows

$$\deg(\mathrm{Res}_H(Y_{t-1})) = \deg(\mathrm{Res}_H(\mathrm{Res}_H(X_t))) \geq \alpha_t - \binom{t+2}{2} - \binom{t+1}{2}$$

and then, since $\binom{t+2}{2} + \binom{t+1}{2} = \binom{t+3}{3} - \binom{t+1}{3}$, we get

$$\deg(\mathrm{Res}_H(Y_{t-1})) \geq \binom{t+1}{3} - \binom{t+3}{3} + \alpha_t \geq \binom{t+1}{3} - \delta_t.$$

So in order to prove $\mathsf{C}(t)$ it is enough to prove that $\mathrm{Res}_H(Y_{t-1})$ has good postulation in degree $t-2$. $\square$

Now we are in position to prove our main result. In the following diagram we sketch the steps of the proof:

$$\begin{tikzcd}
\boxed{\text{Given } Y_t} \arrow[r, "Y_t \text{ of type II}"] \arrow[d, "Y_t \text{ of type I}"] & \boxed{\mathsf{A}(t) \text{ true by Claim 10}} \\
\boxed{\mathsf{B}(t) \Rightarrow \mathsf{A}(t) \text{ by Claim 11}} \arrow[d] & \\
\boxed{\mathsf{A}(t-1) + \mathsf{C}(t) \Rightarrow \mathsf{B}(t) \text{ by Claim 12}} \arrow[r, "Y_{t-1}", "\text{of type II}"'] \arrow[d, "Y_{t-1} \text{ of type I}"] & \boxed{\begin{array}{c} \mathsf{B}(t-1) \Rightarrow \mathsf{C}(t) \text{ by Claim 13} \\ \mathsf{B}(t-1) \text{ and } \mathsf{A}(t-1) \text{ true by Claim 10} \end{array}} \\
\boxed{\begin{array}{c} \mathsf{B}(t-1) \Rightarrow \mathsf{A}(t-1) \text{ by Claim 11} \\ \mathsf{B}(t-1) \Rightarrow \mathsf{C}(t) \text{ by Claim 13} \end{array}} \arrow[d] & \\
\boxed{\mathsf{A}(t-2) + \mathsf{C}(t-1) \Rightarrow \mathsf{B}(t-1) \text{ by Claim 12}} \arrow[r, "Y_{t-2}", "\text{of type II}"'] \arrow[d, "Y_{t-2} \text{ of type I}"] & \cdots \\
\cdots &
\end{tikzcd}$$

Proof of Theorem 1. Fix an integer $d \geq 41$ and a plane $H \subset \mathbf{P}^3$. For all non negative integers d, x, y, z, w, set

$$\epsilon(d, x, y, z) := \binom{d+3}{3} - 20x - 10y - 4z.$$

Notice that $\epsilon(d, x, y, z+1) = \epsilon(d, x, y, z) - 4$, $\epsilon(d, x, y+1, 0) = \epsilon(d, x, y, 0) - 10$ and $\epsilon(d, x+1, 0, 0) = \epsilon(d, x, 0, 0) - 20$. Hence to prove our statement for all triples (x, y, z) it is sufficient to check it for all triples (x, y, z) such that $-19 \leq \epsilon(d, x, y, z) \leq 3$.

We fix any such triple and a general union Y of x 4-points, y 3-points and z 2-points. We also set $\epsilon = \epsilon(d, x, y, z)$. We want to prove that Y has good postulation.

Notice that

$$x+y+z \geq \left\lceil \frac{1}{20}\left(\binom{d+3}{3}-3\right)\right\rceil \geq \frac{1}{20}\binom{d+3}{3}-\frac{3}{20}, \tag{7}$$

i.e. the scheme Y has at least $\lceil \frac{1}{20}(\binom{d+3}{3}-3)\rceil$ connected components.

Now we proceed by induction following the steps sketched in the diagram above. Set $Y_d = Y$. We can assume by generality that $\deg(Y_d \cap H) \leq \binom{d+2}{2}$, hence we can apply Lemma 9, thus specializing the scheme Y_d to a scheme X_d. If Y_d is of type (II), then we conclude by Claim 10, since $d \geq 13$.

Hence we can assume that Y_d is of type (I), and so, since $d \geq 12$, by Claim 11 it is enough to check that the scheme $\mathrm{Res}_H(X_d)$ has good postulation in degree $d-1$. Now we write $\mathrm{Res}_H(X_d) = Y_{d-1} \cup Z_{d-1}$, where Y_{d-1} is the union of all unreduced components of $\mathrm{Res}_H(X_d)$ and $Z_{d-1} = \mathrm{Res}_H(X_d) \setminus Y_{d-1}$.

By Claim 12, it is enough to prove that $\mathsf{A}(d-1)$ and $\mathsf{C}(d)$ are true.

Notice that by (5) we get $\deg(Y_{d-1} \cap H) \leq \deg(\mathrm{Res}_H(X_d) \cap H) \leq \binom{d+1}{2}$. Hence Y_{d-1} satisfies condition (4) in degree $d-1$, then we can apply again Lemma 9. We have now two alternatives: either Y_{d-1} is of type (I) or of type (II). In both cases, we note that by Claim 13 the statement $\mathsf{C}(d)$ follows from $\mathsf{B}(d-1)$, since Y_d is of type (I).

Now assume that Y_{d-1} is of type (II). Then by Claim 10, since $d-1 \geq 13$ we know that $\mathsf{B}(d-1)$ and $\mathsf{A}(d-1)$ are true and this concludes the proof. It remains to consider the case Y_{d-1} of type (I). We apply again Claim 12 and we go on iterating the same steps.

Now in order to prove our statement we need to show that the number of steps in the procedure described above is finite. Moreover we need to check that every time we apply Claim 10 we are in degree ≥ 13 and every time we apply Claim 11 we are in degree ≥ 12.

In order to satisfy all these requirements it is enough to show that in a finite number of steps we arrive at a scheme of type (II) in degree ≥ 13.

Recall that we denote, for any integer t, by X_t the specialization described in Lemma 9, we write $\mathrm{Res}_H(X_t) = Y_{t-1} \cup Z_{t-1}$ and we set $z_t := \sharp(Z_t)$ and $\alpha_t := \deg(Y_t) = \deg(X_t)$.

Now we want to estimate the number of simple points we obtain iterating the steps above.

Let us assume that starting from the scheme Y_d we arrive in w steps at a scheme X_{d-w} in such a way that the case (II) never occurs. Assume also that in these w steps we apply γ times Lemma 4 with respect to sequences of type $(1,10,6,3), (1,6,3)$ or $(1,3)$. Since $g \leq 2$, by Lemma 9, we have $\gamma \leq 2w$. Notice also that the scheme X_{d-w} does not contain simple points, hence it contains at most $\frac{1}{3}\deg(X_{d-w}) = \frac{1}{3}\alpha_{d-w}$ irreducible components. Hence it follows that

$$\sum_{t=d-w}^{d-1} z_t \geq x+y+z-2w-\frac{\alpha_{d-w}}{3}. \tag{8}$$

Notice also that

$$\alpha_{d-w} = \binom{d-w+3}{3} - \epsilon - \sum_{t=d-w}^{d-1} z_t. \tag{9}$$

and so (8) implies

$$\sum_{t=d-w}^{d-1} z_t \geq \frac{3}{2}(x+y+z) - 3w - \frac{1}{2}\binom{d-w+3}{3} + \frac{\epsilon}{2}. \tag{10}$$

Moreover, setting $w = d$ in (10) and using (7), we get

$$\sum_{t=0}^{d-1} z_t \geq \frac{3}{40}\binom{d+3}{3} - 3d - 10 - \frac{9}{40}. \tag{11}$$

Assume now that v is the maximal integer w such that for w steps the case (II) is not verified. Now we prove that we must have $v \leq d$. Indeed the assumption $v > d$ would imply that after d steps we obtain as a residual a scheme X_0 of positive degree (at least 3). Hence we have

$$3 \leq \deg(X_0) = \alpha_0 = -\epsilon + 1 - \sum_{t=0}^{d-1} z_t \leq 20 - \sum_{t=0}^{d-1} z_t \leq 20$$

and so $\sum_{t=0}^{d-1} z_t \leq 17$, which contradicts (11) for $d \geq 17$.

Then let us assume $v \leq d$. Now we want to prove that $d - v \geq 13$, and this will conclude the proof.

From the assumption $d \geq 41$ we easily get the following inequality

$$\frac{1}{20}\binom{d+3}{3} - \frac{3}{20} - 19 \geq 2(d-13) + \binom{16}{3}, \tag{12}$$

and then, using (8) and (7), it follows

$$0 \leq \deg(X_{d-v}) = \deg(X_d) - \left(\binom{d+3}{3} - \binom{d+3-v}{3}\right) - \sum_{t=d-v}^{d-1} z_t \leq$$

$$\leq -\epsilon + \binom{d+3-v}{3} - \frac{1}{20}\binom{d+3}{3} + \frac{3}{20} + 2v + \frac{\deg(X_{d-v})}{3}.$$

From this we get, by using $\epsilon \geq -19$, $\deg(X_{d-v}) \geq 0$ and inequality (12)

$$0 \leq -\epsilon + \binom{d+3-v}{3} - \frac{1}{20}\binom{d+3}{3} + \frac{3}{20} + 2v - \frac{2}{3}\deg(X_{d-v}) \leq$$

$$\leq 19 + \binom{d+3-v}{3} - \frac{1}{20}\binom{d+3}{3} + \frac{3}{20} + 2v \leq$$

$$\leq \binom{d+3-v}{3} + 2(v-d+13) - \binom{16}{3} =: f(d-v).$$

It is easy to see that $f(d-v)$ is a nondecreasing function in the interval $d-v \geq 0$, such that $f(13) = 0$. Hence since $f(d-v) \geq 0$, it follows that $d - v \geq 13$, as we wanted. This concludes the proof of the theorem. $\square$

4. Cases of low degree: computer aided proofs

Here we give the results concerning the cases of low degree, that we obtained via numerical computations.

Theorem 14. *Assume that* $\mathbf{K}$ *is an algebraically closed field of characteristic* 0*. Fix non-negative integers* d, x, y, z *such that* $9 \le d \le 13$*. Let* $Y \subset \mathbf{P}^3$ *be a general union of* x *4-points,* y *3-points and* z *2-points. Then* Y *has good postulation.*

The proof is computer aided and uses the program Macaulay2 [9]. Basically we have to check that some matrices, randomly chosen, have maximal rank. For similar computations see also [6].

With the same tools, it is not difficult to check that Theorem 14 is false for $d \le 8$. For example if we consider $d = 8$ and Y given by 9 quartuple points, we expect that there is no hypersurfaces of degree 8 through Y, but we find that one such hypersurface exists, since the rank of the corresponding matrix is not maximal. Other counterexamples we have found are 8 quartuple points and 1 triple point, 8 quartuple points and 1 or 2 double points, 7 quartuple points and 2 triple points and 1 double point. Some of this cases are explained in [11, Example 7.7].

On the other hand, in order to prove Theorem 14 one has to check a huge number of cases. As an example we list below the Macaulay2 script which concerns the case $d = 12$. Running the script the computer checks more than 3000 cases, without founding exceptions. Clearly, with the same method, it is possible to check the remaining cases $14 \le d \le 40$. We did not perform this computation because they need too long time.

Notice that these computations are performed in characteristic 31991, and the result follows in characteristic zero too. Indeed an integer matrix has maximal rank in characteristic zero, if it has maximal rank in positive characteristic. Furthermore Theorem 14 holds for all positive characteristics with the possible exception of a finite number of values of the characteristic.

```
----------------------------------
KK=ZZ/31991;
E=KK[e_0..e_3];
d=12 --degree
N=binomial(d+3,3)

f=ideal(e_0..e_3);
fd=f^d;
T=gens gb(fd)

J=jacobian(T); Jd=J;
--matrix of first derivatives
JJ=jacobian(J);
Jt=submatrix(JJ,{0,1,2,3,5,6,7,10,11,15},{0..N-1});
--matrix of second derivatives: we choose the independent columns
JJJ=jacobian(Jt);
Jq=submatrix(JJJ,{0,2,5,7,9,10,11,14,15,17,19,23,27,30,31,34,35,37,38,39},{0..N-1});
--matrix of third derivatives: we choose the independent columns

mat=random(E^1,E^N)*0
h=1;
for z from 0 to ceiling(N/4) do
for y from 0 to ceiling(N/10) do
for x from 0 to ceiling(N/20) do
```

```
(
if ((20*x+10*y+4*z>N-4)and(20*x+10*y+4*z<N+20))
then (print(h,x,y,z), h=h+1,
mat=random(E^1,E^N)*0,
for i from 1 to z do (q=random(E^1,E^4),mat=(mat||sub(Jd,q))),
for i from 1 to y do (q=random(E^1,E^4),mat=(mat||sub(Jt,q))),
for i from 1 to x do (q=random(E^1,E^4),mat=(mat||sub(Jq,q))),
r=rank mat,
if ((20*x+10*y+4*z<N+1)and(r!=20*x+10*y+4*z)) then (print (x,y,z,20*x+10*y+4*z,r)),
if ((20*x+10*y+4*z>N)and(r!=N)) then (print (x,y,z,N,r))))
---------------------------------
```

4.1. **The higher multiplicity cases.** It is not difficult to modify the script above in order to perform some numerical experiments related to the higher multiplicity cases. Here we list some results obtained for schemes of multiplicity 5. We denote by c_i the number of i-points we consider.

$m = 5$, $d = 8$:

(c_5, c_4, c_3, c_2)	good postulation
$(5, 1, 0, 0)$	yes
$(4, 2, 0, 0)$	no
$(3, 3, 0, 0)$	no
$(3, 4, 0, 0)$	yes
$(2, 5, 0, 0)$	no
$(2, 6, 0, 0)$	yes
$(1, 7, 0, 0)$	yes
$(0, 9, 0, 0)$	no

$m = 5$, $d = 9$:

(c_5, c_4, c_3, c_2)	good postulation
$(7, 0, 0, 0)$	yes
$(6, 2, 0, 0)$	yes
$(5, 3, 0, 0)$	yes
$(4, 5, 0, 0)$	yes
$(3, 6, 0, 0)$	no
$(3, 7, 0, 0)$	yes
$(6, 0, 1, 0)$	no
$(6, 0, 2, 0)$	yes

$m = 5$, $d = 10$:

(c_5, c_4, c_3, c_2)	good postulation
$(9, 0, 0, 0)$	no
$(8, 1, 0, 0)$	no
$(7, 2, 0, 0)$	no
$(8, 2, 0, 0)$	yes
$(7, 3, 0, 0)$	yes
$(6, 4, 0, 0)$	yes
$(6, 5, 0, 0)$	yes
$(8, 0, 1, 0)$	no

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