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SIMPLICITY OF VECTOR BUNDLES ON $\mathbb{P}^n$ AND EXCEPTIONAL BUNDLES

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Introduction

Steiner bundles on a complex projective space $\mathbb{P}^{N-1} = \mathbb{P}(V)$ were defined by Dolgachev and Kapranov in [DK93] as the bundles F that admit a linear resolution of the form

$$0 \to I \otimes \mathcal{O}(-1) \to W \otimes \mathcal{O} \to F \to 0, \tag{0.0.1}$$

where I and W are vector spaces. Steiner bundles form the simplest and more natural family of vector bundles on projective spaces. As it is well known, they have rank $t - s \geq N - 1$, where $t = \dim W$ and $s = \dim I$. When the equality holds true, it can be proved that Steiner bundles are stable (see [AO94] or [BS92]) and hence, in particular, simple. The first example of a Steiner bundle with rank $N-1$ is the tangent bundle on $\mathbb{P}^{N-1}$, which is actually stable. More precisely, if we consider $s = 1$ and $t = N$ in (0.0.1), then we get $F \cong T_{\mathbb{P}^{N-1}} \otimes \mathcal{O}(-1)$. In this thesis we are interested in Steiner bundles with higher rank. Our main purpose is to characterize simplicity and stability of the generic Steiner bundle on $\mathbb{P}^{N-1}$, for $N \geq 3$.

As a first remark, we observe that F is the cokernel of a sheaf morphism

$$m : I \otimes \mathcal{O}(-1) \to W \otimes \mathcal{O}.$$

When a basis is fixed in each of the vector spaces V, I and W, the morphism m is represented by a $(s \times t)$-matrix, whose entries are linear forms in N variables, or, equivalently, by a three-dimensional matrix M of complex numbers of size $(s \times t \times N)$, i.e.

$$M \in \mathrm{Hom}(I \otimes \mathcal{O}(-1), W \otimes \mathcal{O}) \cong I^\vee \otimes W \otimes V =: H.$$

We say that F is a *generic* bundle if the corresponding matrix M is generic in the space H.

Our first result characterizes the simplicity of generic Steiner bundles. More precisely (see Theorem 3.2.1), we have the following

Theorem A. *Let F be a generic Steiner bundle on $\mathbb{P}(V)$, where $\dim V = N \geq 3$. Then*

$$F \text{ is simple} \qquad \Leftrightarrow \qquad \chi(\operatorname{End} F) \leq 1.$$

We point out that the genericity assumption cannot be dropped, because when $\operatorname{rk} F > N-1$ it is always possible to construct Steiner bundles which are decomposable, hence in particular non-simple. Moreover, it is easy to check that if E is simple then $\chi(\operatorname{End} E) \leq 1$, and this implication is true for every bundle on $\mathbb{P}^2$, even non-Steiner. On the other hand, the converse is not true in general: for example the generic bundle G on $\mathbb{P}^2$ with resolution

$$0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^4 \to \mathcal{O}^{16} \to G \to 0$$

satisfies $\chi(\operatorname{End} G) = -3$, and it can be shown that $\mathrm{h}^0(\operatorname{End} G) = 5$, therefore G is not simple.

When F is a Steiner bundle with resolution (0.0.1), we obtain $\chi(\operatorname{End} F) = s^2 + t^2 - Nst$. Since $\chi(\operatorname{End} F)$ is an integer, the condition $\chi(\operatorname{End} F) \leq 1$ splits in two cases: either

(i) $\chi(\operatorname{End} F) = s^2 + t^2 - Nst = 1$, or

(ii) $\chi(\operatorname{End} F) \leq 0$, i.e. $t \leq \frac{N+\sqrt{N^2-4}}{2}s$.

In the first case we have a hyperbola in the (s,t) plane, and we prove (see Lemma 3.1.4) that all the integer points on the hyperbola are of the form $(s,t) = (a_{k-1}, a_k)$, where

$$a_k = \frac{\left(\frac{N+\sqrt{N^2-4}}{2}\right)^k - \left(\frac{N-\sqrt{N^2-4}}{2}\right)^k}{\sqrt{N^2-4}}.$$

We observe that when $N = 3$, the sequence $\{a_k\}$ is exactly the odd part of the Fibonacci sequence. For any $N > 3$, we call the numbers $\{a_k\}$ *generalized Fibonacci numbers* and we verify that they satisfy a recurrence relation. We denote by E_k the bundles on $\mathbb{P}^{N-1}$ with resolution

$$0 \to \mathcal{O}(-1)^{a_{k-1}} \to \mathcal{O}^{a_k} \to E_k \to 0,$$

and we prove that they are exceptional bundles.

Exceptional bundles were introduced by Drézet and Le Potier in [DLP85] as a class of bundles on $\mathbb{P}^2$ without deformations. Exceptional bundles appeared as some sort of exceptional points in the study of the problem of stability of bundles on $\mathbb{P}^2$. Drézet and Le Potier showed that these vector bundles are uniquely determined by their slopes, and they described the set of possible slopes. Later, the school of Rudakov (see for example [Rud90]) generalized the definition of exceptional bundles on $\mathbb{P}^n$ and on other varieties. They developed a general axiomatic presentation of exceptional objects: the theory of helices.

Following the definition of Gorodentsev and Rudakov ([GR87]), we say that a bundle E on $\mathbb{P}^n$ is *exceptional* if $\mathrm{h}^0(\operatorname{End} E) = 1$ and $\mathrm{h}^i(\operatorname{End} E) = 0$, for all $i > 0$. We remark that the condition $\mathrm{h}^1(\operatorname{End} E) = 0$ implies that E has no infinitesimal deformations.

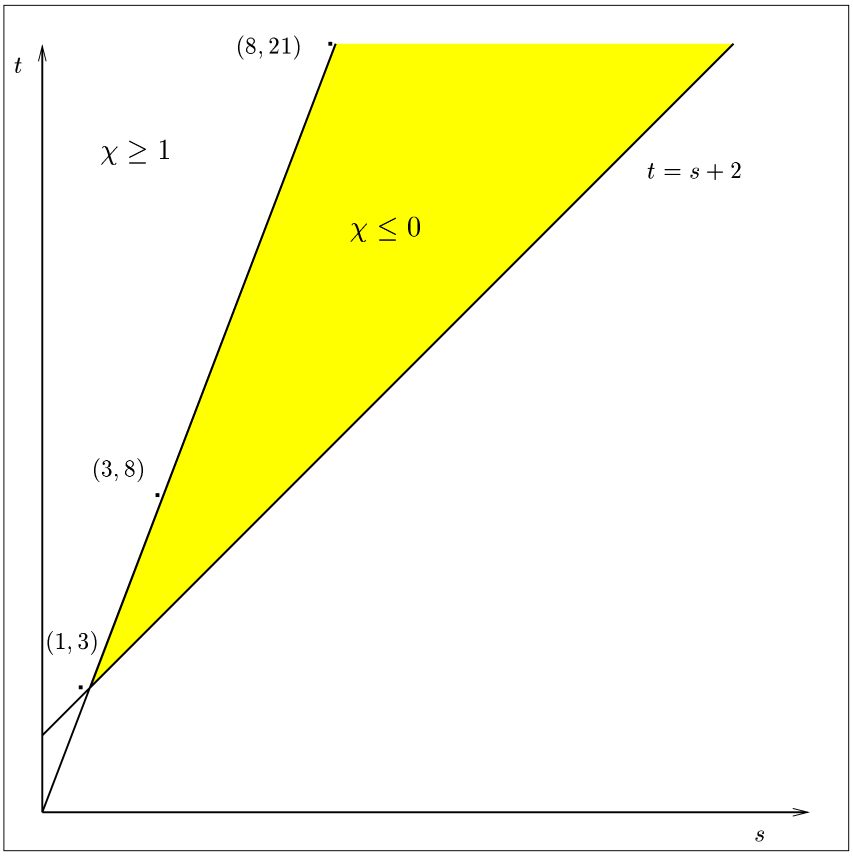

Figure 1: Region of simple and stable Steiner bundles on $\mathbb{P}^2$.

In the case $\chi(\operatorname{End} F) \leq 0$, we can prove the simplicity of generic Steiner bundles by using the natural action of the group $\mathrm{GL}(I) \times \mathrm{GL}(W)$ on the vector space $H = I^\vee \otimes W \otimes V$. With this perspective the problem is reduced to prove that the stabilizer of the generic matrix in H has dimension 1. Hence, we solve it by studying in detail the action. We underline that the proof of Theorem A is self-contained and independent from [DLP85].

In Figure 1 we represent the situation in the case $N = 3$: simple non-exceptional bundles live in the shaded region and the three points represent the first three exceptional bundles that live on the hyperbola $s^2 + t^2 - 3st = 1$.

A second result concerns the case of the Steiner bundles that are not simple. A non-simple bundle is, in general, not decomposable, but in this case we find that any Steiner non-simple bundle is decomposable. Furthermore, it is always a sum of exceptional Steiner bundles.

More precisely, (see Theorem 3.3.5), we state the following

Theorem B. *If $t > \frac{N+\sqrt{N^2-4}}{2}s$, then a generic Steiner bundle with resolution*

$$0 \to \mathcal{O}(-1)^s \longrightarrow \mathcal{O}^t \longrightarrow F \to 0,$$

is isomorphic to a bundle $E_k^n \oplus E_{k+1}^m$, where E_k, E_{k+1} are the exceptional Steiner bundles previously defined and n, m are suitable natural numbers.

Also the proof of Theorem B is independent from [DLP85]. It is interesting to reformulate this result in the setting of matrices. We say that a matrix M in the space $H = \mathbb{C}^s \otimes \mathbb{C}^t \otimes \mathbb{C}^N$ is a *canonical form of type* (n, m, k), if

$$M \in ((\mathbb{C}^{a_{k-1}} \otimes \mathbb{C}^{a_k})^n \oplus (\mathbb{C}^{a_k} \otimes \mathbb{C}^{a_{k+1}})^m) \otimes \mathbb{C}^N \subset H,$$

for a suitable triple of natural numbers (n, m, k). We remark that acting with $\mathrm{GL}(s) \times \mathrm{GL}(t)$ on H is equivalent to performing the Gaussian elimination. Then, Theorem B can be reformulated in the following way:

Corollary C. *If $t > \frac{N+\sqrt{N^2-4}}{2}s$, then the generic $(s \times t \times N)$-matrix $M \in H$ is equivalent, by the $\mathrm{GL}(s) \times \mathrm{GL}(t)$-action, to a matrix of the canonical form.*

We emphasize that the sizes of the blocks in a canonical matrix are always multiple of the generalized Fibonacci numbers considered above. This result is quite surprising. An example of the block shape of the canonical form is shown in Figure 2.

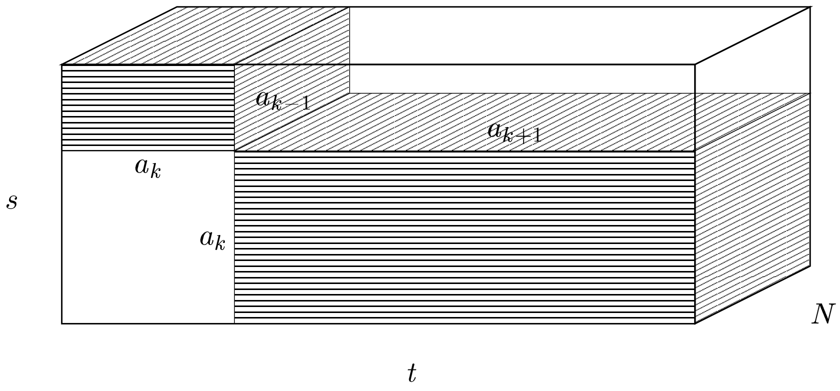

Figure 2: Canonical matrix of type $(1, 1, k)$

We easily obtain a generalization of Theorem A and Theorem B to the case of bundles F on $\mathbb{P}^{N-1}$ with resolution

$$0 \to \mathcal{O}(-k)^s \xrightarrow{M} \mathcal{O}^t \longrightarrow F \to 0,$$

for all $1 \leq k \leq n$. It is easy to check that such bundles F are exceptional when $k \leq n$: this motivates the above inequality. In this case the entries of the matrix M are homogeneous forms in N variables.

There exist several results of Drézet and Le Potier about stability of bundles on $\mathbb{P}^2$. In particular, in [DLP85], they found a criterion to check the stability of a generic bundle, given its rank and Chern classes. This criterion is very complicated to apply. Moreover, from another result of Drézet ([Dré99]), we deduce that on $\mathbb{P}^2$ the stability is equivalent to the generic simplicity. Therefore, our Theorem 3.2.1 provides a criterion for the stability of generic Steiner bundles on $\mathbb{P}^2$ which is very straightforward to apply. More explicitly, we get the following

Corollary D. *Let E be a generic Steiner bundle on $\mathbb{P}^2$. Then, E is stable if and only if $\chi(\operatorname{End} E) \leq 1$.*

As we already pointed out, it is easy to prove that the stability implies $\chi(\operatorname{End} E) \leq 1$. On the other hand, we are not able to deduce directly from [DLP85] the other implication. We do not know yet if Corollary D is true on $\mathbb{P}^{N-1}$ for $N \geq 4$.

The first interesting generalization of the Steiner bundles is the family of bundles with resolution

$$0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^s \to \mathcal{O}^t \to F \to 0. \tag{0.0.2}$$

We undertook the study of bundles of this form only on $\mathbb{P}^2$, by using strongly the Drézet-Le Potier criterion.

The first difference with respect to the Steiner case is that simple and non-simple bundles are not separated by a line, even if we exclude the case of exceptional bundles. Nevertheless, we can give a numerical characterization of generic simplicity i.e. of stability. To prove it we use both approaches: on one hand we apply the results of Drézet, simplifying them in this particular case, and on the other hand we complete the description by proving directly the simplicity.

The result we obtain is summarized in Theorem 5.2.3. Explicitly we have

Theorem E. *Let F be a generic bundle on $\mathbb{P}^2$ with resolution* (0.0.2). *Then F is simple (and stable) if and only if the pair (s,t) lies either on a particular polygonal p (defined in the statement of Theorem 5.1.1), or between the polygonal p and the line $t = s+3$. Vice versa, F is not simple (and not stable) if and only if (s,t) is above the polygonal p.*

We represent graphically the situation in Figure 3, where the beginning part of the polygonal p is shown.

Furthermore, we obtain a nice characterization of the exceptional bundles of the form (0.0.2). More precisely, we have (see Corollary 4.2.8):

Theorem F. *Let E be a bundle on $\mathbb{P}^2$ with resolution* (0.0.2). *Then the following are equivalent:*

(1) E is exceptional,

(2) $\chi(\operatorname{End} E) = 1$,

(3) $(s,t) = (3r_k a_{k-2}, 3r_k a_{k-1})$, *where* $r_k = a_k - a_{k-1}$.

Notice that, in Figure 3, the exceptional bundle corresponds to the first vertex $(15,45)$. In fact, all exceptional bundles are vertices of the polygonal p.

The general case

$$0 \to \mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s \to \mathcal{O}^t \to F \to 0,$$

for any $q \in \mathbb{N}$, seems very hard to approach. First of all, for some fixed q, e.g. for $q = 2$, there exist no exceptional bundles. Moreover if $q > 1$, the condition $\chi(\operatorname{End} F) = 1$

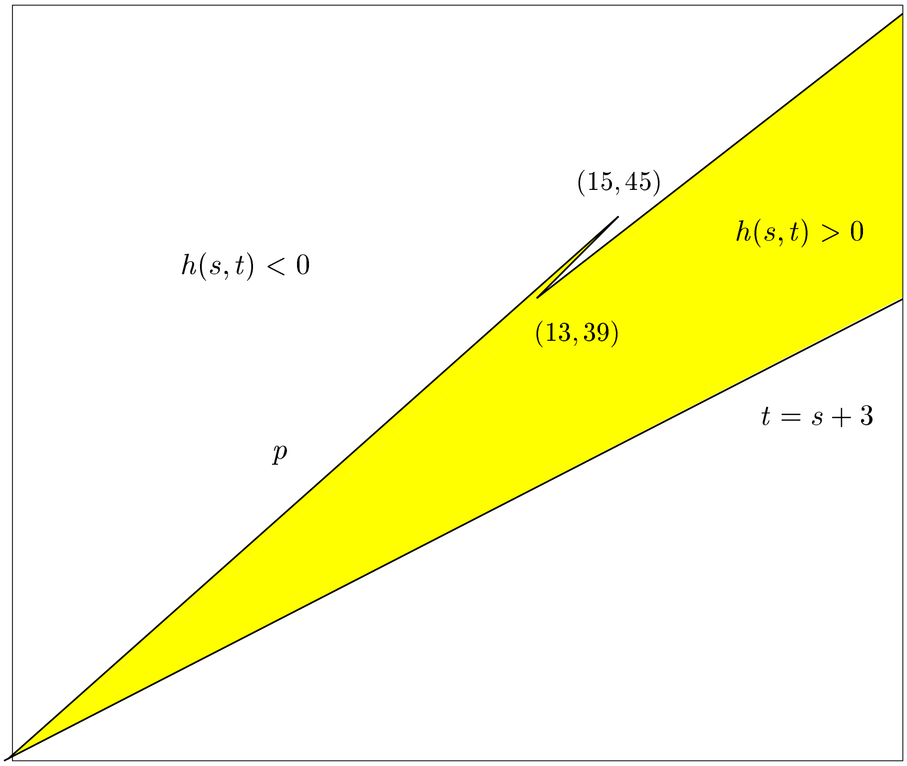

Figure 3: Region of simple and stable bundles on $\mathbb{P}^2$ in the case $q = 1$.

does not imply that F is exceptional. For example the generic bundle G with the following resolution

$$0 \to \mathcal{O}(-3)^3 \oplus \mathcal{O}(-2)^{24} \to \mathcal{O}(-1)^{80} \to G \to 0$$

has $\chi(\operatorname{End} G) = 1$, but G is not exceptional. This suggests that the two cases studied in detail in this thesis, i.e. $q = 0$ (Steiner bundles) and $q = 1$, are very particular. This fact can be understood by looking at the inductive construction of exceptional bundles on $\mathbb{P}^2$ (for more details see Remark 4.3.6).

It is remarkable to note that all the exceptional bundles on $\mathbb{P}^2$ can be constructed by the theory of helices; in particular there exists a correspondence between the exceptional bundles on the projective plane and the solutions of the Markov equation $x^2 + y^2 + z^2 = 3xyz$ (see [Rud88]).

The thesis is structured as follows: in Chapter 1, we give some preliminary definitions and notations about multidimensional matrices and group actions. In Chapter 2, we introduce the fundamental notion of exceptional vector bundle and we describe the basic results and tools of this theory. As main references see [DLP85] and [Rud90]. In Chapter 3, we study the case of Steiner bundles on $\mathbb{P}^n$. In particular we prove Theorem A and Theorem B and we give the reformulation of these results in terms of matrices. The first part of this chapter is contained in the preprint [Bra03]. In Chapter 4, we analyze and classify all the possible resolutions of the exceptional bundles on $\mathbb{P}^2$. In particular we study the case of the resolution (0.0.2) and we prove Theorem F. In Chapter 5, we provide the criterion (Theorem E) for the generic simplicity and stability of bundles with resolution (0.0.2). As a basic reference for bundles on $\mathbb{P}^n$, see [OSS80].

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Chapter 1

Matrices and group actions

1.1 Multidimensional matrices

A general theory of multidimensional matrices is explained by Gelfand, Kapranov and Zelevinsky in the book [GKZ94]. In this section, we recall only the particular case we are interested in.

Definition 1.1.1. *A three-dimensional matrix is an array*

$$A = (a_{i_1,i_2,i_3})$$

of complex numbers, where any index i_j, for $j = 1,2,3$, ranges over some finite set.

The set of three-dimensional matrices is isomorphic to

$$V_1 \otimes V_2 \otimes V_3,$$

where V_j is a complex vector space of dimension l_j, for $j = 1,2,3$, and l_j is the cardinality of the corresponding set of indices. We denote by $x_1^{(j)}, \ldots, x_{l_j}^{(j)}$ the coordinates in V_j, for $j = 1,2,3$.

Then, the matrix A has the following different descriptions:

(i) a multilinear form

$$\sum_{(i_1,i_2,i_3)} a_{i_1,i_2,i_3} x_{i_1} \otimes x_{i_2} \otimes x_{i_3};$$

(ii) an ordinary matrix $M = (m_{i_1,i_2})$ of size $l_1 \times l_2$, whose entries are linear forms

$$m_{i_1,i_2} = \sum_{i_3=1}^{l_3} a_{i_1,i_2,i_3} x_{i_3};$$

(iii) a sheaf morphism ϕ_A on the projective space $X = \mathbb{P}^{l_3-1}$:

$$\mathcal{O}(-1)_X^{l_1} \quad \xrightarrow{\phi_A} \quad \mathcal{O}_X^{l_2}.$$

In this thesis, after introducing the concept of Steiner bundle on $\mathbb{P}^{N-1}$, we will use these alternative descriptions. In particular we will focus on the second point of view.

1.2 Group actions

It is interesting to consider the natural action of the group $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ on the space $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}$.

The classical case is when $n_3 = 1$, i.e. when we act on the space of ordinary $(n_1 \times n_2)$-matrices. In this case the number of the orbits of $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ is $s + 1 = \min(n_1, n_2) + 1$. In fact, the only invariant with respect to the action is the rank of the matrices in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$. Hence, the $s + 1$ orbits are $\{0 = O_0, O_1, \dots, O_s\}$, where O_j is the set of matrices of rank j. The orbit O_s is the open orbit of matrices with maximal rank.

When $n_3 \geq 2$, we act on the space of three-dimensional matrices, i.e. on matrices of linear forms in n_3 variables. In this case it is not true that there exists always an open orbit. Indeed, a necessary condition to have an open orbit is that

$$\dim(\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)) \geq \dim(\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \mathbb{C}^{n_3}).$$

We will see that this condition is sufficient too, and, when it holds, we can find a canonical form for matrices which are in the open orbit (see Section 3.4).

Remark 1.2.1. In [Par01], Parfenov studied the action of $\mathrm{GL}(n_1) \times \ldots \times \mathrm{GL}(n_p)$ on the space $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_p}$. He found that the orbits are infinitely many, except for the following two cases:

- $p = 3$ and $(n_1, n_2, n_3) = (2, 2, n)$, with $n \geq 3$,
- $p = 3$ and $(n_1, n_2, n_3) = (2, 3, n)$, with $n \geq 3$.

Let us observe that

$$\mathrm{GL}(n_1) \times \mathrm{GL}(n_2) \cong \mathrm{GL}(n_1) \times \mathrm{GL}(n_2) \times \mathrm{Id} \subset \mathrm{GL}(n_1) \times \mathrm{GL}(n_2) \times \mathrm{GL}(n_3).$$

Obviously, when the $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2) \times \mathrm{GL}(n_3)$-orbits are infinitely many, also the orbits with respect to the subgroup $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ are infinitely many.

1.3 Steiner bundles

Definition 1.3.1. *According to [DK93], a Steiner bundle on* $\mathbb{P}(V) = \mathbb{P}^{N-1}$ *is the cokernel of a linear map of the form*

$$0 \longrightarrow \mathcal{O}(-1)^s \longrightarrow \mathcal{O}^t \longrightarrow E \longrightarrow 0, \tag{1.3.1}$$

where $0 \leq s \leq t - N + 1$.

Obviously, from the definition it follows that $\mathrm{rk}(E) = t - s \geq N - 1$. This inequality is necessary in order to have a vector bundle E. Indeed, if $\mathrm{rk}(E) = t - s < N - 1$, then we would have $c_{t-s+1}(E(1)) = 0$. But this is impossible, since from (1.3.1) we can easily compute that

$$c_{t-s+1}(E(1)) = c_{t-s+1}(O(1)^t) = \binom{t}{t-s+1} \neq 0.$$

In [AO01], Ancona and Ottaviani studied the action of $\mathrm{SL}(V_0) \times \mathrm{SL}(V_1) \times \mathrm{SL}(V_2)$ on a particular class of multidimensional matrices. As a consequence they obtained some results on Steiner bundles with rank $N - 1$, i.e. with minimum rank. In this thesis we use in a similar way the correspondence between Steiner bundles and multidimensional matrices.

Now, we explain this correspondence more precisely. Let E be a Steiner bundle defined by the sequence

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \xrightarrow{m} W \otimes \mathcal{O} \longrightarrow E \longrightarrow 0,$$

where V, I and W are complex vector spaces of dimension $N \geq 2$, s and t respectively. If we fix a basis in each of the vector spaces V, I and W, the morphism m can be represented by a $t \times s$ matrix M whose entries are linear forms, or equivalently by a three-dimensional matrix of size $t \times s \times N$. The natural action of $\mathrm{GL}(I) \times \mathrm{GL}(W)$ on the space

$$H := \mathrm{Hom}(I \otimes \mathcal{O}(-1), W \otimes \mathcal{O}) \cong V \otimes I^{\vee} \otimes W,$$

is the following

$$\mathrm{GL}(I) \times \mathrm{GL}(W) \times H \to H$$

$$(A, B, M) \mapsto A^{-1} M B.$$

The action is represented in the following commutative diagram, where we denote $M' = A^{-1}MB$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes \mathcal{O}(-1) & \xrightarrow{M} & W \otimes \mathcal{O} & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle B} & & & & \\ 0 & \longrightarrow & I \otimes \mathcal{O}(-1) & \xrightarrow{M'} & W \otimes \mathcal{O} & \longrightarrow & E' & \longrightarrow & 0 \end{array}$$

Remark 1.3.2. Let us observe that considering the action of $\mathrm{GL}(V)$ on H is equivalent to make a change of coordinates in the space V. Hence it is natural to consider the action of $\mathrm{PGL}(V)$, the group of automorphisms of $\mathbb{P}(V)$. This approach is used by Karnik to study bundles on $\mathbb{P}^2$ (see [Kar02]).

Definition 1.3.3. *We say that a Steiner bundle E on $\mathbb{P}^{N-1}$ with the following resolution*

$$0 \longrightarrow \mathcal{O}(-1)^s \xrightarrow{m} \mathcal{O}^t \longrightarrow E \longrightarrow 0,$$

is generic if m is generic in the space $\mathrm{Hom}(\mathcal{O}(-1)^s, \mathcal{O}^t) \cong \mathbb{C}^s \otimes \mathbb{C}^t \otimes \mathbb{C}^N$.

Chapter 2

Exceptional bundles

2.1 Logarithmic invariants

Let X be a projective smooth variety of dimension n and F be a vector bundle on X of rank $r > 0$. The *logarithmic invariants* were introduced by Drézet and Le Potier in [DLP85] and then formally defined in [Dré95] by the following formula:

$$\log(\mathrm{ch}(F)) = \log(r) + \sum_{i=1}^{n} (-1)^{i+1} \Delta_i(F),$$

where $\mathrm{ch}(F)$ is the Chern character of F and $\Delta_i(F) \in A^i(F) \otimes \mathbb{Q}$.
The next properties easily follow from the definition. Let E, F be vector bundles on $\mathbb{P}^n$, then

(i) $\Delta_i(E) = 0$ for all $i > \operatorname{rk} E$, in particular if $\operatorname{rk} E = 1$, we have $\Delta_i(E) = 0$ for all $i > 1$;

(ii) $\Delta_i(E \otimes F) = \Delta_i(E) + \Delta_i(F)$ for $1 \leq i \leq n$, and consequently

$$\Delta_i(E \otimes L) = \Delta_i(E),$$

for any line bundle L and $i \geq 2$;

(iii) $\Delta_i(E^*) = (-1)^i \Delta_i(E)$, for any $1 \leq i \leq n$.

When we work on $\mathbb{P}^2$, we are interested in the first two invariants, respectively called *slope* of F

$$\Delta_1(F) = \mu(F) = \frac{c_1(F)}{\mathrm{rk}(F)},$$

and *discriminant* of F

$$\Delta_2(F) = \Delta(F) = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2).$$

In the following propositions, we give some properties of the logarithmic invariants.

Proposition 2.1.1. *If*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of bundles (or torsion free sheaves), then

$$\mu(A) < \mu(B) \qquad (\textit{respectively} \;=\; \textit{or} \;>)$$

if and only if

$$\mu(B) < \mu(C) \qquad (\textit{respectively} \;=\; \textit{or} \;>).$$

Proof. Let us show that if $\mu(A) < \mu(B)$ then $\mu(B) < \mu(C)$. In fact the first Chern class c_1 and the rank are additive with respect to the exact sequence, i.e. $c_1(C) = c_1(B) - c_1(A)$ and $\mathrm{rk}(C) = \mathrm{rk}(B) - \mathrm{rk}(A)$. Then by hypotesis

$$\frac{c_1(A)}{\mathrm{rk}(A)} < \frac{c_1(B)}{\mathrm{rk}(B)},$$

i.e. $c_1(A)\,\mathrm{rk}(B) - c_1(B)\,\mathrm{rk}(A) < 0$, from which we have $c_1(B)\,\mathrm{rk}(B) - c_1(A)\,\mathrm{rk}(B) > c_1(B)\,\mathrm{rk}(B) - c_1(B)\,\mathrm{rk}(A)$, i.e.

$$\frac{c_1(C)}{\mathrm{rk}(C)} = \frac{c_1(B) - c_1(A)}{\mathrm{rk}(B) - \mathrm{rk}(A)} > \frac{c_1(B)}{\mathrm{rk}(B)}.$$

The other inequalities are obtained with the same computations. □

Proposition 2.1.2. *If*

$$0 \to A \to B \to C \to 0$$

is an exact sequence of bundles (or torsion free sheaves) and $\mu(A) = \mu(B) = \mu(C)$ *then*

$$\Delta(A) < \Delta(B) \qquad (\textit{respectively} = \;\textit{or} \;>)$$

if and only if

$$\Delta(B) < \Delta(C) \qquad (\textit{respectively} = \;\textit{or} \;>).$$

Proof. Let us prove that $\Delta(A) < \Delta(B)$ implies $\Delta(B) < \Delta(C)$. We know that $c_1(C) = c_1(B) - c_1(A)$ and $c_2(C) = c_2(B) - c_2(A) - c_1(A)c_1(C)$. Then by definition of Δ, we can compute

$$\Delta(C) = \frac{1}{r(B) - r(A)}\Big(c_2(B) - c_2(A) - c_1(A)c_1(B) + c_1(A)^2 +$$

$$-\frac{r(B) - r(A) - 1}{2(r(B) - r(A))}(c_1(B)^2 + c_1(A)^2 - 2c_1(A)c_1(B))\Big).$$

Now, from the hypotesis $\mu(A) = \mu(B)$, we know that

$$c_1(A) = \frac{c_1(B)r(A)}{r(B)}$$

and, from $\Delta(A) < \Delta(B)$, we get

$$c_2(A) < \frac{r(A)}{r(B)}\Big(c_2(B) - \frac{r(B)-1}{2r_B}c_1(B)^2\Big) + \frac{r(A)-1}{2r(A)}c_1(A).$$

Then, by substituting $c_1(A)$ and $c_2(A)$ in the expression of $\Delta(C)$ and by simplifying the computations, we get:

$$\Delta(C) > \frac{c_2(B)}{r(B)} - c_1(B)^2\Big(\frac{r(B)-1}{2r(B)}\Big) = \Delta(B).$$

The other inequalities are obtained in the same way. □

2.2 Moduli spaces of semistable sheaves

In this section, we recall some basic definitions about bundles.

Definition 2.2.1 (Mumford-Takemoto). *A bundle (or a torsion free coherent sheaf) E over $\mathbb{P}^n$ is semistable if, for any coherent subsheaf $\mathcal{F}$ of E (with $0 \neq \mathcal{F}$), we have*

$$\mu(\mathcal{F}) \leq \mu(E).$$

Moreover, if for any coherent subsheaves $\mathcal{F}$ of E with $0 < \operatorname{rk}\mathcal{F} < \operatorname{rk}E$, we have

$$\mu(\mathcal{F}) < \mu(E),$$

then we say that E is stable.

Definition 2.2.2. *A bundle E is simple if and only if* $\mathrm{h}^0(\mathrm{End}\, E) = 1$, *i.e. the endomorphisms of E are the homotheties.*

It is a well known fact that stable bundles are always simple (see for example [OSS80]). Moreover, in the case of rank 2 bundles on $\mathbb{P}^n$, simplicity implies stability, but in general this implication is false.

We will denote by $M(r, c_1, c_2)$ the moduli space (of Maruyama) of semistable sheaves on $\mathbb{P}^2$ of rank r and Chern classes c_1, c_2.

2.3 Riemann-Roch formula

Following [DLP85], we denote by P the polynomial

$$P(x) = \frac{x^2 + 3x + 2}{2} = \binom{x+2}{2}.$$

Given a bundle (or a coherent algebraic sheaf) F, with rank r, slope μ and discriminant Δ, we can write the well-known Riemann-Roch formula as follows:

$$\chi(F) = \sum_i (-1)^i \dim \mathrm{H}^i(F) = r(P(\mu) - \Delta).$$

If we define

$$\chi(F_1, F_2) = \sum_i (-1)^i \dim \mathrm{Ext}^i(F_1, F_2),$$

we get

$$\chi(F_1, F_2) = r_1 r_2 (P(\mu(F_2) - \mu(F_1)) - \Delta(F_1) - \Delta(F_2)).$$

In particular we have

$$\chi(\mathrm{End}\, F) = \chi(F, F) = r^2(1 - 2\Delta).$$

2.4 Exceptional bundles on $\mathbb{P}^2$

Exceptional bundles on $\mathbb{P}^2$ were introduced in [DLP85] as follows:

Definition 2.4.1. *A vector bundle E on $\mathbb{P}^2$ is called exceptional if it is stable and* $\Delta(E) < \frac{1}{2}$.

It is easily seen that if E is stable, then the following conditions are equivalent

(1) E is exceptional,

(2) $\chi(\mathrm{End}(F)) = 1$,

(3) E is rigid, i.e. $\mathrm{Ext}^1(\mathrm{End}(F)) = 0$.

Definition 2.4.2. *A vector bundle E on $\mathbb{P}^2$ is called semi-exceptional if it is semi-stable and $\Delta(E) < \frac{1}{2}$.*

Proposition 2.4.3 ([DLP85]). *Any semi-exceptional vector bundle with slope μ is a direct sum of exceptional bundles with slope μ.*

The first examples of exceptional bundles on $\mathbb{P}^2$ are the linear bundles $\mathcal{O}(k)$ and the tangent bundle $T_{\mathbb{P}^2}$. Notice that if E is exceptional, then $E(k) = E \otimes \mathcal{O}(k)$ is exceptional for any $k \in \mathbb{Z}$, and the dual bundle E^* is exceptional. All the possible slopes of an exceptional bundle have been characterized very precisely by Drézet and Le Potier. In order to give their result, we define a map ε from the set of binary rational numbers $\mathcal{D}$ to $\mathbb{Q}$ which satisfies the following two properties:

(i) $\varepsilon(n) = n$, for all $n \in \mathbb{Z}$,

(ii) if $\varepsilon(a/2^b) = k_1/r_1$ and $\varepsilon((a+1)/2^b) = k_2/r_2$, then

$$\varepsilon\left(\frac{2a+1}{2^{b+1}}\right) = \frac{1}{2}\left(\frac{k_1}{r_1} + \frac{k_2}{r_2}\right) + \frac{1/(2r_1^2) - 1/(2r_2^2)}{3 + k_1/r_1 - k_2/r_2}.$$

Theorem 2.4.4 ([DLP85]). *If E is an exceptional bundle on $\mathbb{P}^2$, then $\mu(E) \in \varepsilon(\mathcal{D})$. Furthermore for each $\mu \in \varepsilon(\mathcal{D})$, there exists a unique exceptional bundle E such that $\mu(E) = \mu$.*

As a consequence of this theorem, the exceptional bundles are uniquely determined by their slopes. Moreover, Tyurin conjectured that any exceptional bundle with $0 \leq \mu \leq \frac{1}{2}$ is uniquely determined by its rank. In [Rud88], Rudakov proves that Tyurin's conjecture is equivalent to an important conjecture of Number Theory: the so called conjecture of "uniqueness of the Markov numbers".

The Markov equation is

$$x^2 + y^2 + z^2 = 3xyz.$$

It is known that all the integral solutions of the Markov equation can be obtained from the solution $(1, 1, 1)$ by two standard transformations, which allows us to place the solutions at the vertices of a tree. The conjecture of "uniqueness of the Markov numbers" says that any solution of the Markov equation is determined, up to order, by its maximal number.

2.5 Helices of exceptional bundles on $\mathbb{P}^n$

In [Rud90] Rudakov gives an axiomatic presentation of the theory of helices of exceptional objects; this subject was introduced and investigated by the Russian school. Here, we give some definitions and results that we will recall later. The following concepts are defined for objects of any category, but we consider only the case of vector bundles on $\mathbb{P}^n$, studied in particular in [GR87].

Definition 2.5.1 ([GR87]). *A vector bundle E on $\mathbb{P}^n$ is called exceptional if*

$$\operatorname{Hom}(E,E)=\mathbb{C} \qquad \text{and} \qquad \operatorname{Ext}^i(E,E)=0 \qquad \text{for } i\geq 1.$$

Remark 2.5.2. Obviously if E is exceptional

$$\chi(\operatorname{End}E)=\sum_i(-1)^i\,\mathrm{h}^i(\operatorname{End}E)=1.$$

An ordered *collection* of bundles $(E_1,\dots,E_k)$ is called *exceptional* if all its bundles are exceptional and, whenever $1\leq l<m\leq k$, the following conditions hold

$$\operatorname{Ext}^i(E_l,E_m)=0 \text{ for all } i\geq 1$$

and

$$\operatorname{Ext}^i(E_m,E_l)=0 \text{ for all } i\geq 0.$$

It is trivial to see that if $\varepsilon=(E_1,\dots,E_k)$ is an exceptional collection, then the collections $\varepsilon^*=(E_1^*,\dots,E_k^*)$ and $\varepsilon(i)=(E_1(i),\dots,E_k(i))$ are the same.

Now, we define the fundamental concept of mutation. Let $\varepsilon=(A,B)$ be an ordered pair of bundles. Assume that there exist bundles A' and B' such that we have the following exact sequences

$$0\longrightarrow B'\longrightarrow A\otimes\operatorname{Hom}(A,B)\xrightarrow{can}B\longrightarrow 0,$$

$$0\longrightarrow A\xrightarrow{can}\operatorname{Hom}(A,B)^\vee\otimes B\longrightarrow A'\longrightarrow 0,$$

where the morphism *can* is the canonical map. In the above situation we call B' a *left shift* of B and we denote it by L_AB. Analogously we call A' a *right shift* of A in ε and we denote it by R_BA. We say that (B',A) is a *left mutation* of ε and (B,A') is a *right mutation*, and we denote them respectively $L\varepsilon$ and $R\varepsilon$. It is easy to prove that if $\varepsilon=(A,B)$ is an ordered pair of bundles, then $(L\varepsilon)^*=R\varepsilon^*$ and $(R\varepsilon)^*=L\varepsilon^*$.

A pair of vector bundles (A,B) is *left admissible* if the left mutation L_AB is defined and is *right admissible* if the right mutation R_BA is defined.

Definition 2.5.3. *A collection* $\{A_i, i \in \mathbb{Z}\}$ *of exceptional vector bundles on* $\mathbb{P}^n$ *is called helix of period* $(n+1)$ *if*

$$A_{i+n+1} = A_i(n+1), \qquad \textit{for all } i \in \mathbb{Z},$$

the pairs (A_{s-1}, A_s), $(A_{s-1}, L^{(1)}A_s), \ldots, (A_{s-n}, L^{(n-1)}A_s)$ *are all left admissible pairs of exceptional bundles, and* $L^{(n)}A_s = A_{s-n}$.

A mutation of a collection $(\ldots, A, B, \ldots)$ is the collection obtained by substituting the pair A, B by its right or left mutation.

Proposition 2.5.4 ([GR87]). *The collection* $\{\mathcal{O}(i)\}$, *for* $i \in \mathbb{Z}$ *is a helix.*

In [Rud88], Rudakov showed that all the exceptional bundles on $\mathbb{P}^2$ can be obtained by starting from the helix $\{(\mathcal{O}(i)\}_{i\in\mathbb{Z}}$ and by applying several times right and left mutations. In particular, if we start from the collection $(\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1))$, this method gives triples of exceptional bundles, called *adjacent bundles*, which are associated to the solutions of the Markov equation. Moreover, we can place these triples at the vertices of the Markov tree. From the general theory we get the following remark:

Remark 2.5.5. If (A, B, C) is a triple of adjacent exceptional bundles on $\mathbb{P}^2$, then

$$\mu(A) \leq \mu(B) \leq \mu(C).$$

2.6 Drézet-Le Potier criterion

Let E be an exceptional vector bundle on $\mathbb{P}^2$ with rank r_E, slope μ_E and discriminant Δ_E. Clearly, since E is exceptional, $\Delta_E = \frac{1}{2}(\frac{r^2-1}{r^2})$. Denote x_+ and x_- the roots of the equation

$$x^2 - 3r_E x + 1 = 0$$

and define

$$x_E = \frac{x_-}{r_E} = \frac{3r_E - \sqrt{9r_E^2 - 4}}{2r_E}.$$

Denote by I_E the interval $(\mu_E - x_E, \mu_E + x_E) \cap \mathbb{Q}$. Recall that we denote $P(x) = \frac{x^2+3x+2}{2}$.

Theorem 2.6.1 ([Dré87]). *The family* $\{I_E : E \textit{ exceptional}\}$ *is a partition of* $\mathbb{Q}$.

Definition 2.6.2. *Let δ be a map $\mathbb{Q} \to \mathbb{Q}$ defined as follows: for any $\mu \in I_E$,*

$$\delta(\mu) = P(-|\mu - \mu_E|) - \Delta_E.$$

Definition 2.6.3. *If F is a bundle with rank r and Chern classes c_1 and c_2, we call height of F the value*

$$h(r, c_1, c_2) = \Delta(r, c_1, c_2) - \delta(\frac{c_1}{r}).$$

Now, we introduce some properties of the function δ:

Proposition 2.6.4. *The function $\delta(\mu) : \mathbb{Q} \to \mathbb{Q}$ defined in 2.6.2 has the following properties:*

(i) $\delta(\mu + n) = \delta(\mu)$, for all $n \in \mathbb{Z}$ and $\mu \in \mathbb{Q}$

(ii) $\delta(\mu) \leq 1$ for all $\mu \in \mathbb{Q}$ and $=$ holds if and only if $\mu \in \mathbb{Z}$

(iii) $\delta(\mu) > \frac{1}{2}$, for all $\mu \in \mathbb{Q}$.

Proof. Claim *(i)* is easily seen. Now, suppose that $\mu \in I_E$, where E is an exceptional bundle with slope μ_E and rank r_E. We know that

$$I_E = (\mu_E - x_E, \mu_E + x_E) \cap \mathbb{Q}$$

where $x_E = \frac{3}{2} - \frac{\sqrt{9r_E^2-4}}{2r_E}$. Since E is exceptional, we have $\Delta(E) = \frac{1}{2}(1 - \frac{1}{r_E^2})$. We denote $0 \leq x = |\mu - \mu_E| < x_E$, and we compute

$$\delta(\mu) = P(-x) - \frac{1}{2}(1 - \frac{1}{r_E^2}) = \frac{r_E^2(x^2 - 3x + 1) + 1}{2r_E^2}.$$

Now, we prove claim *(ii)*. By solving the following inequality

$$\delta(\mu) = \frac{r_E^2(x^2 - 3x + 1) + 1}{2r_E^2} \leq 1,$$

we have

$$\frac{3}{2} - \frac{\sqrt{13r_E^2 - 4}}{2r_E} \leq x \leq \frac{3}{2} + \frac{\sqrt{13r_E^2 - 4}}{2r_E}. \tag{2.6.1}$$

By hypotesis, we have $0 \leq x < x_E$, then the inequality (2.6.1) holds, since $\frac{3}{2} - \frac{\sqrt{13r_E^2-4}}{2r_E} \leq 0$ and $x_E < \frac{3}{2} + \frac{\sqrt{13r_E-4}}{2r_E}$. Moreover, if $\delta(\mu) = 1$ it follows $r_E = 1$ and

$x = 0$, hence $\mu = \mu_E \in \mathbb{Z}$.
Now, we prove claim *(iii)*. By solving the inequality

$$\delta(\mu) = \frac{r_E^2(x^2 - 3x + 1) + 1}{2r_E^2} > \frac{1}{2},$$

we get $x < \frac{3}{2} - \frac{\sqrt{9r_E^2-4}}{2r_E}$ or $x > \frac{3}{2} + \frac{\sqrt{9r_E^2-4}}{2r_E}$. This is obviously true, since $x < x_E = \frac{3}{2} - \frac{\sqrt{9r_E^2-4}}{2r_E}$. $\square$

Remark 2.6.5. It is easily seen that if $\mu \notin I_E$, for an exceptional bundle E, then

$$P(-|\mu - \mu_E|) - \Delta_E < \frac{1}{2}.$$

Recall that we denote by $M(r, c_1, c_2)$ the variety of moduli of semistable algebraic coherent sheaves on $\mathbb{P}^2$, with rank r and Chern classes c_1 and c_2. Now, we give the following fundamental **Drézet-Le Potier criterion**:

Theorem 2.6.6 ([DLP85],[Dré87]). *Given $r, c_1, c_2 \in \mathbb{Z}$ and $r > 0$, the variety of moduli $M(r, c_1, c_2)$ has strictly positive dimension if and only if*

$$\delta(\frac{c_1}{r}) \leq \Delta(r, c_1, c_2) = \frac{1}{r}(c_2 - \frac{r-1}{2r}c_1^2),$$

i.e. if and only if $h(r, c_1, c_2) \geq 0$.

This implies that if $h(r, c_1, c_2) \geq 0$, then there exists a semistable bundle with rank r and Chern classes c_1 and c_2. Furthermore Drézet and Le Potier prove that there exists a stable bundle in $M(r, c_1, c_2)$.

From another result of Drézet (Theorem 3.1, [Dré99]) we know that when a generic bundle on $\mathbb{P}^2$ is not stable, then it is not simple. Then we state the following important proposition.

Proposition 2.6.7 ([Dré99]). *A generic bundle F on $\mathbb{P}^2$ is stable if and only if it is simple.*

Chapter 3

Steiner bundles on $\mathbb{P}^n$

In this chapter we study the main properties of Steiner bundles on $\mathbb{P}^n$. We prove that exceptional Steiner bundles are characterized by the property $\chi(\operatorname{End} F) = 1$, and we describe their resolutions. Then we study conditions for simplicity of Steiner bundles and finally we give a complete description of the case of non-simple Steiner bundles.

By Definition 1.3.1, a Steiner bundle on $\mathbb{P}(V) = \mathbb{P}^{N-1}$ is the cokernel of a linear map of the form

$$0 \to \mathcal{O}(-1)^s \to \mathcal{O}^t \to E \to 0, \tag{3.0.1}$$

where $0 \leq s \leq t - N + 1$.

Remark 3.0.8. From the sequence (3.0.1), it follows that $\chi(E) = t$ and $\chi(E(1)) = (Nt - s)$. Dualizing (3.0.1) and tensoring by E, we get

$$0 \to \operatorname{End} E \to E^t \to E(1)^s \to 0,$$

therefore

$$\chi(\operatorname{End} E) = t\chi(E) - s\chi(E(1)) = t^2 - s(Nt - s) = t^2 - Nst + s^2.$$

Now, we introduce the sequence $\{a_k\}$ which is of fundamental importance for the characterization of Steiner bundles on $\mathbb{P}^{N-1}$. For any fixed $N \geq 3$ and $k \geq 0$, let

$$a_k = \frac{\left(\frac{N+\sqrt{N^2-4}}{2}\right)^k - \left(\frac{N-\sqrt{N^2-4}}{2}\right)^k}{\sqrt{N^2-4}}.$$

Notice that the sequence $\{a_k\}$ can be also defined recursively by

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{k+1} = Na_k - a_{k-1}. \end{cases}$$

In the case $N = 3$, this sequence is exactly the odd part of the well known Fibonacci sequence, i.e. the sequence $\{f_k\}$ defined recursively as

$$\begin{cases} f_0 = 0, \\ f_1 = 1, \\ f_{k+1} = f_k + f_{k-1}. \end{cases}$$

For this reason, for any fixed $N > 3$, we call *generalized Fibonacci numbers* the elements of the sequence $\{a_k\}$.

3.1 Exceptional Steiner bundles

In Section 2.5 we introduced the general theory of helices of exceptional bundles. The following result is a particular case of this theory. We give the proof in order to show that the result is elementary, in fact we only need standard cohomology sequences. The proof is partially inspired by that given by Rudakov in [Rud88] regarding exceptional bundles on $\mathbb{P}^2$. Let us recall that a bundle E on $\mathbb{P}^n$ is exceptional if $\mathrm{h}^0(\operatorname{End} E) = 1$ and $\mathrm{h}^i(\operatorname{End} E) = 0$ for all $i > 0$.

Theorem 3.1.1. *If E_k is a generic Steiner bundle on $\mathbb{P}^{N-1}$, with $N \geq 3$, defined by the exact sequence*

$$0 \to \mathcal{O}(-1)^{a_{k-1}} \to \mathcal{O}^{a_k} \to E_k \to 0,$$

where

$$a_k = \frac{\left(\frac{N+\sqrt{N^2-4}}{2}\right)^k - \left(\frac{N-\sqrt{N^2-4}}{2}\right)^k}{\sqrt{N^2-4}},$$

then E_k is exceptional.

On $\mathbb{P}(V) = \mathbb{P}^{N-1}$ we define a sequence of vector bundles as follows:

$$F_0 = \mathcal{O}(1), \quad F_1 = \mathcal{O}, \quad F_{n+1} = \operatorname{Ker}(F_n \otimes \operatorname{Hom}(F_n, F_{n-1}) \xrightarrow{\psi_n} F_{n-1}), \tag{3.1.1}$$

where ψ_n is the canonical map.

Lemma 3.1.2. *Let F_n and ψ_n be as in* (3.1.1). *Then, the canonical map ψ_n is an epimorphism, for any $n \geq 1$. Moreover the following properties (A_n), (B_n) and (C_n) are satisfied for any $n \geq 1$:*

$$\begin{array}{lll}
(A_n) & \mathrm{Hom}(F_n, F_n) \cong \mathbb{C}, \quad \mathrm{Ext}^i(F_n, F_n) = 0, & \textit{for any} \quad i \geq 1, \\
(B_n) & \mathrm{Hom}(F_{n-1}, F_n) = 0, \quad \mathrm{Ext}^i(F_{n-1}, F_n) = 0, & \textit{for any} \quad i \geq 1, \\
(C_n) & \mathrm{Hom}(F_n, F_{n-1}) \cong V, \quad \mathrm{Ext}^i(F_n, F_{n-1}) = 0, & \textit{for any} \quad i \geq 1.
\end{array}$$

Note that (A_n) means that every F_n is an exceptional bundle.

Proof. We prove the lemma by induction on n. If $n = 1$, $F_1 = \mathcal{O}$ is exceptional because it is a line bundle, therefore (A_1) holds. Moreover, since $F_0 = \mathcal{O}(1)$, we have

$$\begin{array}{ll}
(B_1) & \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}) = \mathrm{H}^0(\mathcal{O}(-1)) = 0, \quad \mathrm{Ext}^i(\mathcal{O}(1), \mathcal{O}) = \mathrm{H}^i(\mathcal{O}(-1)) = 0, \\
(C_1) & \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = \mathrm{H}^0(\mathcal{O}(1)) \cong V, \quad \mathrm{Ext}^i(\mathcal{O}, \mathcal{O}(1)) = \mathrm{H}^i(\mathcal{O}(1)) = 0,
\end{array}$$

for any $i \geq 1$. Finally, we know that $\mathcal{O} \otimes \mathrm{H}^0(\mathcal{O}(1)) \xrightarrow{\psi_1} \mathcal{O}(1)$ is surjective, since it is contained in the well known Euler exact sequence.

Now, we suppose that ψ_k is an epimorphism for all $k \leq n$. Then, we can consider for all $k \leq n$ the following exact sequence of bundles

$$0 \to F_{k+1} \to F_k \otimes \mathrm{Hom}(F_k, F_{k-1}) \to F_{k-1} \to 0,$$

in particular if $k = n$ we get

$$0 \to F_{n+1} \to F_n \otimes \mathrm{Hom}(F_n, F_{n-1}) \to F_{n-1} \to 0. \tag{3.1.2}$$

We also suppose that (A_k), (B_k) and (C_k) are true for all $k \leq n$. We want to prove (A_{n+1}), (B_{n+1}) and (C_{n+1}). By applying the functor $\mathrm{Hom}(-, F_n)$ to the exact sequence (3.1.2), we get

$$\begin{array}{l}
0 \to \mathrm{Hom}(F_{n-1}, F_n) \to \mathrm{Hom}(F_n, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Hom}(F_{n+1}, F_n) \to \\
\to \mathrm{Ext}^1(F_{n-1}, F_n) \to \mathrm{Ext}^1(F_n, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^1(F_{n+1}, F_n) \to \\
\to \mathrm{Ext}^2(F_{n-1}, F_n) \to \mathrm{Ext}^2(F_n, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^2(F_{n+1}, F_n) \to \dots
\end{array}$$

and it is easy to check that (C_{n+1}) holds, because (A_n) and (B_n) are true. Now, by applying $\mathrm{Hom}(F_n, -)$ to the sequence (3.1.2) we get

$$
\begin{aligned}
&0 \to \mathrm{Hom}(F_n, F_{n+1}) \to \mathrm{Hom}(F_n, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \xrightarrow{\alpha} \mathrm{Hom}(F_n, F_{n-1}) \to \\
&\to \mathrm{Ext}^1(F_n, F_{n+1}) \to \mathrm{Ext}^1(F_n, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^1(F_n, F_{n-1}) \to \\
&\to \mathrm{Ext}^2(F_n, F_{n+1}) \to \mathrm{Ext}^2(F_n, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^2(F_n, F_{n-1}) \to \dots
\end{aligned}
$$

Since α is the natural map and $\mathrm{Hom}(F_n, F_n) \cong \mathbb{C}$, it follows that α is an isomorphism, hence $\mathrm{Hom}(F_n, F_{n+1}) = 0$ and $\mathrm{Ext}^1(F_n, F_{n+1}) = 0$. Moreover $\mathrm{Ext}^i(F_n, F_{n+1}) = 0$ for all $i \geq 2$ because (A_n) and (C_n) hold. Therefore (B_{n+1}) is true. Now we have to prove (A_{n+1}). First we apply $\mathrm{Hom}(F_{n-1}, -)$ to (3.1.2), and we have

$$
\begin{aligned}
&0 \to \mathrm{Hom}(F_{n-1}, F_{n+1}) \to \mathrm{Hom}(F_{n-1}, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Hom}(F_{n-1}, F_{n-1}) \to \\
&\to \mathrm{Ext}^1(F_{n-1}, F_{n+1}) \to \mathrm{Ext}^1(F_{n-1}, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^1(F_{n-1}, F_{n-1}) \to \\
&\to \mathrm{Ext}^2(F_{n-1}, F_{n+1}) \to \mathrm{Ext}^2(F_{n-1}, F_n) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^2(F_{n-1}, F_{n-1}) \to \dots
\end{aligned}
$$

then (B_n) and (A_{n-1}) imply that $\mathrm{Ext}^1(F_{n-1}, F_{n+1}) \cong \mathbb{C}$ and $\mathrm{Ext}^i(F_{n-1}, F_{n+1}) = 0$ for all $i \geq 2$. Now by applying $\mathrm{Hom}(-, F_{n+1})$ to (3.1.2), we get

$$
\begin{aligned}
&0 \to \mathrm{Hom}(F_{n-1}, F_{n+1}) \to \mathrm{Hom}(F_n, F_{n+1}) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Hom}(F_{n+1}, F_{n+1}) \to \\
&\to \mathrm{Ext}^1(F_{n-1}, F_{n+1}) \to \mathrm{Ext}^1(F_n, F_{n+1}) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^1(F_{n+1}, F_{n+1}) \to \\
&\to \mathrm{Ext}^2(F_{n-1}, F_{n+1}) \to \mathrm{Ext}^2(F_n, F_{n+1}) \otimes \mathrm{Hom}(F_n, F_{n-1}) \to \mathrm{Ext}^2(F_{n+1}, F_{n+1}) \to \dots
\end{aligned}
$$

and, using (B_{n+1}), we obtain that $\mathrm{Hom}(F_{n+1}, F_{n+1}) \cong \mathrm{Ext}^1(F_{n-1}, F_{n+1}) \cong \mathbb{C}$ and, for all $i \geq 1$, $\mathrm{Ext}^i(F_{n+1}, F_{n+1}) \cong \mathrm{Ext}^{i+1}(F_{n-1}, F_{n+1}) = 0$, then (A_{n+1}) holds.

Finally, we prove that ψ_{n+1} is an epimorphism. From the proof of the property (C_{n+1}) we have an isomorphism $\phi_n : \mathrm{Hom}(F_{n+1}, F_n) \cong \mathrm{Hom}(F_n, F_{n-1})$ for any $n \leq 1$. Therefore we obtain the following commutative diagram (in which we denote

$\mathrm{Hom}(F_{n+1}, F_n)$ by V_{n+1}):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_{n+2} \arrow[r] \arrow[d] & F_{n+1} \otimes V_{n+1} \arrow[r, "\psi_{n+1}"] \arrow[d, "i\otimes\phi_n"] & F_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{n+1} \otimes V_{n+1} \arrow[r, "i\otimes\phi_n"] \arrow[d] & F_n \otimes V_n \otimes V_n \arrow[r, "\psi_n\otimes\phi_{n-1}"] \arrow[d, "\psi_n\otimes\phi_{n-1}"] & F_{n-1} \otimes V_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_n \arrow[r, "i\otimes\phi_{n-1}"] \arrow[d] & F_{n-1} \otimes V_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d] & F_{n-2} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From this diagram it easily follows that ψ_{n+1} is an epimorphism. $\square$

Remark 3.1.3. Following the notations introduced in Section 2.5, the previous lemma means that (F_n, F_{n-1}) is a left admissible pair and (F_{n+1}, F_n) is the left mutation of (F_n, F_{n-1}), and that the sequence (F_n) forms an exceptional collection generated by the helix $(\mathcal{O}(i))$ by left mutations.

Proof of Theorem 3.1.1. Lemma 3.1.2 states that, for all $n \geq 0$, the bundles F_n, defined as in (3.1.1), are exceptional. Obviously their dual bundles, F_n^*, are exceptional too. Now we will prove that, for every $n \geq 1$, the bundle F_n^* admits the following resolution

$$0 \to \mathcal{O}(-1)^{a_{n-1}} \to \mathcal{O}^{a_n} \to F_n^* \to 0, \tag{3.1.3}$$

where $\{a_n\}$ is the sequence defined in the statement. This implies that a generic bundle with this resolution is exceptional. Recalling that the sequence $\{a_n\}$ is also defined recursively by

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{n+1} = Na_n - a_{n-1}, \end{cases}$$

we can prove (3.1.3) by induction on n. In fact if $n = 1$ the sequence (3.1.3) is $0 \to \mathcal{O}(-1)^{a_0} \to \mathcal{O}^{a_1} \to F_1^* \to 0$, i.e. $0 \to \mathcal{O} \to F_1^* \to 0$, and the claim is true because $F_1 \cong \mathcal{O}$. Now, we suppose that every F_k^* admits a resolution of the form (3.1.3) for all $k \leq n$ and we prove the same assertion for F_{n+1}^*. By dualizing the sequence

$$0 \to F_{n+1} \to F_n \otimes \mathrm{Hom}(F_n, F_{n-1}) \to F_{n-1} \to 0,$$

and by induction hypothesis, we have:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & F_{n-1}^* \arrow[u] \arrow[r] & F_n^* \otimes V^* \arrow[u] \arrow[r] & F_{n+1}^* \arrow[r] & 0 \\
 & \mathcal{O}^{a_{n-1}} \arrow[u] & \mathcal{O}^{a_n} \otimes V^* \arrow[u] & & \\
 & \mathcal{O}(-1)^{a_{n-2}} \arrow[u] & \mathcal{O}(-1)^{a_{n-1}} \otimes V^* \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

We define the map $\alpha : \mathcal{O}^{a_{n-1}} \to F_n^* \otimes V^*$ as the composition of the known maps. Since $\mathrm{Ext}^1(\mathcal{O}^{a_{n-1}}, \mathcal{O}(-1)^{a_{n-1}} \otimes V^*) \cong \mathrm{H}^1(\mathcal{O}(-1))^{a_{n-1}^2} \otimes V^* = 0$, the map α induces a map $\widetilde{\alpha} : \mathcal{O}^{a_{n-1}} \to \mathcal{O}^{a_n} \otimes V^*$ such that the following diagram commutes:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & F_{n-1}^* \arrow[u] \arrow[r, "f"] & F_n^* \otimes V^* \arrow[u] \arrow[r] & F_{n+1}^* \arrow[r] & 0 \\
 & \mathcal{O}^{a_{n-1}} \arrow[u] \arrow[ur, "\alpha"] \arrow[r, dashed, "\widetilde{\alpha}"] & \mathcal{O}^{a_n} \otimes V^* \arrow[u] & & \\
 & \mathcal{O}(-1)^{a_{n-2}} \arrow[u] & \mathcal{O}(-1)^{a_{n-1}} \otimes V^* \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

We observe that $\widetilde{\alpha}$ is injective if and only if $\mathrm{H}^0(\widetilde{\alpha})$ is injective and, since $\mathrm{H}^0(\widetilde{\alpha}) = \mathrm{H}^0(f)$, they are injective. Obviously, the cokernel of $\widetilde{\alpha}$ is $\mathcal{O}^{Na_n - a_{n-1}} = \mathcal{O}^{a_{n+1}}$. Let $\widetilde{\beta}$ be the restriction of $\widetilde{\alpha}$ to $\mathcal{O}(-1)^{a_{n-2}}$. Then, we can check that $\widetilde{\beta}$ is injective, its cokernel is

$\mathcal{O}(-1)^{Na_{n-1}-a_{n-2}} = \mathcal{O}(-1)^{a_n}$ and the following diagram commutes:

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & F_{n-1}^* \arrow[r] \arrow[u] & F_n^* \otimes V^* \arrow[r] \arrow[u] & F_{n+1}^* \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}^{a_{n-1}} \arrow[r, dashed, "\widetilde{\alpha}"] \arrow[u] \arrow[ur, "\alpha"] & \mathcal{O}^{a_n} \otimes V^* \arrow[r] \arrow[u] & \mathcal{O}^{a_{n+1}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{O}(-1)^{a_{n-2}} \arrow[r, dashed, "\widetilde{\beta}"] \arrow[u] & \mathcal{O}(-1)^{a_{n-1}} \otimes V^* \arrow[r] \arrow[u] & \mathcal{O}(-1)^{a_n} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

It follows that F_{n+1}^* has the resolution $0 \to \mathcal{O}(-1)^{a_n} \to \mathcal{O}^{a_{n+1}} \to F_{n+1}^* \to 0$ and this completes the proof of our theorem. $\square$

Now, we prove that the bundles with resolutions (3.1.3) are the only exceptional Steiner bundles. If F is an exceptional Steiner bundle, we have obviously $\chi(\operatorname{End} F) = s^2 - Nst + t^2 = 1$. Thus we have only to prove that $s^2 - Nst + t^2 = 1$ implies $(t, s) = (a_{k+1}, a_k)$, where a_k has been defined above. This will follow from the following lemma.

Lemma 3.1.4. *All integer solutions of $s^2 - Nst + t^2 = 1$, when $t > s$, are exactly $s = a_k, t = a_{k+1}$, where $a_k = \dfrac{\left(\frac{N+\sqrt{N^2-4}}{2}\right)^k - \left(\frac{N-\sqrt{N^2-4}}{2}\right)^k}{\sqrt{N^2-4}}$.*

Proof. We already know that the sequence $\{a_k\}$ is defined recursively by

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{k+1} = Na_k - a_{k-1}. \end{cases}$$

So we prove by induction on k that $(s = a_k, t = a_{k+1})$ is a solution of

$$s^2 - Nst + t^2 = 1. \tag{3.1.4}$$

If $k = 0$, obviously $s = 0, t = 1$ is a solution. If the pair (a_{k-1}, a_k) satisfies (3.1.4), then, using the recursive definition, we check that (a_k, a_{k+1}) is a solution too. Hence, we have to prove that there are no other solutions. By the change of coordinates $\{r = 2t - Ns, s = s\}$, our equation becomes the following Pell-Fermat equation

$$r^2 - (N^2 - 4)s^2 = 4. \tag{3.1.5}$$

(see for example [Sam67], page 77, or [Len02]). Notice that all integer solutions of (3.1.4) are integer solutions of (3.1.5). By some Number Theory results, we know that all the solutions (r, s) of (3.1.5) are given by the sequence (r_k, s_k) defined by

$$r_k + s_k\sqrt{N^2 - 4} = \frac{1}{2^{k-1}}(N + \sqrt{N^2 - 4})^k.$$

This sequence is also defined by

$$\begin{cases} r_0 = 2, \\ s_0 = 0, \\ r_{k+1} = \frac{(N^2-4)s_k + Nr_k}{2}, \\ s_{k+1} = \frac{Ns_k + r_k}{2}. \end{cases}$$

By a change of coordinates, we define $t_k = \frac{Ns_k + r_k}{2}$ and we check that the sequence (s_k, t_k) is exactly (a_k, a_{k+1}). In fact $(s_0, t_0) = (0, 1) = (a_0, a_1)$ and moreover $t_k = s_{k+1}$ and $t_{k+1} = \frac{Ns_{k+1} + r_{k+1}}{2} = \frac{(N^2-2)s_k + Nr_k}{2} = \frac{(N^2-2)s_k + N(2t_k - Ns_k)}{2} = Nt_k - t_{k-1}$. □

Remark 3.1.5. From the previous lemma, it follows that in the case of a Steiner bundle F on $\mathbb{P}^n$, we have $\chi(\operatorname{End} F) = 1$ if and only if F is exceptional. Notice that this is false in general. Indeed we can find bundles G on $\mathbb{P}^2$ which satisfy $\chi(\operatorname{End} G)) = 1$, but which are not exceptional. For example, consider a bundle G with the following resolution

$$0 \to \mathcal{O}(-3)^3 \oplus \mathcal{O}(-2)^{24} \to \mathcal{O}(-1)^{80} \to G \to 0.$$

Then, it is easy to compute that $\chi(\operatorname{End} G) = 1$. On the other hand, G is not exceptional, since its rank is 53, which is not admissible for an exceptional bundle on $\mathbb{P}^2$.

3.2 Simple Steiner bundles

We investigate now simplicity of Steiner bundles. It is well known that Steiner bundles with rank $N - 1$ are simple because they are stable (see [AO94] or [BS92]). We are interested in bundles with higher rank. We state now the following main theorem, the remaining part of the section being devoted to its proof.

Theorem 3.2.1. *Let E be a Steiner bundle on $\mathbb{P}^{N-1}$, with $N \geq 3$, defined by the exact sequence*

$$0 \to \mathcal{O}(-1)^s \stackrel{m}{\to} \mathcal{O}^t \to E \to 0,$$

where m is a generic morphism in $\mathrm{Hom}(\mathcal{O}(-1)^s, \mathcal{O}^t)$. Then, the following statements are equivalent:

(i) *E is simple, i.e.* $\mathrm{h}^0(\mathrm{End}\, E) = 1$,

(ii) $s^2 - Nst + t^2 \leq 1$ *i.e.* $\chi(\mathrm{End}\, E) \leq 1$,

(iii) *either E is exceptional or $s^2 - Nst + t^2 \leq 0$ i.e. $t \leq (\frac{N+\sqrt{N^2-4}}{2})s$.*

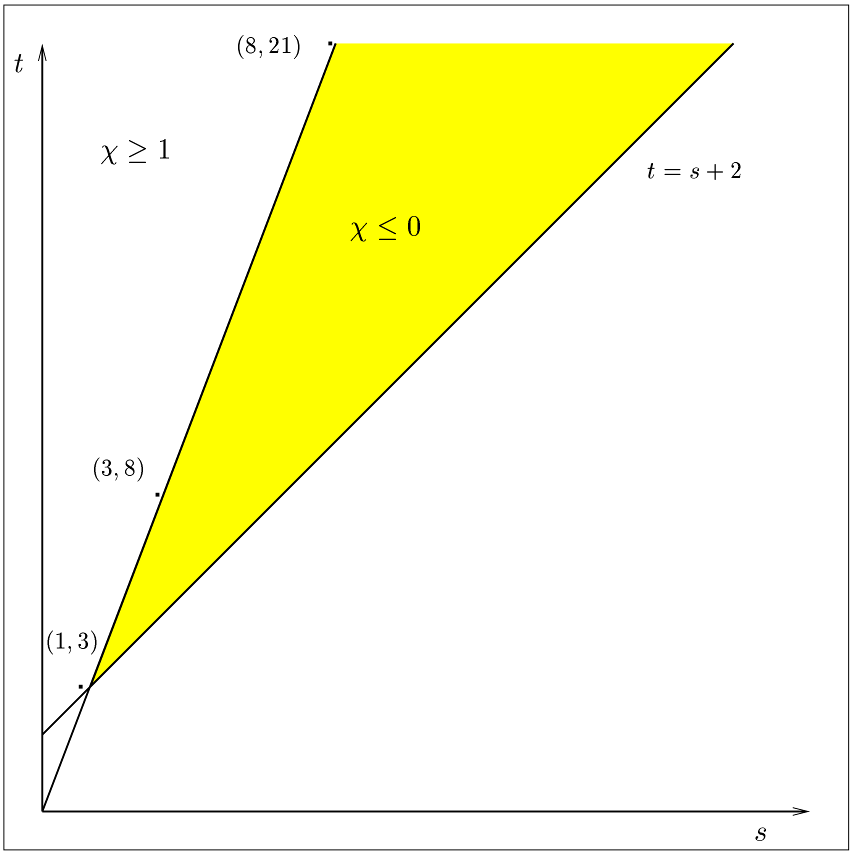

Figure 3.1: Case $N = 3$, the shaded region represents simple bundles.

Remark 3.2.2. In Figure 3.1 we represent the situation in the case $N = 3$: simple non-exceptional bundles live in the shaded region and the three points represent the first three exceptional bundles, that live on the hyperbola of equation $s^2+t^2-3st = 1$.

In order to prove Theorem 3.2.1, we recall the notation introduced in Chapter 1. Let E be given by the exact sequence on $\mathbb{P}^{N-1} = \mathbb{P}(V)$

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \xrightarrow{m} W \otimes \mathcal{O} \longrightarrow E \longrightarrow 0, \tag{3.2.1}$$

where V, I and W are complex vector spaces of dimension $N \geq 3$, s and t respectively and m is a generic morphism. If we fix a basis in each of the vector spaces V, I and W, the morphism m is represented by a $t \times s$ matrix M whose entries are linear forms. Let us consider the natural action of $\mathrm{GL}(I) \times \mathrm{GL}(W)$ on the space

$$H = \mathrm{Hom}(I \otimes \mathcal{O}(-1), W \otimes \mathcal{O}) \cong V \otimes I^\vee \otimes W,$$

i.e. the action

$$\mathrm{GL}(I) \times \mathrm{GL}(W) \times H \to H$$

$$(A, B, M) \mapsto A^{-1}MB.$$

When the pair (A, B) belongs to the stabilizer of M, it induces a morphism $\phi : E \to E$, such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & I \otimes \mathcal{O}(-1) \arrow[r, "M"] \arrow[d, "A"] & W \otimes \mathcal{O} \arrow[r] \arrow[d, "B"] & E \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & I \otimes \mathcal{O}(-1) \arrow[r, "M"] & W \otimes \mathcal{O} \arrow[r] & E \arrow[r] & 0
\end{tikzcd} \tag{3.2.2}$$

Now we prove the first part of the theorem. Let us recall that from (3.2.1) we get the following sequence

$$0 \longrightarrow \mathrm{End}\, E \longrightarrow W^\vee \otimes E \longrightarrow I^\vee \otimes E(1) \longrightarrow 0. \tag{3.2.3}$$

Lemma 3.2.3. *If E is a simple Steiner bundle, then $\chi(\mathrm{End}\, E) \leq 1$.*

Proof. From sequences (3.2.1) and (3.2.3) it is easy to check that $\mathrm{H}^i(\mathrm{End}\, E) = 0$, for all $i \geq 2$. Moreover, $\mathrm{h}^0(\mathrm{End}\, E) = 1$ because of the simplicity, and consequentely $\chi(\mathrm{End}\, E) = 1 - \mathrm{h}^1(\mathrm{End}\, E) \leq 1$. $\square$

Remark 3.2.4. Notice that the previous lemma is true for every simple bundle on $\mathbb{P}^2$. Nevertheless, it is not true in general. As a counterexample consider the instanton bundles on $\mathbb{P}^5$, which are simple and satisfy the property $\chi(\operatorname{End} E) > 1$ (see [OT94]).

We also give another proof of the same implication consisting in the following two lemmas. In the following, Id denotes the identity matrix.

Lemma 3.2.5. *If $\mathrm{H}^0(E^*) = 0$ and there exist two matrices (A, B) in the stabilizer of M such that $(A, B) \neq (\lambda \operatorname{Id}, \lambda \operatorname{Id})$, then the pair (A, B) induces $\phi \neq \lambda \operatorname{Id}$.*

Proof. Using the cohomology sequence associated to (3.2.1), it is easy to check that $\mathrm{H}^0(E) = W$, $\mathrm{H}^1(E) = 0$ and $\mathrm{H}^0(E(-1)) = 0$. Dualizing (3.2.1), we get

$$0 \longrightarrow E^* \longrightarrow W^\vee \otimes \mathcal{O} \xrightarrow{M^t} I^\vee \otimes \mathcal{O}(1) \longrightarrow 0.$$

and, tensoring by $\mathcal{O}(-1)$, we obtain $\mathrm{H}^1(E^*(-1)) = I^\vee$ and $\mathrm{H}^2(E^*(-1)) = 0$. Tensoring (3.2.1) by E^*, we get

$$0 \longrightarrow I \otimes E^*(-1) \longrightarrow W \otimes E^* \longrightarrow \operatorname{End} E \longrightarrow 0,$$

and the associated cohomology sequence, together with the previous results, gives the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
 & & & \operatorname{End} W \arrow[d, "l_M"] & & \\
 & & & W \otimes \operatorname{Hom}(I, V) = H \arrow[d, "\pi"] & & \\
0 \arrow[r] & \mathrm{H}^0(\operatorname{End} E) \arrow[r, "i"] & \operatorname{End} I \arrow[r] \arrow[ur, "r_M"] & W \otimes \mathrm{H}^1(E^*) \arrow[r] \arrow[d] & \mathrm{H}^1(\operatorname{End} E) \arrow[r] & 0 \\
 & & & 0 & &
\end{tikzcd}$$

where $r_M(A) = AM$ and $l_M(B) = MB$. Let us suppose that $AM = MB$ and $A \neq \lambda \operatorname{Id}$. Then $r_M(A) = AM = MB = l_M(B)$. This implies that $\pi(r_M(A)) = 0$, thus there exists $\phi = i^{-1}(A) \in \mathrm{H}^0(\operatorname{End} E)$ such that $\phi \neq \lambda \operatorname{Id}$, since $i(\lambda \operatorname{Id}) = \lambda \operatorname{Id}$ and i is injective. Now if $A = \lambda \operatorname{Id}$, then $\lambda M = MB$, i.e. $M(B - \lambda I) = 0$. This implies that $B = \lambda \operatorname{Id}$, because $\mathrm{H}^0(E^*) = 0$. $\square$

Lemma 3.2.6. *If E is simple then $s^2 - Nst + t^2 \leq 1$.*

Proof. We suppose by contradiction that $s^2 - Nst + t^2 > 1$. It is obvious that $(\lambda \mathrm{Id}, \lambda \mathrm{Id}) \in \mathrm{Stab}(M)$, so we consider the action of the group $G = \frac{\mathrm{GL}(I)\times\mathrm{GL}(W)}{\mathbb{C}^*}$ on the space $H = V \otimes I^\vee \otimes W$. As

$$\dim \frac{\mathrm{GL}(I)\times\mathrm{GL}(W)}{\mathbb{C}^*} = s^2 + t^2 - 1 > Nst = \dim H,$$

the action of G on H cannot be free, i.e. it exist $(A, B) \neq (\lambda \mathrm{Id}, \lambda \mathrm{Id})$ such that $AM = MB$. Since $\mathrm{H}^0(E) = W$ and E is simple by assumption, we get $\mathrm{H}^0(E^*) = 0$, and by Lemma 3.2.5 we conclude that E is not simple. $\square$

Now, we prove that statement *(ii)* is equivalent to *(iii)*. By Remark 3.1.5, we know that $s^2 - Nst + t^2 = 1$ if and only if the bundle is exceptional. On the other hand, $s^2 - Nst + t^2 \leq 0$ is equivalent to $(\frac{N-\sqrt{N^2-4}}{2})s \leq t \leq (\frac{N+\sqrt{N^2-4}}{2})s$. Since $t > s$ and $N > 2$, this inequality is equivalent to $t \leq (\frac{N+\sqrt{N^2-4}}{2})s$.

Now, we prove the last implication, i.e. *(iii)* implies *(i)*. In the case $(t, s) = (a_{k+1}, a_k)$, the generic E is an exceptional bundle by Theorem 3.1.1, therefore, in particular, E is simple. Suppose $s^2 - Nst + t^2 \leq 0$. Recall that H denotes $\mathrm{Hom}(I \otimes \mathcal{O}(-1), W \otimes \mathcal{O}) \cong V \otimes I^\vee \otimes W$. Let S be the set

$$\{A, B, M : A^{-1}MB = M\} \subset \mathrm{GL}(I) \times \mathrm{GL}(W) \times H$$

and π_1 and π_2 the projections on $\mathrm{GL}(I) \times \mathrm{GL}(W)$ and on H respectively. Notice that, for all $M \in H$, $\pi_1(\pi_2^{-1}(M))$ is the stabilizer of M with respect to the action of $\mathrm{GL}(I) \times \mathrm{GL}(W)$. Obviously, since $(\lambda \mathrm{Id}, \lambda \mathrm{Id}) \in \mathrm{Stab}(M)$, it follows that

$$\dim \mathrm{Stab}(M) \geq 1.$$

Lemma 3.2.7. *If E is defined by the sequence*

$$0 \longrightarrow I \otimes \mathcal{O}(-1) \xrightarrow{M} W \otimes \mathcal{O} \longrightarrow E \longrightarrow 0 \tag{3.2.4}$$

and $\dim \mathrm{Stab}(M) = 1$*, then E is simple.*

Proof. If, by contradiction, E is not simple, then there exists $\phi : E \to E$ non-trivial. Applying the functor $\mathrm{Hom}(-, E)$ to the sequence (3.2.4), we get that ϕ induces $\widetilde{\phi}$ non-trivial in $\mathrm{Hom}(W \otimes \mathcal{O}, E)$. Now, applying the functor $\mathrm{Hom}(W \otimes \mathcal{O}, -)$ again to the same sequence, we get $\mathrm{Hom}(W \otimes \mathcal{O}, W \otimes \mathcal{O}) \cong \mathrm{Hom}(W \otimes \mathcal{O}, E)$, because

$\mathrm{Hom}(W \otimes \mathcal{O}, I \otimes \mathcal{O}(-1)) \cong W \otimes I \otimes \mathrm{H}^0(\mathcal{O}(-1)) = 0$ and $\mathrm{Ext}^1(W \otimes \mathcal{O}, I \otimes \mathcal{O}(-1)) \cong W \otimes I \otimes \mathrm{H}^1(\mathcal{O}(-1)) = 0$. It follows that there exists $\widetilde{\phi}$ non-trivial in $\mathrm{End}(W \otimes \mathcal{O})$, i.e. a matrix $B \neq \mathrm{Id}$ in $\mathrm{GL}(W)$. Restricting $\widetilde{\phi}$ to $I \otimes \mathcal{O}(-1)$ and calling A the corresponding matrix in $\mathrm{GL}(I)$, we get the commutative diagram (3.2.2). Therefore $(A, B) \neq (\lambda \mathrm{Id}, \lambda \mathrm{Id})$ belongs to $\mathrm{Stab}(M)$ and consequently $\dim \mathrm{Stab}(M) > 1$. □

Finally it suffices to prove that for all generic $M \in H$, the dimension of the stabilizer is exactly 1. In other words, we have to prove the following proposition.

Proposition 3.2.8. *Let $H = V \otimes I^\vee \otimes W$ as above and suppose $s^2 - Nst + t^2 \leq 0$. Then the generic orbit in H, with respect to the natural action of $GL(I) \times \mathrm{GL}(W)$, has dimension exactly equal to $(s^2 + t^2 - 1)$.*

We recall that we have defined the following diagram

$$\begin{tikzcd}[column sep=small] & S = \{A, B, M : A^{-1}MB = M\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{GL}(I) \times \mathrm{GL}(W) & & H \end{tikzcd}$$

Let A, B be two fixed Jordan canonical forms in $\mathrm{GL}(I) \times \mathrm{GL}(W)$. We define $G_{AB} \subset \mathrm{GL}(I) \times \mathrm{GL}(W)$ as the set of pairs of matrices respectively similar to A and B. Note that $\pi_2\pi_1^{-1}(G_{AB}) = \{C^{-1}MD : A^{-1}MB = M, C \in \mathrm{GL}(I), D \in \mathrm{GL}(W)\}$. Moreover $G_{\mathrm{Id}\,\mathrm{Id}} = \{(\lambda \mathrm{Id}, \lambda \mathrm{Id}), \lambda \in \mathbb{C}\}$ and $\pi_2\pi_1^{-1}(G_{\mathrm{Id}\,\mathrm{Id}}) = H$.

Lemma 3.2.9. *If $s^2 - Nst + t^2 \leq 0$ and (A, B) are Jordan canonical forms different from $(\lambda \mathrm{Id}, \lambda \mathrm{Id})$ for any λ, then $\pi_2\pi_1^{-1}(G_{AB})$ is contained in a Zariski closed subset strictly contained in H.*

Proof. Suppose that the assertion is false. Then there exist two Jordan canonical forms A and B, different from $(\lambda \mathrm{Id}, \lambda \mathrm{Id})$, such that $\pi_2\pi_1^{-1}(G_{AB})$ is not contained in any closed subset. This implies that we can take a general $M \in H$ such that $AM = MB$ and, in particular, we can suppose the rank of M maximum.

Now, we prove that A and B have the same minimal polynomial. First, when p_B is the minimal polynomial of B, we get $p_B(B) = 0$, then it follows that $p_B(A)M = Mp_B(B) = 0$. Since M is injective, we get $p_B(A) = 0$, hence the minimal polynomial of B divides that of A. Now, if we denote by λ_i (for $1 \leq i \leq q$) the eigenvalues of A and by μ_j (for $1 \leq j \leq q'$) those of B, we obtain that $\mu_j \in \{\lambda_1, \ldots, \lambda_q\}$ for all $1 \leq j \leq q'$. Let us define $A' = (A - x\,\mathrm{Id}_s)$ and $B' = (B - x\,\mathrm{Id}_t)$: obviously we obtain $A'M = MB'$. We denote by $\overline{B'}$ the matrix of cofactors of B', and we know that $B'\overline{B'} = \det(B')\,\mathrm{Id}_t = P_B(x)\,\mathrm{Id}_t$, where P_B is the characteristic polynomial of B. Therefore

$$A'M\overline{B'} = P_B(x)M$$

and, developing this expression, we see that $q' = q$. In fact, if there exists a $\lambda_i \neq \mu_j$ for all $j = 1, \ldots, q'$, then there is a row of zeroes in M and consequently M is not generic. Then, the matrices A and B have the same eigenvalues λ_i $(1 \leq i \leq q)$ with multiplicity respectively $a_i \geq 1$ and $b_i \geq 1$. The assumption $(A, B) \neq (\lambda\,\mathrm{Id}, \lambda\,\mathrm{Id})$ means that either A and B have more than one eigenvalue, or at least one of them is not diagonal.

Now, consider the first case, i.e. $q \geq 2$. Since $\dim I = s$ and $\dim W = t$, obviously $\sum_{i=1}^{q} a_i = s$ and $\sum_{i=1}^{q} b_i = t$. Now we denote $M = (M_{ij})$, where M_{ij} has dimension $a_i \times b_j$. Since $AM = MB$, every block M_{ij} is zero for all $i \neq j$, i.e. it is possible to write M in the form

$$M = \begin{pmatrix} * & 0 & \cdots & 0 \\ 0 & * & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & * \end{pmatrix}.$$

In particular, we can define $n_1 = a_1, n_2 = \sum_{i=2}^{p} a_i, m_1 = b_1, m_2 = \sum_{i=2}^{p} b_i$ and thus the matrix M becomes

$$M = \begin{pmatrix} (*)_{n_1 \times m_1} & (0)_{n_1 \times m_2} \\ (0)_{n_2 \times m_1} & (*)_{n_2 \times m_2} \end{pmatrix} \tag{3.2.5}$$

where $n_1 + n_2 = s$ and $m_1 + m_2 = t$ and $n_i, m_i \geq 1$ for $i = 1, 2$. Thus, it suffices to show that a matrix in the orbit

$$O_M = \{C^{-1}MD : C \in \mathrm{GL}(s), D \in \mathrm{GL}(t), M \text{ with the form (3.2.5)}\}$$

is not generic in H if $s^2 - Nst + t^2 \leq 0$. This fact contradicts our assumption and completes the proof.

Now, in order to show this, we introduce the following diagrams

$$\begin{tikzcd} & \{\phi, I_1, W_1 : \phi(I_1 \otimes V^\vee) \subseteq W_1\} \arrow[dl, "\alpha_1"'] \arrow[dr, "\beta_1"] & \\ H = \mathrm{Hom}(I \otimes V^\vee, W) & & \mathcal{G}_1 = \mathcal{G}(\mathbb{C}^{n_1}, \mathbb{C}^s) \times \mathcal{G}(\mathbb{C}^{m_1}, \mathbb{C}^t)) \end{tikzcd}$$

where $\mathcal{G}(\mathbb{C}^k, \mathbb{C}^h)$ denotes the Grassmannian of $\mathbb{C}^k \subset \mathbb{C}^h$ and

$$\begin{tikzcd}[column sep=tiny] & \{\phi, I_2, W_2 : \phi(I_2 \otimes V^\vee) \subseteq W_2\} \arrow[dl, "\alpha_2"'] \arrow[dr, "\beta_2"] & \\ H = \mathrm{Hom}(I \otimes V^\vee, W) & & \mathcal{G}_2 = \mathcal{G}(\mathbb{C}^{n_2}, \mathbb{C}^s) \times \mathcal{G}(\mathbb{C}^{m_2}, \mathbb{C}^t)) \end{tikzcd}$$

It is easy to check that the matrices of the set O_M live in the subvariety

$$\widetilde{H} = \alpha_1(\beta_1^{-1}(\mathcal{G}_1)) \cap \alpha_2(\beta_2^{-1}(\mathcal{G}_2)) \subseteq H.$$

Thus, in order to prove that these matrices are not generic, it suffices to show that $\dim \widetilde{H} < \dim H$. Since $\dim(\mathcal{G}_i) = (n_1n_2 + m_1m_2)$ for $i = 1, 2$, we obtain

$$\dim(\alpha_1(\beta_1^{-1}(\mathcal{G}_1))) \leq \dim(\beta_1^{-1}(\mathcal{G}_1)) = n_1n_2 + m_1m_2 + N(n_1(m_1 + m_2) + n_2m_2)$$

and

$$\dim(\alpha_2(\beta_2^{-1}(\mathcal{G}_2))) \leq \dim(\beta_2^{-1}(\mathcal{G}_2)) = n_1n_2 + m_1m_2 + N(n_1m_1 + n_2(m_1 + m_2)).$$

Therefore, since $\dim H = Nst = N(n_1 + n_2)(m_1 + m_2)$ we only need to show that either $(n_1n_2 + m_1m_2 - Nn_2m_1) < 0$, or $(n_1n_2 + m_1m_2 - Nn_1m_2) < 0$. In other words, we have to prove that the system

$$\begin{cases} n_1n_2 + m_1m_2 - Nn_1m_2 \geq 0 \\ n_1n_2 + m_1m_2 - Nn_2m_1 \geq 0 \end{cases}$$

has no solutions in our hypotesis $s^2 - Nst + t^2 \leq 0$, i.e. if

$$\frac{N - \sqrt{N^2 - 4}}{2} t \leq s \leq \frac{N + \sqrt{N^2 - 4}}{2} t.$$

This is equivalent to prove that the system

$$\begin{cases} n_1n_2 + m_1m_2 - Nn_1m_2 \geq 0 \\ n_1n_2 + m_1m_2 - Nn_2m_1 \geq 0 \\ n_1 + n_2 \geq \frac{N - \sqrt{N^2 - 4}}{2}(m_1 + m_2) \\ n_1 + n_2 \leq \frac{N + \sqrt{N^2 - 4}}{2}(m_1 + m_2) \end{cases}$$

has no solutions. To see this, consider n_1 and m_1 as parameters and write the previous system as a system of linear inequalities in the two unknowns n_2 and m_2:

$$\begin{cases} n_1n_2 \geq (Nn_1 - m_1)m_2 \\ (n_1 - Nm_1)n_2 \geq -m_1m_2 \\ n_2 \geq \alpha_-m_2 + (\alpha_-m_1 - n_1) \\ n_2 \leq \alpha_+m_2 + (\alpha_+m_1 - n_1) \end{cases}$$

where we denote $\alpha_- = \frac{N-\sqrt{N^2-4}}{2}$ and $\alpha_+ = \frac{N+\sqrt{N^2-4}}{2}$. Notice that $(\alpha_- + \alpha_+) = N$ and $\alpha_-\alpha_+ = 1$, because they are the solutions of the equation $s^2 - Nst + t^2 = 0$. Now we consider three cases:

- if $0 < n_1 - \alpha_+ m_1$ the system
$$\begin{cases} n_2 \geq \frac{(Nn_1-m_1)}{n_1} m_2 \\ n_2 \leq \alpha_+ m_2 + (\alpha_+ m_1 - n_1) \end{cases}$$
has no solutions, because $(\alpha_+ m_1 - n_1) < 0$ and $\alpha_+ < \frac{(Nn_1-m_1)}{n_1}$, since $(N - \alpha_+)n_1 - m_1 = \alpha_- n_1 - m_1 = (\alpha_+)^{-1}(n_1 - \alpha_+ m_1) > 0$;

- if $n_1 - \alpha_+ m_1 < 0 < n_1 - \alpha_- m_1$ the system is
$$\begin{cases} n_2 \geq \frac{(Nn_1-m_1)}{n_1} m_2 \\ n_2 \leq \frac{m_1}{(Nm_1-n_1)} m_2 \end{cases}$$
because $Nm_1 - n_1 > \alpha_+ m_1 - n_1 > 0$ and there is no solution, because $\frac{m_1}{(Nm_1-n_1)} < \frac{(Nn_1-m_1)}{n_1}$, since $N(Nn_1m_1 - m_1^2 - n_1^2) > 0$;

- if $n_1 - \alpha_- m_1 < 0$ then the system
$$\begin{cases} n_2 \leq \frac{m_1}{(Nm_1-n_1)} m_2 \\ n_2 \geq \alpha_- m_2 + (\alpha_- m_1 - n_1) \end{cases}$$
has no solutions, because $(\alpha_- m_1 - n_1) > 0$ and $\alpha_- > \frac{m_1}{(Nm_1-n_1)}$, i.e. $\alpha_+ < \frac{(Nm_1-n_1)}{m_1}$, since $(N - \alpha_+)m_1 - n_1 = \alpha_- m_1 - n_1 > 0$.

This completes the proof in the case $q \geq 2$.

In the second case we consider $q = 1$ and the two matrices are

$$A = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_h \end{pmatrix}, \quad \text{where} \quad J_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix}$$

and c_i denotes the order of J_i, and

$$B = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_k \end{pmatrix}, \quad \text{where} \quad L_i = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix}$$

and d_i is the order of L_i. We suppose that $c_1 \geq 2$ or $d_1 \geq 2$ i.e. $h < s$ or $k < t$. Then, a matrix M such that $AM = MB$ has the form $M = (M_{ij})$ and M_{ij} is a $c_i \times d_j$ matrix such that

$$M_{ij} = \begin{cases} T_c & \text{if} \quad c_i = d_j = c \\ (0|T_c) & \text{if} \quad c = c_i < d_j \\ (\frac{T_d}{0}) & \text{if} \quad c_i > d_j = d \end{cases}$$

and T_c is a $c \times c$ upper-triangular Toeplitz matrix. It is easy to see that M has at least k columns in which there are at least $(c_1 - 1) + (c_2 - 1) + \ldots + (c_h - 1) = (s - h)$ zeroes in such a way that we can order the basis so as to write M in the following form

$$\begin{pmatrix} (*)_{h \times k} & (*)_{h \times (t-k)} \\ (0)_{(s-h) \times k} & (*)_{(s-h) \times (t-k)} \end{pmatrix}.$$

Analogously M has at least h rows with at least $(t - k)$ zeroes, such that it is possible to write the matrix in the form

$$\begin{pmatrix} (*)_{h \times k} & (0)_{h \times (t-k)} \\ (*)_{(s-h) \times k} & (*)_{(s-h) \times (t-k)} \end{pmatrix}.$$

Hence, there exist non-trivial subspaces I_1, I_2, W_1, W_2 such that $M(I_i \otimes V^\vee) \subseteq W_i$, for $i = 1, 2$, and $\dim I_1 = s - h$, $\dim W_1 = k$, $\dim I_2 = h$, $\dim W_2 = t - k$. Therefore, exactly the same argument used in the first case gives that M is not generic and completes the proof. $\square$

The previous lemma proves Proposition 3.2.8 and the main Theorem 3.2.1 follows.

Remark 3.2.10. By Proposition 2.6.7, we know that bundles on $\mathbb{P}^2$ are generically simple if and only if they are stable. Then, Theorem 3.2.1 gives us not only a criterion for the simplicity of the generic Steiner bundle, but also a criterion for the stability on $\mathbb{P}^2$. Obviously this criterion agrees with the Drézet-Le Potier criterion for the stability, when it is applied to the case of Steiner bundles, but it is much easier to handle (and to prove).

3.3 Non-simple Steiner bundles

We study now the case of non-simple Steiner bundles. As in the previous sections, E_k denotes the exceptional Steiner bundle with resolution

$$0 \to \mathcal{O}(-1)^{a_{k-1}} \to \mathcal{O}^{a_k} \to E_k \to 0,$$

where the sequence $\{a_k\}$ is defined recursively by

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{n+1} = Na_n - a_{n-1}. \end{cases}$$

Observe that $(a_k, a_{k-1}) = 1$.

Remark 3.3.1. For all $k \geq 1$, we have $a_k^2 - a_{k+1}a_{k-1} = 1$. It is easy to prove this equality by induction.

Theorem 3.3.2. *If $N \geq 3$ and $\frac{t}{s} > \left(\frac{N+\sqrt{N^2-4}}{2}\right)$, then there exist $k, n, m \in \mathbb{N}$ (where n and m are not both 0) such that the bundle $E_k^n \oplus E_{k+1}^m$ on $\mathbb{P}^{N-1}$ has resolution*

$$0 \to \mathcal{O}(-1)^s \to \mathcal{O}^t \to E_k^n \oplus E_{k+1}^m \to 0, \tag{3.3.1}$$

where E_k and E_{k+1} are exceptional Steiner bundles on P^{N-1}.

Proof. Let E_k be the exceptional Steiner bundle defined as above. It is easy to check that the sequence $\{\frac{a_{k+1}}{a_k}\}$ is decreasing to $\frac{N+\sqrt{N^2-4}}{2}$. It follows that there exists $k \geq 1$ such that either

$$\frac{a_k}{a_{k-1}} = \frac{t}{s},$$

or

$$\frac{a_{k+1}}{a_k} < \frac{t}{s} < \frac{a_k}{a_{k-1}}.$$

In the first case, since $(a_k, a_{k-1}) = 1$, there exists $n > 1$ such that $t = na_k, s = na_{k-1}$, i.e. the bundle E_k^n admits resolution (3.3.1), with $m = 0$. In the second case, we solve the following system

$$\begin{cases} t = na_k + ma_{k+1}, \\ s = na_{k-1} + ma_k. \end{cases}$$

This system has discriminant $\Delta = a_k^2 - a_{k+1}a_{k-1} = 1$, by the previous remark, then it admits integer solutions (n, m). In particular, $n > 0$ because $\frac{t}{s} > \frac{a_{k+1}}{a_k}$, and $m > 0$ because $\frac{t}{s} < \frac{a_k}{a_{k-1}}$. It follows that $E_k^n \oplus E_{k+1}^m$ has resolution (3.3.1). $\square$

Lemma 3.3.3. *Let E_k, E_{k+1} be exceptional Steiner bundles on $\mathbb{P}^{N-1}$ as above. Then, for all $n, m \geq 0$,*

$$\chi(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = \mathrm{h}^0(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = n^2 + m^2 + Nnm.$$

Proof. By Lemma (3.1.2) and by the proof of Theorem (3.1.1), using the same notations, we know that

$$\mathrm{Hom}(E_k, E_k) = \mathrm{Hom}(F_k^*, F_k^*) = \mathrm{Hom}(F_k, F_k) \cong \mathbb{C},$$

$$\mathrm{Hom}(E_k, E_{k-1}) = \mathrm{Hom}(F_k^*, F_{k-1}^*) = \mathrm{Hom}(F_{k-1}, F_k) = 0,$$

$$\mathrm{Hom}(E_{k-1}, E_k) = \mathrm{Hom}(F_{k-1}^*, F_k^*) = \mathrm{Hom}(F_k, F_{k-1}) \cong V,$$

and, analogously,

$$\mathrm{Ext}^i(E_k, E_k) = 0, \qquad \mathrm{Ext}^i(E_k, E_{k-1}) = 0, \qquad \mathrm{Ext}^i(E_{k-1}, E_k) = 0,$$

for all $i > 0$. Then,

$$\mathrm{H}^0(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = \mathrm{Hom}(E_k, E_k)^{n^2} \oplus \mathrm{Hom}(E_{k+1}, E_{k+1})^{m^2} \oplus$$

$$\oplus \mathrm{Hom}(E_{k+1}, E_k)^{nm} \oplus \mathrm{Hom}(E_k, E_{k+1})^{nm} = n^2 + m^2 + Nnm$$

and, for all $i > 0$,

$$\mathrm{H}^i(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = \mathrm{Ext}^i(E_k, E_k)^{n^2} \oplus \mathrm{Ext}^i(E_{k+1}, E_{k+1})^{m^2} \oplus$$

$$\oplus \mathrm{Ext}^i(E_{k+1}, E_k)^{nm} \oplus \mathrm{Ext}^i(E_k, E_{k+1})^{nm} = 0.$$

Therefore,

$$\chi(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = \mathrm{h}^0(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = n^2 + m^2 + Nnm,$$

as claimed. $\square$

Consider the space $H = \mathrm{Hom}(\mathcal{O}(-1)^s, \mathcal{O}^t)$ that parameterizes the bundles on $\mathbb{P}^{N-1}$ with resolution

$$0 \to \mathcal{O}(-1)^s \to \mathcal{O}^t \to F \to 0.$$

Consider the following action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$ on H:

$$\mathrm{GL}(s) \times \mathrm{GL}(t) \times H \to H$$

$$(A, B, M) \mapsto A^{-1}MB.$$

For $M \in H$, we denote by $(\mathrm{GL}(s) \times \mathrm{GL}(t))M$ the orbit of M and by $\mathrm{Stab}(M)$ the stabilizer of M, with respect to the action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$.

Lemma 3.3.4. *If F has resolution*

$$0 \to I \otimes \mathcal{O}(-1) \xrightarrow{M} W \otimes \mathcal{O}^t \to F \to 0 \tag{3.3.2}$$

and $\mathrm{H}^0(F^*) = 0$, *then* $\dim \mathrm{Stab}(M) = \dim \mathrm{End}(F)$.

Proof. Dualizing (3.3.2), we get

$$0 \to F^* \to W^\vee \otimes \mathcal{O} \to I^\vee \otimes \mathcal{O}(1) \to 0 \tag{3.3.3}$$

and tensoring it by $I \otimes \mathcal{O}(-1)$, we obtain

$$\mathrm{End}\, I = \mathrm{H}^0(\mathrm{End}\, I) = \mathrm{H}^0(I \otimes I^\vee) = \mathrm{H}^1(I \otimes F^*(-1)).$$

Tensoring (3.3.3) by W and using the hypothesis $\mathrm{H}^0(F^*) = 0$, we get

$$0 \to \mathrm{End}\, W \to \mathrm{H}^0(W \otimes I^\vee \otimes \mathcal{O}(1)) \to W \otimes \mathrm{H}^1(F^*) \to 0.$$

Tensoring (3.3.2) by F^*, we get

$$0 \longrightarrow I \otimes F^*(-1) \longrightarrow W \otimes F^* \longrightarrow \mathrm{End}\, F \longrightarrow 0,$$

and the associated cohomology sequence, together with the previous results, gives the following commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & & & \mathrm{End}\, W \arrow[d, "r_M"] \\
 & & & W \otimes I^\vee \otimes \mathrm{H}^0(\mathcal{O}(1)) = H \arrow[d, "\pi"] \\
0 \arrow[r] & \mathrm{H}^0(\mathrm{End}\, F) \arrow[r, "i"] & \mathrm{End}\, I \arrow[ur, "l_M"] \arrow[r] & W \otimes \mathrm{H}^1(F^*) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

where $l_M(A) = MA$ and $r_M(B) = BM$. Notice that the tangent space to the stabilizer of M is

$$\mathrm{T}(\mathrm{Stab}(M)) = \{(A, B) \in \mathrm{End}\, I \times \mathrm{End}\, W \,|\, l_M(A) = r_M(B)\}.$$

We have to prove that $\dim \mathrm{Stab}(M) = \dim \mathrm{T}(\mathrm{Stab}(M)) = \mathrm{h}^0(\mathrm{End}(F))$. Let us suppose that $A \in \mathrm{End}\, I$ satisfies $l_M(A) \in \mathrm{Im}(r_M)$. Since the map r_M is injective, there exists a unique $B \in \mathrm{End}\, W$ such that (A, B) is in the stabilizer. Moreover $\pi(l_M(A)) = 0$, so, since the diagram is commutative, there exists $\phi = i^{-1}(A) \in \mathrm{H}^0(\mathrm{End}\, F)$ and it is unique, since i is injective. Vice versa, to every $\phi \in \mathrm{H}^0(\mathrm{End}\, F)$ we associate a unique $A = i(\phi)$. Since the sequences are exact and the diagram commutes, we have $l_M(A) \in \mathrm{Ker}\, \pi = \mathrm{Im}\, r_M$, i.e. there exists B such that the pair (A, B) is in the stabilizer. Moreover, B is unique, since $\mathrm{H}^0(F^*) = 0$. $\square$

We denote by F_M the Steiner bundle associated to the matrix M in $\mathrm{Hom}(\mathcal{O}(-1)^s, \mathcal{O}^t) = H$.

Theorem 3.3.5. *If $N \geq 3$ and $\frac{t}{s} > \left(\frac{N+\sqrt{N^2-4}}{2}\right)$, then the space of matrices M such that $F_M \cong E_k^n \oplus E_{k+1}^m$ is a dense subset in H.*

Proof. By Theorem 3.3.2, we can fix $N \in H$ such that

$$0 \to \mathcal{O}(-1)^s \xrightarrow{N} \mathcal{O}^t \to E_k^n \oplus E_{k+1}^m \to 0.$$

Consider the action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$ on H, as above. The space of matrices M such that $F_M \cong E_k^n \oplus E_{k+1}^m$ is the orbit of N, with respect to this action. We denote by $S_N = \mathrm{Stab}(N)$ the stabilizer of N. Notice that $H^0(E_k^{*n} \oplus E_{k+1}^{*m}) = 0$. In fact since the bundle E_k is simple for all k, then $H^0(E_k) \neq 0$ implies $H^0(E_k^*) = 0$. Therefore, by Lemma 3.3.4, we obtain

$$\dim S_N = \dim \mathrm{Hom}(E_k^n \oplus E_{k+1}^m, E_k^n \oplus E_{k+1}^m) = \mathrm{h}^0(\mathrm{End}(E_k^n \oplus E_{k+1}^m)),$$

and now, using Lemma 3.3.3, we get

$$\dim(\mathrm{GL}(s) \times \mathrm{GL}(t))N = \dim(\mathrm{GL}(s) \times \mathrm{GL}(t)) - \dim S_N =$$

$$= s^2 + t^2 - \mathrm{h}^0(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = s^2 + t^2 - (n^2 + m^2 + Nmn).$$

Finally, since

$$\chi(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = s^2 + t^2 - Nst,$$

and, by Lemma 3.3.3,

$$\chi(\mathrm{End}(E_k^n \oplus E_{k+1}^m)) = n^2 + m^2 + Nmn,$$

we get

$$\dim(\mathrm{GL}(s) \times \mathrm{GL}(t))N = s^2 + t^2 - (n^2 + m^2 + Nmn) = Nst,$$

i.e. the space of matrices M such that $F_M \cong E_k^n \oplus E_{k+1}^m$ has dimension $Nst = \dim H$, hence it is dense in H. $\square$

3.4 Reformulation in terms of matrices

Using the language of matrices and recalling the notations introduced in Chapter 1, we can reformulate the results of this chapter in a nice way. Recall that we denote by

$$H = \mathrm{Hom}(I \otimes \mathcal{O}(-1), W \otimes \mathcal{O}) \cong I^\vee \otimes W \otimes V \cong \mathbb{C}^s \otimes \mathbb{C}^t \otimes \mathbb{C}^N,$$

the space of $(s \times t)$-matrices whose entries are linear forms in N variables or, alternatively, the space of $(s \times t \times N)$-matrices of numbers. Then, Theorem 3.2.1 can be reformulated as follows:

Theorem 3.4.1. *Let $M \in H$ and $N \geq 3$. Let us consider the system*

$$XM = MY, \tag{3.4.1}$$

where $X \in \mathrm{GL}(s)$ and $Y \in \mathrm{GL}(t)$ are the unknowns. Then, if $s^2 + t^2 - Nst \leq 1$, there is a dense subset of the space H, where M lives, such that the only solutions of (3.4.1) are trivial, i.e. $(X, Y) = (\lambda\,\mathrm{Id}, \lambda\,\mathrm{Id}) \in \mathrm{GL}(s) \times \mathrm{GL}(t)$ for $\lambda \in \mathbb{C}$. Conversely, if $s^2 + t^2 - Nst \geq 2$, then for all M there are non-trivial solutions.

We say that two matrices $M, M' \in H$ are *$GL(s) \times \mathrm{GL}(t)$-equivalent* if there exist $A \in \mathrm{GL}(s)$ and $B \in \mathrm{GL}(t)$ such that $M' = A^{-1}MB$. In other words, the matrices M and M' are in the same orbit with respect to the action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$ on H. This is equivalent to perform Gaussian elimination on the $(s \times t)$-matrix with linear entries (see description (ii) in Section 1.1).

Definition 3.4.2. *If n, m, k are such that*

$$\mathbb{C}^s = (\mathbb{C}^{a_{k-1}})^n \oplus (\mathbb{C}^{a_k})^m$$

and

$$\mathbb{C}^t = (\mathbb{C}^{a_k})^n \oplus (\mathbb{C}^{a_{k+1}})^m$$

we call "canonical form of type (n, m, k)" a matrix $M \in H$ such that

$$M \in ((\mathbb{C}^{a_{k-1}} \otimes \mathbb{C}^{a_k})^n \oplus (\mathbb{C}^{a_k} \otimes \mathbb{C}^{a_{k+1}})^m) \otimes \mathbb{C}^N \subset H.$$

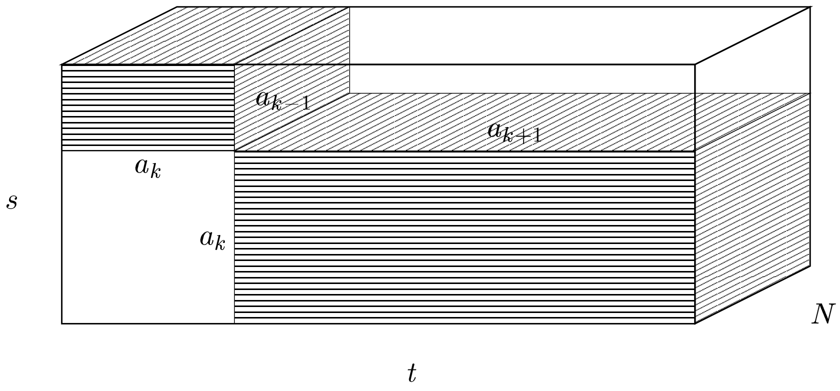

Figure 3.2: Example of canonical matrix of type $(1, 1, k)$

Notice that the matrices in canonical form represent bundles of the form $E_k^n \oplus E_{k+1}^m$, which appear in Theorem 3.3.2. Indeed Theorem 3.3.2 means that when $t > \left(\frac{N+\sqrt{N^2-4}}{2}\right)s$, there exists a unique triple $(\overline{n}, \overline{m}, \overline{k})$ such that $s = a_{\overline{k}-1}\overline{n} + a_{\overline{k}}\overline{m}$ and $t = a_{\overline{k}}\overline{n} + a_{\overline{k}+1}\overline{m}$. Therefore, Theorem 3.3.5 can be reformulated in the following way.

Theorem 3.4.3. *If $t > \frac{N+\sqrt{N^2-4}}{2}s$, then the $\mathrm{GL}(s) \times \mathrm{GL}(t)$-orbit of a canonical matrix of type $(\overline{n}, \overline{m}, \overline{k})$ is a dense subset of H.*

In other words, if $t > \frac{N+\sqrt{N^2-4}}{2}s$, then the set of matrices in H which are $\mathrm{GL}(s) \times \mathrm{GL}(t)$-equivalent to a canonical matrix of type $(\overline{n}, \overline{m}, \overline{k})$ is a dense subset of H. This means that the generic matrix can be transformed into the canonical form $(\overline{n}, \overline{m}, \overline{k})$ by the action of $\mathrm{GL}(s) \times \mathrm{GL}(t)$.

3.4.1 The case of $(s \times t \times 2)$-matrices

In this chapter we have always supposed $N \geq 3$, because the case $N = 2$, corresponding to bundles on $\mathbb{P}^1$, is trivial. In fact, it is easily seen that the exceptional Steiner bundles on $\mathbb{P}^1$ are the line bundles: $E_k \cong \mathcal{O}(k-1)$. Moreover, if $N = 2$, then the sequence $\{a_k\}$ is exactly the sequence $\{k\}$ and the numerical condition $t \geq \frac{N+\sqrt{N^2-4}}{2}s$ is reduced to $t \geq s$.

In fact, it is interesting to note that also in the case $N = 2$ the results of Section 3.3 are true. More precisely, given a pair of natural numbers (s, t), there exists a unique k such that $\frac{k+1}{k} \leq \frac{t}{s} < \frac{k}{k-1}$. The system

$$\begin{cases} t = nk + m(k+1) \\ s = n(k-1) + mk \end{cases}$$

has the explicit solutions $(\overline{n} = kt - (k+1)s, \overline{m} = ks - (k-1)t)$. Therefore, the canonical form of a matrix of size $(s \times t \times 2)$, with $t \geq s$, is a matrix formed by two block with sizes respectively $(\overline{n}(k-1) \times \overline{n}k \times 2)$ and $(\overline{m}k \times \overline{m}(k+1) \times 2)$ and we get the following result.

Theorem 3.4.4. *If $t \geq s$ and $N = 2$, the generic matrix in H is $\mathrm{GL}(s) \times \mathrm{GL}(t)$-equivalent to a matrix of the canonical form.*

3.5 Generalization of Steiner bundles

We easily obtain a generalization of the theorems concerning Steiner bundles to the case of bundles on $\mathbb{P}^{N-1}$ with resolution

$$0 \to \mathcal{O}(-h)^s \to \mathcal{O}^t \to F \to 0,$$

for all $1 \leq h \leq N-1$.

Remark 3.5.1. As in the case of Steiner bundles, we require that $t - s \geq N - 1$.

Remark 3.5.2. The condition $1 \leq h \leq N-1$ is necessary, because of cohomological computations. It is easy to check that if $h \geq N$, then there exist some $i > 0$ such that $\mathrm{H}^i(\mathrm{End}\, F) \neq 0$. More precisely, we can compute e.g. that

$$\dim \mathrm{H}^{N-2}(\mathrm{End}\, F) = \begin{pmatrix} h-1 \\ h-N \end{pmatrix} \neq 0.$$

Now we give only the results, because the corresponding proves are simply the natural generalizations of the proves in the case of Steiner bundles. The generalization of Theorem 3.1.1 is the following.

Theorem 3.5.3. *Let $N \geq 3$, $1 \leq h \leq N-1$ and $L = \dim \mathrm{H}^0(\mathcal{O}(h)) = \begin{pmatrix} N+h-1 \\ h \end{pmatrix}$. Let H_k be a generic bundle on $\mathbb{P}^{N-1}$ defined by the exact sequence*

$$0 \to \mathcal{O}(-h)^{b_{k-1}} \to \mathcal{O}^{b_k} \to H_k \to 0, \tag{3.5.1}$$

where

$$b_k = \frac{\left(\frac{L+\sqrt{L^2-4}}{2}\right)^k - \left(\frac{L-\sqrt{L^2-4}}{2}\right)^k}{\sqrt{L^2-4}},$$

then H_k is exceptional.

Let us observe that the sequence $\{b_k\}$ is defined recursively by

$$\begin{cases} b_0 = 0, \\ b_1 = 1, \\ b_{k+1} = Lb_k - b_{k-1}, \end{cases}$$

i.e. the numbers $\{b_k\}$, as well as $\{a_k\}$, are generalized Fibonacci numbers. Indeed if H is a bundle on $\mathbb{P}^{N-1}$ corresponding to a matrix of homogeneous forms of degree h, then the sequence $\{b_k\}$ is exactly the generalized Fibonacci sequence associated to the integer

$$L = \binom{N+h-1}{h}.$$

If $h = 1$, then $L = N$ and we find again the case of Steiner bundles.

The generalization of Theorem 3.2.1 is the following.

Theorem 3.5.4. *Let $N \geq 3$, $1 \leq h \leq N-1$ and $L = \dim \mathrm{H}^0(\mathcal{O}(h))$. Let F be a generic bundle on $\mathbb{P}^{N-1}$ defined by the exact sequence*

$$0 \to \mathcal{O}(-h)^s \to \mathcal{O}^t \to F \to 0.$$

Then the following statements are equivalent:

(i) *F is simple, i.e.* $\mathrm{h}^0(\mathrm{End}\, F) = 1$,

(ii) $s^2 - Lst + t^2 \leq 1$ *i.e.* $\chi(\mathrm{End}\, F) \leq 1$,

(iii) *either F is exceptional or $s^2 - Lst + t^2 \leq 0$ i.e. $t \leq (\frac{L+\sqrt{L^2-4}}{2})s$.*

Finally Theorem 3.3.2 and Theorem 3.3.5 are generalized as follows.

Theorem 3.5.5. *Let $N \geq 3$, $1 \leq h \leq N-1$ and $L = \dim \mathrm{H}^0(\mathcal{O}(h))$.
If $t > \left(\frac{L+\sqrt{L^2-4}}{2}\right)s$, then there exist $k, n, m \in \mathbb{N}$ (where n and m are not both 0) such that the bundle $H_k^n \oplus H_{k+1}^m$ on $\mathbb{P}^{N-1}$ has resolution*

$$0 \to \mathcal{O}(-h)^s \to \mathcal{O}^t \to H_k^n \oplus H_{k+1}^m \to 0,$$

where H_k and H_{k+1} are exceptional bundles on P^{N-1} with resolution (3.5.1).

Theorem 3.5.6. *Let $N \geq 3$, $1 \leq h \leq N-1$ and $L = \dim \mathrm{H}^0(\mathcal{O}(h))$.
If $t > \left(\frac{L+\sqrt{L^2-4}}{2}\right)s$, then the space of matrices M such that the bundle associated to M is isomorphic to $H_k^n \oplus H_{k+1}^m$ is a dense subset of the vector space* $\mathrm{Hom}(\mathcal{O}(-h)^s, \mathcal{O}^t)$.

Chapter 4

Resolutions of exceptional bundles on $\mathbb{P}^2$

4.1 Prioritary bundles and resolutions

The prioritary bundles on $\mathbb{P}^2$ were introduced by Hirschowitz and Laszlo in [HL93] and Dionisi and Maggesi studied their resolutions in [DM03].

A vector bundle (or a coherent torsionfree sheaf) is called *prioritary* when

$$\mathrm{Ext}^2(F, F(-1)) = 0.$$

It is easily seen that any semi-stable vector bundle is prioritary.

Proposition 4.1.1 ([HL93]). *If F is a generic prioritary bundle on $\mathbb{P}^2$, it has resolution either of the form*

$$0 \to \mathcal{O}(k-2)^a \oplus \mathcal{O}(k-1)^b \to \mathcal{O}(k)^c \to F \to 0, \tag{4.1.1}$$

or

$$0 \to \mathcal{O}(k-2)^a \to \mathcal{O}(k-1)^b \oplus \mathcal{O}(k)^c \to F \to 0, \tag{4.1.2}$$

for some $k \in \mathbb{Z}$, $a, b \geq 0$ and $c > 0$.

Lemma 4.1.2. *Given a generic prioritary bundle F with resolution either* (4.1.1)*, or* (4.1.2)*, the integer k which appears in the resolutions satisfies the following conditions*

$$\chi(F(-k-1)) \leq 0 \quad \textit{and} \quad \chi(F(-k)) > 0.$$

Vice versa, if k satisfies these two conditions, then it appears in resolution (4.1.1) *or* (4.1.2) *of F. Furthermore, $\chi(F(-k-1)) = 0$ holds if and only if $a = 0$ in both cases* (4.1.1) *and* (4.1.2).

Proof. In the case (4.1.1), the Hilbert polynomial of F

$$\chi(F(t)) = \Big(\frac{c-b-a}{2}\Big)t^2 + \Big(\frac{3c-b+a}{2} + (c-b-a)k\Big)t+ \\ +\Big(\frac{c-b-a}{2}k^2 + \frac{3c-b+a}{2}k + c\Big),$$

has two roots. The largest root is

$$t_0 = -k + \frac{-3c+b-a+\sqrt{b^2-2ab+a^2+c^2+2bc+14ac}}{2(c-b-a)}$$

and it is easy to check that

$$-k-1 \leq t_0 < -k.$$

Moreover, $t_0 = -k-1$ holds true if and only if $a = 0$. Thus, since $y = \chi(F(t))$ is a convex parabola, then when F has a resolution of the form (4.1.1) with $a \neq 0$, we get $\chi(F(-k-1)) < 0$ and $\chi(F(-k)) > 0$. On the other hand, if G is a bundle of the form

$$0 \to \mathcal{O}(k-1)^b \to \mathcal{O}(k)^c \to G \to 0$$

then $\chi(G(-k-1)) = 0$ and $\chi(G(-k)) > 0$.

In the case (4.1.2), the Hilbert polynomial of F is

$$\chi(F(t)) = \Big(\frac{c+b-a}{2}\Big)t^2 + \Big(\frac{3c+b+a}{2} + (c+b-a)k\Big)t+ \\ +\Big(\frac{c+b-a}{2}k^2 + \frac{3c+b+a}{2}k + c\Big).$$

The largest root is

$$t_0 = -k + \frac{-3c-b-a+\sqrt{b^2+2ab+a^2+c^2-2bc+14ac}}{2(c+b-a)}$$

and we get

$$-k-1 \leq t_0 < -k,$$

in particular $t_0 = -k-1$ if and only if $a = 0$. It follows that a bundle with resolution (4.1.2) and $a \neq 0$ has $\chi(F(-k-1)) < 0$ and $\chi(F(-k)) > 0$. Moreover, a bundle of the form

$$G = \mathcal{O}(k-1)^b \oplus \mathcal{O}(k)^c$$

and $c \neq 0$ satisfies $\chi(G(-k-1)) = 0$ and $\chi(G(-k)) > 0$. □

Lemma 4.1.3. *Let E be a normalized exceptional bundle different from $\mathcal{O}$ and $\mathcal{O}(-1)$. Then E has a resolution of the form either*

$$0 \to \mathcal{O}(-3)^a \oplus \mathcal{O}(-2)^b \to \mathcal{O}(-1)^c \to E \to 0, \tag{4.1.3}$$

or

$$0 \to \mathcal{O}(-3)^a \to \mathcal{O}(-2)^b \oplus \mathcal{O}(-1)^c \to E \to 0. \tag{4.1.4}$$

Proof. Since E is exceptional, then it is prioritary and rigid and it has a resolution of the form either (4.1.1), or (4.1.2). We have only to prove that $k = -1$ or equivalently, by Lemma 4.1.2, that $\chi(E) \leq 0$ and $\chi(E(1)) > 0$.

Now, we show that $\chi(E) \leq 0$. We know that

$$\chi(E) = \chi(\mathcal{O}, E) = \dim \operatorname{Hom}(\mathcal{O}, E) - \dim \operatorname{Ext}^1(\mathcal{O}, E) + \dim \operatorname{Ext}^2(\mathcal{O}, E).$$

Since E is stable and its slope is negative, then $\operatorname{Hom}(\mathcal{O}, E) = \operatorname{Ext}^2(\mathcal{O}, E) = 0$, hence $\chi(E) \leq 0$.

Now, let us compute $\chi(E(1))$: by Riemann-Roch formula, we have that

$$\chi(E(1)) = r_E\Big(\frac{(\mu+1)^2 + 3(\mu+1) + 2}{2} - \Delta(E)\Big).$$

Since E is exceptional, then $\Delta(E) = \frac{1}{2}(1 - \frac{1}{r^2})$, which implies that

$$\chi(E(1)) = r_E\Big(\frac{(\mu+1)^2 + 3(\mu+1) + 1}{2} + \frac{1}{2r^2}\Big) > 0.$$

$\square$

Remark 4.1.4. Let E be a bundle with resolution (4.1.3) or (4.1.4). It is easy to compute that in both cases $a = -\chi(E) \geq 0$ and $c = \chi(E(1)) > 0$. We distinguish the different sequences by means of the value $l = 3\chi(E) - \chi(E(-1))$ as follows: if $l > 0$ then E has resolution (4.1.3) with $b = l$, if $l \leq 0$ then E has resolution (4.1.4) with $b = -l$.

Since the dual of an exceptional bundle is exceptional too, there exist two admissible right resolutions for a normalized exceptional bundle, which are

$$0 \to F \to \mathcal{O}^c \to \mathcal{O}(1)^b \oplus \mathcal{O}(2)^a \to 0, \tag{4.1.5}$$

and

$$0 \to F \to \mathcal{O}^c \oplus \mathcal{O}(1)^b \to \mathcal{O}(2)^a \to 0. \tag{4.1.6}$$

By gluing a left and a right resolution, we get a long exact sequence, for example from (4.1.3) and (4.1.6) we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-3)^a \oplus \mathcal{O}(-2)^b \arrow[r] & \mathcal{O}(-1)^c \arrow[rr] \arrow[dr] & & \mathcal{O}^d \oplus \mathcal{O}(1)^e \arrow[r] & \mathcal{O}(2)^f \arrow[r] & 0 \\
 & & & F \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

$$0 \to \mathcal{O}(-3)^a \oplus \mathcal{O}(-2)^b \xrightarrow{M} \mathcal{O}(-1)^c \to \mathcal{O}^d \oplus \mathcal{O}(1)^e \xrightarrow{N} \mathcal{O}(2)^f \to 0,$$

where $F = \operatorname{Coker} M = \operatorname{Ker} N$. We say that F is the *middle syzygy bundle* of the long exact sequence.

Lemma 4.1.5. *If E is a normalized exceptional bundle with slope $\mu \geq -\frac{1}{2}$ different from $\mathcal{O}$, then E has left resolution of the form* (4.1.3).

Proof. Denote $r = \operatorname{rk}(E)$, then by Riemann-Roch formula we compute

$$\begin{aligned}
l &= 3\chi(E) - \chi(E(-1)) = \\
&= 3r\Big(\frac{\mu^2 + 3\mu + 2}{2} - \Delta(E)\Big) - r\Big(\frac{(\mu-1)^2 + 3(\mu-1) + 2}{2} - \Delta(E)\Big).
\end{aligned}$$

Since $\Delta(E) = \frac{1}{2}(1 - \frac{1}{r^2})$, then

$$l = r(\mu^2 + 4\mu + 2) + \frac{1}{r},$$

which implies that

$$l \geq 0 \qquad \text{iff} \qquad \mu \geq -2 + \frac{\sqrt{2r^2 - 1}}{r}.$$

In particular if

$$\mu \geq -\frac{1}{2} > -2 + \frac{\sqrt{2r^2 - 1}}{r},$$

then $l > 0$ and so, by Remark 4.1.4, E has a resolution of the form (4.1.3). $\square$

In general by gluing the admissible resolutions (4.1.3), (4.1.4) and (4.1.5), (4.1.6) four different cases arise, but as far as exceptional bundles are concerned, we will see that only two of them are possible. More precisely, in Remark 4.4.3 of Section 4.4, we will prove that the following proposition holds.

Proposition 4.1.6. *Let E be a normalized exceptional bundle with* $\mathrm{rk}(E) > 1$ *and* $\mu(E) \geq -\frac{1}{2}$. *Then E has left resolution of the form* (4.1.3) *and right resolution of the form* (4.1.6), *i.e. E is the middle syzygy bundle of the following exact sequence*

$$0 \to \mathcal{O}(-3)^a \oplus \mathcal{O}(-2)^b \to \mathcal{O}(-1)^c \to \mathcal{O}^d \oplus \mathcal{O}(1)^e \to \mathcal{O}(2)^f \to 0.$$

By duality, if $\mu(E) < -\frac{1}{2}$, *then E has resolutions* (4.1.4) *and* (4.1.5), *i.e. E is the middle syzygy bundle of the sequence*

$$0 \to \mathcal{O}(-3)^a \to \mathcal{O}(-2)^b \oplus \mathcal{O}(-1)^c \to \mathcal{O}^d \to \mathcal{O}(1)^e \oplus \mathcal{O}(2)^f \to 0.$$

4.2 Exceptional bundles on $\mathbb{P}^2$ with only one quadratic form

Here, we study the case of exceptional bundles F on $\mathbb{P}^2$ with resolution

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \xrightarrow{M} \mathcal{O}(-1)^t \to F \to 0.$$

This is the first interesting generalization of the case of Steiner bundles, because in this case the entries of the matrix M are all linear forms except those which lie on one row and are quadratic homogeneous forms. In this section we will see that, also in this case, some good properties hold, which are not true in general. In particular, Corollary 4.2.8 states that the exceptional bundles of this form are characterized by the property $\chi(\mathrm{End}\, F) = 1$ and they are all classified. Furthermore, any exceptional bundle of this form comes from a left mutation of the exceptional pair $(E_k, \mathcal{O})$, where E_k is a Steiner exceptional bundle (see Remark 4.3.6).

As in the previous chapter, we denote

$$a_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2k} - \left(\frac{1-\sqrt{5}}{2}\right)^{2k}}{\sqrt{5}},$$

$$r_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2k-1}}{\sqrt{5}},$$

i.e. $\{a_k\}$ is the even part of the Fibonacci sequence and $\{r_k\}$ is the odd part. Notice that

$$r_k = a_k - a_{k-1},$$

and that the sequences $\{a_k\}$ and $\{r_k\}$ are defined recursively as follows:

$$\begin{cases} a_0 = 0, \\ a_1 = 1, \\ a_{k+1} = 3a_k - a_{k-1}, \end{cases} \quad \text{and} \quad \begin{cases} r_1 = 1, \\ r_2 = 2, \\ r_{k+1} = 3r_k - r_{k-1}. \end{cases}$$

Remark 4.2.1. For all $k \geq 1$ the following equalities hold

$$a_k - 2a_{k-1} = a_{k-1} - a_{k-2} = r_{k-1},$$

and

$$a_k^2 + a_{k-1}^2 - 3a_{k-1}a_k = a_k^2 - a_{k+1}a_{k-1} = 1,$$

and the same relations hold for r_k. It is easy to prove them by induction.

According to the notations of the previous chapter, we denote by E_k the exceptional Steiner bundle on $\mathbb{P}^2$ defined by the exact sequence

$$0 \to \mathcal{O}(-1)^{a_{k-1}} \to \mathcal{O}^{a_k} \to E_k \to 0.$$

It is easily seen that, for any $k \geq 2$, the normalized exceptional Steiner bundle is $\widehat{E}_k = E_k(-1)$ and has resolution

$$0 \to \mathcal{O}(-2)^{a_{k-1}} \to \mathcal{O}(-1)^{a_k} \to \widehat{E}_k \to 0. \tag{4.2.1}$$

Now, consider the sequence of bundles D_k given by a left mutation of the exceptional pair $(\widehat{E}_k, \mathcal{O})$ as follows:

$$D_k = \ker(\widehat{E}_k \otimes \mathrm{Hom}(\widehat{E}_k, \mathcal{O}) \xrightarrow{\phi_k} \mathcal{O}),$$

where ϕ_k is the canonical map, i.e.

$$0 \to D_k \to \mathrm{Hom}(\widehat{E}_k, \mathcal{O}) \otimes \widehat{E}_k \to \mathcal{O} \to 0.$$

From the theory of mutations, we know that the bundles D_k are exceptional for all k.

Lemma 4.2.2. *The bundles D_k admit the resolution*

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{3r_k a_{k-2}} \to \mathcal{O}(-1)^{3r_k a_{k-1}} \to D_k \to 0,$$

for all $k \geq 3$.

Proof. In order to simplify the computations, we will prove that the bundles $D_k(1)$ have resolution

$$0 \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{3r_k a_{k-2}} \to \mathcal{O}^{3r_k a_{k-1}} \to D_k(1) \to 0,$$

where $D_k(1)$ is defined by the sequence

$$0 \to D_k(1) \to \operatorname{Hom}(\widehat{E}_k, \mathcal{O}) \otimes E_k \to \mathcal{O}(1) \to 0. \tag{4.2.2}$$

Notice that $\dim \operatorname{Hom}(\widehat{E}_k, \mathcal{O}) = \mathrm{h}^0(\widehat{E}_k^*) = 3r_{k-1}$, hence from (4.2.1) and from (4.2.2) we can compute

$$\operatorname{rk}(D_k(1)) = 3r_{k-1}(a_k - a_{k-1}) - 1,$$

$$c_1(D_k(1)) = 3r_{k-1}a_{k-1} - 1,$$

$$c_2(D_k(1)) = \frac{1}{2}(9r_{k-1}^2 a_{k-1}^2 - 3r_{k-1}a_{k-1} + 2).$$

We observe that, for all $k \geq 3$, we get

$$\mu(D_k(1)) = \frac{3r_{k-1}a_{k-1} - 1}{3r_{k-1}(a_k - a_{k-1}) - 1} > \frac{1}{2}.$$

In fact, $2(3r_{k-1}a_{k-1} - 1) - (3r_{k-1}(a_k - a_{k-1}) - 1) > 0$, since $3a_{k-1}r_{k-2} - 4 > 0$ for all $k \geq 3$. Therefore, we have obviously

$$\mu(D_k) = \mu(D_k(1)) - 1 > -\frac{1}{2}.$$

By Lemma 4.1.5, we know that D_k has a resolution of the form

$$0 \to \mathcal{O}(-3)^a \oplus \mathcal{O}(-2)^b \to \mathcal{O}(-1)^c \to D_k \to 0,$$

i.e. $D_k(1)$ has a resolution

$$0 \to \mathcal{O}(-2)^a \oplus \mathcal{O}(-1)^b \to \mathcal{O}^c \to D_k(1) \to 0, \tag{4.2.3}$$

for some $a, b, c \in \mathbb{N}$. If we compute rank, c_1 and c_2 of $D_k(1)$ using the resolution (4.2.3), then we get the following system:

$$\begin{cases} c - b - a = 3r_{k-1}(a_k - a_{k-1}) - 1, \\ b + 2a = 3r_{k-1}a_{k-1} - 1, \\ \frac{1}{2}(b^2 + 4a^2 + 4ab + 4a + b) = \frac{1}{2}(9r_{k-1}^2 a_{k-1}^2 - 3r_{k-1}a_{k-1} + 2). \end{cases}$$

This system admits the following unique solution:

$$\begin{cases} a = 1, \\ b = 3(r_{k-1}a_{k-1} - 1) = 3r_k a_{k-2}, \\ c = 3(r_{k-1}a_k - 1) = 3r_k a_{k-1}, \end{cases}$$

and this is exactly our claim. $\square$

Furthermore, the bundles D_k are the only exceptional bundles of the form

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0,$$

for some $s, t \in \mathbb{N}$. In order to prove this fact, we have to use some techniques from Number Theory.

4.2.1 Preliminary techniques on Number Theory

In this section, we recall some definitions and results of Number Theory; as a reference see [BS66].

A *quadratic field* K is an extension of the rational field $\mathbb{Q}$ of degree 2. In a quadratic field $K = \mathbb{Q}[\sqrt{d}]$, the *norm* of an element is defined as

$$N(a + b\sqrt{d}) = a^2 - db^2.$$

We call *unit* of K an invertible element of the *integers ring* of K (i.e. of the ring of elements of K whose minimum polynomial has integer coefficients). We say that ε is a positive unit, if $N(\varepsilon) = 1$. The set of positive units of a quadratic field $\mathbb{Q}[\sqrt{d}]$, with $d > 0$, is a group isomorphic to $\mathbb{Z}$. We say that two elements x, y in K are *associate* if there exists a unit ε such that $x = \varepsilon y$.

For any $\alpha, \beta \in K$, the *module* $\{\alpha, \beta\}$ is defined as the set of all linear combinations of α and β with coefficients in $\mathbb{Z}$. Two modules M, N are called *similar* if there exists $0 \neq \gamma \in K$ such that $M = \gamma N$ (and *strictly similar* if $N(\gamma) > 0$).

Lemma 4.2.3 ([BS66]). *Two modules $\{1, \alpha\}$ and $\{1, \beta\}$ are similar if and only if there exist some integers k, l, m, n such that*

$$\alpha = \frac{k\beta + l}{m\beta + n}$$

and $kn - ml = 1$; more precisely we have

$$\{1, \alpha\} = \frac{1}{m\beta + n}\{m\beta + n, k\beta + l\} = \frac{1}{m\beta + n}\{1, \beta\}.$$

Any element δ such that $\delta M \subset M$ is called *coefficient* of the module M and it is known that the set of coefficients $\mathcal{D}$ is a ring. We define the *norm of a module* M as the determinant of the transition matrix from a basis of the ring $\mathcal{D}$, to a basis of M. The set of classes of similar modules in a quadratic field, with given coefficient ring, forms a finite Abelian (multiplicative) group.

The *discriminant* of a binary quadratic form $Ax^2 + Bxy + cy^2$ is the integer $B^2 - 4AC$. Any quadratic form $F(x, y)$ can be represented by a module $M = \{\alpha, \beta\}$ in the following way:

$$F(x, y) = \frac{N(\alpha x + \beta y)}{N(M)}.$$

The mapping $(x, y) \mapsto \xi = \alpha x + \beta y$ establishes a one-to-one correspondence between the solutions of the equation

$$F(x, y) = m$$

and the numbers $\xi \in M$ with norm $mN(M)$. Two solutions are called *associate* if the corresponding numbers in M are associate.

Theorem 4.2.4 ([BS66]). *Let $F(x, y)$ be a quadratic form represented by a class C of strictly similar modules with coefficient ring $\mathcal{D}$. Given the equation*

$$F(x, y) = m,$$

with $0 < m \in \mathbb{Z}$, then the set of classes of associate solutions of this equation is in a one-to-one correspondence with the set of modules A which are in the class (of strictly similarity) C^{-1}, are contained in $\mathcal{D}$ and have norm m.

In order to find the modules A satisfying the conditions of this theorem, we can apply the following algorithm:

- find the integers $s > 0, a > 0, b, c$ such that

  $$m = a^2 s, \qquad b^2 - 4ac = D, \qquad (a, b, c) = 1, \qquad -a \leq b \leq a$$

  where D is the discriminant of F;

- set $\gamma = \frac{-b+\sqrt{D}}{2a}$ and obtain the module $A = \{as, as\gamma\}$.

4.2.2 Simplifying the equation

Now, we easily compute that

$$\chi(\operatorname{End} F) = t^2 + s^2 - 3st - 6t + 3s + 1.$$

We know that if F is exceptional then $\chi(\operatorname{End} F) = 1$. Hence, if F has resolution

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0,$$

the numbers s, t are integer solutions of the following equation

$$t^2 + s^2 - 3st - 6t + 3s = 0. \tag{4.2.4}$$

Remark 4.2.5. For each $k \geq 2$, the pair $(t_k, s_k) = (3r_k a_{k-1}, 3r_k a_{k-2})$ is a solution of equation (4.2.4). Observe that the sequence

$$(t_k, s_k) = (6, 0), (45, 15), (312, 117), (2142, 816), \ldots$$

is defined recursively as

$$\begin{cases} t_2 = 6, \\ s_2 = 0, \\ t_{k+1} = 8t_k - 3s_k - 3, \\ s_{k+1} = 3t_k - s_k - 3. \end{cases} \tag{4.2.5}$$

Now, by the following change of coordinates

$$\begin{cases} q = 5s + 12, \\ p = 2t - 3s - 6, \end{cases} \qquad \text{i.e.} \qquad \begin{cases} t = \frac{1}{10}(3q + 5p - 6), \\ s = \frac{1}{5}(q - 12), \end{cases}$$

we obtain the following equation

$$5p^2 - q^2 = 36. \tag{4.2.6}$$

This diophantine equation is not equivalent to (4.2.4), because, even if any integral solution of (4.2.4) is also an integral solution of (4.2.6), the converse is false. So, in order to solve our problem, we need to list all the solutions of (4.2.6) and then determine those which come from an integral solution of (4.2.4). Finally, we will check that the solutions that we find are exactly those already known, defined by (4.2.5).

4.2.3 Finding the solutions of $5p^2 - q^2 = 36$.

Let K be the quadratic field $\mathbb{Z}[\sqrt{5}]$. Let $\{\varepsilon_n\}$ be the group of positive units of K: it is known that $\varepsilon_n = (\frac{3+\sqrt{5}}{2})^n$, for any $n \geq 1$. To each solution (Q, P) of (4.2.6), we associate the number $\xi = Q + \sqrt{5}P$ in K. For any n, the number $\xi\varepsilon_n$ corresponds to another solution of (4.2.6). The smallest integer positive solution of (4.2.6) is $(3, 3)$, then by multiplying $3 + 3\sqrt{5}$ by ε_n, we obtain the sequence of solutions (Q_n, P_n), defined as follows:

$$\begin{cases} Q_1 = 3, \\ P_1 = 3, \\ Q_{n+1} = \frac{1}{2}(3Q_n + 5P_n), \\ P_{n+1} = \frac{1}{2}(Q_n + 3P_n), \end{cases}$$

i.e.

$$\{(Q_n, P_n)\} = \{(3,3), (12,6), (33,15), (87,39), (228,102), (597,267), \ldots\}.$$

4.2.4 Uniqueness of the solutions of $5p^2 - q^2 = 36$.

In order to prove that these are all the solutions of (4.2.6), we apply Theorem 4.2.4 and the corresponding algorithm. Then, we have to consider all modules A with norm 36, contained in the coefficient ring of A and with discriminant 20. In order to ensure these conditions, we have to find some integers numbers $s > 0, a > 0, b, c$ such that

$$36 = as^2, \qquad b^2 - 4ac = 20, \qquad (a, b, c) = 1, \qquad -a \leq b \leq a.$$

By solving computations, we obtain three solutions

$$s = 3, a = 4, b = -2, c = -1,$$

$$s = 3, a = 4, b = 2, c = -1,$$

$$s = 6, a = 1, b = 0, c = -5,$$

which correspond to the modules

$$A_1 = \{12, 3 + 3\sqrt{5}\}, A_2 = \{12, -3 + 3\sqrt{5}\}, A_3 = \{6, 6\sqrt{5}\}.$$

Now, we have to check the strictly similarity of these modules, with respect to the module associated to the equation, i.e.

$$M = M^{-1} = \{\sqrt{5}, 1\}.$$

To do this, we use Lemma 4.2.3 and then we compute that

$$A_1 = \{12, 3+3\sqrt{5}\} = 12\{1, \frac{1+\sqrt{5}}{4}\} = 12\frac{5}{-\sqrt{5}+5}\{\frac{-\sqrt{5}+5}{5}, \frac{\sqrt{5}}{5}\} =$$

$$= 12\frac{5}{-\sqrt{5}+5}\{1, \frac{\sqrt{5}}{5}\} = 12\frac{5}{-\sqrt{5}+5}\frac{\sqrt{5}}{5}\{\sqrt{5}, 1\} = (3\sqrt{5}+3)\{\sqrt{5}, 1\},$$

and analogously

$$A_2 = \{12, -3+3\sqrt{5}\} = 12\{1, \frac{-1+\sqrt{5}}{4}\} = 12\frac{5}{\sqrt{5}+5}\{\frac{\sqrt{5}+5}{5}, \frac{\sqrt{5}}{5}\} =$$

$$= 12\frac{5}{\sqrt{5}+5}\{1, \frac{\sqrt{5}}{5}\} = 12\frac{5}{\sqrt{5}+5}\frac{\sqrt{5}}{5}\{\sqrt{5}, 1\} = (3\sqrt{5}-3)\{\sqrt{5}, 1\},$$

and

$$A_3 = \{6, 6\sqrt{5}\} = 6\{1, \sqrt{5}\} = 6\frac{5}{2\sqrt{5}+5}\{\frac{2\sqrt{5}+5}{5}, \sqrt{5}+1\} =$$

$$= 6\frac{5}{2\sqrt{5}+5}\{1, \frac{\sqrt{5}}{5}\} = 6\frac{5}{2\sqrt{5}+5}\frac{\sqrt{5}}{5}\{\sqrt{5}, 1\} = (6\sqrt{5}-12)\{\sqrt{5}, 1\}.$$

The positive integer solutions (q, p) associated to these modules are $(3, 3)$ and $(12, 6)$. Notice that they are associate, because

$$(3+3\sqrt{5})\frac{3+\sqrt{5}}{2} = 12+6\sqrt{5}$$

and, thus, we have only one independent solution $(3, 3)$ of our equation (4.2.6).

So, we have proved the following lemma.

Lemma 4.2.6. *All the integer solutions of $5p^2 - q^2 = 36$ are given by the sequence (Q_n, P_n), where*

$$\begin{cases} Q_1 = 3, \\ P_1 = 3, \\ Q_{n+1} = \frac{1}{2}(3Q_n + 5P_n), \\ P_{n+1} = \frac{1}{2}(Q_n + 3P_n). \end{cases}$$

4.2.5 Solutions of $t^2 + s^2 - 3st - 6t + 3s = 0$.

By changing coordinates from q, p to t, s we get, for example:

$$(3, 3) \to (9/5, -9/5), \qquad (12, 6) \to (6, 0),$$

$$(33,15) \to (84/5, 21/5), \qquad (87,39) \to (45,15).$$

In general, we have

$$\begin{cases} T_1 = 9/5, \\ S_1 = -9/5, \\ T_{n+1} = 3T_n - S_n - 6/5, \\ S_{n+1} = T_n - 9/5. \end{cases}$$

We observe that the solutions with odd index are not integer, and those with even index are integers and they are exactly (4.2.5). In fact, denoting $A_k = 5T_k$ and $B_k = 5S_k$, we get

$$A_{2k-1} \equiv 4 \mod 5 \qquad \text{and} \qquad B_{2k+1} \equiv 1 \mod 5,$$

for all $k \geq 1$, and

$$A_{2k} \equiv 0 \mod 5 \qquad \text{and} \qquad B_{2k} \equiv 0 \mod 5,$$

for all $k \geq 1$. We prove this claim by induction: first of all, we check that $A_1 = 9 \equiv 4 \mod 5$, $B_1 = -9 \equiv 1 \mod 5$, $A_2 = 30 \equiv 0 \mod 5$ and $B_2 = 0 \equiv 0 \mod 5$. Then

$$A_{2k} = 5T_{2k} = 5(3T_{2k-1} - S_{2k-1} - 6/5) = 3A_{2k-1} - B_{2k-1} - 6 \equiv 0 \mod 5,$$

and analogously

$$B_{2k} = 5S_{2k} = 5(T_{2k-1} - 9/5) = A_{2k-1} - 9 \equiv 0 \mod 5.$$

It follows that T_{2k} and S_{2k} are integer numbers for all $k \geq 1$. Now, notice that

$$A_{2k+1} = 5T_{2k+1} = 5(3T_{2k} - S_{2k} - 6/5) = 3A_{2k} - B_{2k} - 6 \equiv 4 \mod 5,$$

and analogously

$$B_{2k+1} = 5S_{2k+1} = 5(T_{2k} - 9/5) = A_{2k} - 9 \equiv 1 \mod 5.$$

Therefore, we deduce that, for all $k \geq 1$,

$$5T_{2k+1} \equiv 4 \mod 5, \qquad 5S_{2k+1} \equiv 1 \mod 5.$$

Finally the pairs (T_{2k}, S_{2k}) are exactly our solutions (4.2.5). In fact $(T_2, S_2) = (6,0)$ and, by composing twice the recursive formula, we get

$$\begin{cases} T_{n+2} = 8T_n - 3S_n - 3, \\ S_{n+2} = 3T_n - S_n - 3, \end{cases}$$

and this is exactly the sequence (4.2.5), as claimed.

In conclusion, we have proved the following lemma.

Lemma 4.2.7. *All the integer solutions of the equation*

$$t^2 + s^2 - 3st - 6t + 3s = 0$$

are given by the sequence (t_k, s_k)*, where*

$$\begin{cases} t_2 = 6, \\ s_2 = 0, \\ t_{k+1} = 8t_k - 3s_k - 3, \\ s_{k+1} = 3t_k - s_k - 3. \end{cases}$$

Finally from the results of this section, we immediately deduce the following theorem:

Theorem 4.2.8. *Let F be a bundle with resolution*

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0. \tag{4.2.7}$$

Then the following are equivalent

(1) *F is exceptional,*

(2) $\chi(\operatorname{End} F) = 1$,

(3) $s = 3r_k a_{k-2}, t = 3r_k a_{k-1}$.

Remark 4.2.9. We have proved that the property

$$F \text{ is exceptional} \qquad \Leftrightarrow \qquad \chi(\operatorname{End} F) = 1$$

holds when F is Steiner and when F has resolution (4.2.7). Recall that in general this property is false: for a counterexample, see Remark 3.1.5.

4.3 The other exceptional bundles on $\mathbb{P}^2$

Now, in order to understand how to characterize the resolutions of all exceptional bundles on $\mathbb{P}^2$, we study the case of bundles G_k, coming from two left mutations of the exceptional pair $(\widehat{E}_k, \mathcal{O})$.

Lemma 4.3.1. *Let G_k be an exceptional bundle obtained by two left mutations from a Steiner exceptional bundle $\widehat{E}_k$. Then G_k has resolution*

$$0 \to \mathcal{O}(-3)^{q_k} \oplus \mathcal{O}(-2)^{s_k} \to \mathcal{O}(-1)^{t_k} \to G_k \to 0, \tag{4.3.1}$$

with

$$\begin{cases} q_k = 3r_{k-1}, \\ s_k = 9r_k r_{k-1} a_{k-2} - a_{k-1}, \\ t_k = 9r_k r_{k-1} a_{k-1} - a_k, \end{cases}$$

where $\{a_k\}$ is the even part of sequence of Fibonacci numbers and $\{r_k\}$ is the odd part.

Remark 4.3.2. Notice that in the case of bundles D_k, which come from only one left mutation, one coefficient in the resolution is fixed. Conversely, in this case all coefficients are not fixed, then it is impossible to iterate the previous approach in order to classify all the resolutions with one fixed coefficient.

We recall the following relations

$$r_k = a_k - a_{k-1}, \qquad a_{k+1} = 3a_k - a_{k-1}, \qquad r_{k+1} = 3r_k - r_{k-1}.$$

Proof of Lemma 4.3.1. Let D_k be the exceptional bundle on $\mathbb{P}^2$ obtained by a left mutation of the exceptional pair $(\widehat{E}_k, \mathcal{O})$, where $\widehat{E}_k$ is a Steiner exceptional bundle, i.e. let D_k be defined by the following exact sequence

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{3r_k a_{k-2}} \to \mathcal{O}(-1)^{3r_k a_{k-1}} \to D_k \to 0.$$

Consider the left mutation of the exceptional pair $(D_k, \widehat{E}_k)$ as follows:

$$0 \to G_k \to \mathrm{Hom}(D_k, \widehat{E}_k) \otimes D_k \to \widehat{E}_k \to 0. \tag{4.3.2}$$

We want to prove that the bundles G_k admit the resolution (4.3.1), for all $k \geq 3$. As before, in order to simplify the computations, we will consider $G_k(1)$ instead of G_k. We recall that $\widehat{E}_k(1) = E_k$. We denote $H = \mathrm{Hom}(D_k, \widehat{E}_k)$ and we easily compute that $\dim H = 3r_{k-1}$. Then, by the following diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & G_k(1) \arrow[r] & H \otimes D_k(1) \arrow[r] \arrow[u] & E_k \arrow[r] \arrow[u] & 0 \\
 & & H \otimes \mathcal{O}^{3r_k a_{k-1}} \arrow[u] & \mathcal{O}^{a_k} \arrow[u] & \\
 & & H \otimes (\mathcal{O}(-2) \oplus \mathcal{O}(-1)^{3r_k a_{k-2}}) \arrow[u] & \mathcal{O}(-1)^{a_{k-1}} \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

we can compute the invariants rk, c_1 and c_2 of $D_k(1)$. First of all, we observe that

$$\mu(G_k(1)) = \frac{9r_k r_{k-1} a_{k-2} + 6r_{k-1} - a_{k-1}}{9r_{k-1}^2 r_k - r_k - 3r_{k-1}} > \frac{1}{2},$$

since this inequality is equivalent to $9r_k r_{k-1} a_{k-3} + 15a_{k-1} - 16a_{k-2} > 0$, which is true. Therefore, by Lemma 4.1.5, we know that $G_k(1)$ has a resolution of the form

$$0 \to \mathcal{O}(-2)^a \oplus \mathcal{O}(-1)^b \to \mathcal{O}^c \to G_k(1) \to 0, \tag{4.3.3}$$

for some $a, b, c \in \mathbb{N}$. Therefore, if we suppose that rank, c_1 and c_2 of $G_k(1)$ coincide with those obtained by the mutation, we obtain the following system:

$$\begin{cases} c - b - a = (9r_{k-1}^2 - 1)r_k - 3r_{k-1}, \\ b + 2a = 9r_k r_{k-1} a_{k-2} + 6r_{k-1} - a_{k-1}, \\ \frac{1}{2}(b^2 + 4a^2 + 4ab + 4a + b) = \frac{1}{2}(a_{k-1}^2 - a_{k-1} + 36r_{k-1}^2 + 12r_{k-1} + 81r_k^2 r_{k-1}^2 a_{k-2}^2 + \\ +9r_k r_{k-1} a_{k-2} + 108 r_k r_{k-1}^2 a_{k-2} - 12a_{k-1} r_{k-1} - 18a_{k-1} r_k r_{k-1} a_{k-2}). \end{cases}$$

This system admits the following unique solution:

$$\begin{cases} a = 3r_{k-1}, \\ b = 9r_k r_{k-1} a_{k-2} - a_{k-1}, \\ c = 9r_k r_{k-1} a_{k-2} - a_{k-1} + 9r_k r_{k-1}^2 - r_k = 9r_k r_{k-1} a_{k-1} - a_k, \end{cases}$$

and this is exactly our claim. □

Now, we can generalize the proof of Lemma 4.3.1 in order to obtain the resolution of all the other exceptional bundles on $\mathbb{P}^2$. Following the general theory of helices (see

Section 2.5), we will denote by (A, L_BC, B) the left mutation of the exceptional triple (A, B, C) and by (B, R_BA, C) the right mutation of (A, B, C).

Lemma 4.3.3. *Let (A, F, B) be an exceptional triple such that: $r = \mathrm{rk}(A)$, $\mu(A) \geq \frac{1}{2}$, F is defined by the exact sequence*

$$0 \to \mathcal{O}(-2)^d \oplus \mathcal{O}(-1)^e \to \mathcal{O}^f \to F \to 0,$$

and B by

$$0 \to \mathcal{O}(-2)^g \oplus \mathcal{O}(-1)^h \to \mathcal{O}^i \to B \to 0.$$

Then, by a left mutation, we obtain the triple (A, L_FB, F) and the bundle $C = L_FB$ has resolution

$$0 \to \mathcal{O}(-2)^{3rd-g} \oplus \mathcal{O}(-1)^{3re-h} \to \mathcal{O}^{3rf-i} \to C \to 0.$$

Proof. Consider the left mutation of the exceptional pair (F, B) as follows:

$$0 \to C \to \mathrm{Hom}(F, B) \otimes F \to B \to 0, \tag{4.3.4}$$

and denote $H = \mathrm{Hom}(F, B)$. From the theory of mutations, we have that $r_C = 3r_Ar_F - r_B = \dim H r_F - r_B$, thus $\dim H = 3r_A = 3r$. Then, using the following diagram

$$\begin{tikzcd}
 & 0 & & 0 & \\
0 \arrow[r] & C \arrow[r] & H \otimes F \arrow[u] \arrow[r] & B \arrow[u] \arrow[r] & 0 \\
 & & H \otimes \mathcal{O}^f \arrow[u] & \mathcal{O}^i \arrow[u] & \\
 & & H \otimes (\mathcal{O}(-2)^d \oplus \mathcal{O}(-1)^e) \arrow[u] & \mathcal{O}(-2)^g \oplus \mathcal{O}(-1)^h \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

we can compute the invariant rk, c_1 and c_2 of C. By the theory of helices we know that (A, C, F) is an exceptional triple and in particular, by Remark 2.5.5, we get

$\mu(A) \le \mu(C) \le \mu(F)$. Since $\mu(C) \ge \mu(A) \ge \frac{1}{2}$, by Lemma 4.1.5, we know that C has a resolution of the form

$$0 \to \mathcal{O}(-2)^a \oplus \mathcal{O}(-1)^b \to \mathcal{O}^c \to C \to 0, \tag{4.3.5}$$

for some $a, b, c \in \mathbb{N}$. Therefore if we suppose that rank, c_1 and c_2 of C coincide with those obtained by the mutation, we obtain the following system:

$$\begin{cases} c - b - a = 3r(f - e - d) - (i - h - g), \\ b + 2a = -2g - h + 6rd + 3re, \\ \frac{1}{2}(b^2 + 4a^2 + 4ab + 4a + b) = \frac{1}{2}(4g^2 - 4g + h^2 - h + 36r^2d^2 + \\ +12rd + 9r^2e^2 + 3re + 36r^2ed - 12hrd - 6hre + 4gh - 24grd - 12gre). \end{cases}$$

This system admits the following unique solution

$$\begin{cases} a = 3rd - g, \\ b = 3re - h, \\ c = 3rf - i, \end{cases}$$

and this is exactly our claim. $\square$

Lemma 4.3.4. *Let (A, F, B) be an exceptional triple such that: $r = \mathrm{rk}(B)$, $\mu(F) \ge \frac{1}{2}$, F is defined by the exact sequence*

$$0 \to \mathcal{O}(-2)^d \oplus \mathcal{O}(-1)^e \to \mathcal{O}^f \to F \to 0,$$

and A by

$$0 \to \mathcal{O}(-2)^g \oplus \mathcal{O}(-1)^h \to \mathcal{O}^i \to A \to 0.$$

Then, by a right mutation we obtain the triple $(F, R_F A, B)$ and the bundle $D = R_F A$ has resolution

$$0 \to \mathcal{O}(-2)^{3rd-g} \oplus \mathcal{O}(-1)^{3re-h} \to \mathcal{O}^{3rf-i} \to D \to 0.$$

Proof. Consider the right mutation of the exceptional pair (A, F) as follows:

$$0 \to A \to \mathrm{Hom}(A, F)^\vee \otimes F \to D \to 0, \tag{4.3.6}$$

and denote $H = \mathrm{Hom}(A, F)$. From the theory of mutations, we have that $r_D = 3r_B r_F - r_A = \dim H r_F - r_A$, therefore $\dim H = 3r_B = 3r$, Then, using the diagram

$$\begin{tikzcd}
 & 0 & & 0 & & \\
0 \arrow[r] & A \arrow[u] \arrow[rr] & & H^\vee \otimes F \arrow[u] \arrow[rr] & & D \arrow[r] & 0 \\
 & \mathcal{O}^i \arrow[u] & & H^\vee \otimes \mathcal{O}^f \arrow[u] & & \\
 & \mathcal{O}(-1)^h \oplus \mathcal{O}(-2)^g \arrow[u] & & H^\vee \otimes (\mathcal{O}(-2)^d \oplus \mathcal{O}(-1)^e) \arrow[u] & & \\
 & 0 \arrow[u] & & 0 \arrow[u] & &
\end{tikzcd}$$

we can compute the invariant rk, c_1 and c_2 of D. Notice that (F, D, B) is an exceptional triple, hence, by Remark 2.5.5, $\mu(F) \le \mu(D) \le \mu(B)$. Since $\mu(D) \ge \mu(F) \ge \frac{1}{2}$, by Lemma 4.1.5, we know that D has a resolution of the form

$$0 \to \mathcal{O}(-2)^a \oplus \mathcal{O}(-1)^b \to \mathcal{O}^c \to D \to 0, \tag{4.3.7}$$

for some $a, b, c \in \mathbb{N}$. Therefore if we suppose that rank, c_1 and c_2 of D coincide with those obtained by the mutation, we obtain the following system:

$$\begin{cases} c - b - a = 3r(f - e - d) - (i - h - g), \\ b + 2a = -2g - h + 6rd + 3re, \\ \frac{1}{2}(b^2 + 4a^2 + 4ab + 4a + b) = \frac{1}{2}(4g^2 - 4g + h^2 - h + 36d^2r^2 + 12dr + \\ +9r^2e^2 + 3re + 36r^2ed - 12hdr - 6hre + 4gh - 24gdr - 12gre), \end{cases}$$

This system admits the following unique solution

$$\begin{cases} a = 3rd - g, \\ b = 9re - h, \\ c = 9rf - i, \end{cases}$$

and this is exactly our claim. $\square$

We can represent graphically the previous two lemmas as follows

$$\begin{tikzcd}
& A(r) \arrow[dr] & & B(g,h,i) \arrow[dl] \\
& & F(d,e,f) \arrow[dl] & \\
& L_FB = C(3rd-g, 3re-h, 3rf-i) & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
A(g,h,i) \arrow[dr] & & B(r) \arrow[dl] & \\
& F(d,e,f) \arrow[dr] & & \\
& & R_FA = D(3rd-g, 3re-h, 3rf-i) &
\end{tikzcd}$$

Let us show some numerical examples:

$$\begin{tikzcd}
(0,1,3) \arrow[dr] & & (0,3,8) \arrow[dl] \\
& (1,15,45) \arrow[dl] \arrow[dr] & \\
(6,87,262) & & (15,224,672)
\end{tikzcd}$$

$$\begin{tikzcd}
(0,3,8) \arrow[dr] & & (0,8,21) \arrow[dl] \\
& (1,117,312) \arrow[dl] \arrow[dr] & \\
(15,1747,4659) & & (39,4560,12160)
\end{tikzcd}$$

$$\begin{tikzcd}
(0,1,3) \arrow[dr] & & (1,15,45) \arrow[dl] \\
& (6,87,262) \arrow[dl] \arrow[dr] & \\
(35,507,1526) & & (522,7568,22791)
\end{tikzcd}$$

$$
\begin{tikzcd}
(1,15,45) \arrow[dr] & & (0,3,8) \arrow[dl] \\
& (15,224,672) \arrow[dl] \arrow[dr] & \\
(1305,19485,58456) & & (224,3345,10035)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(0,3,8) \arrow[dr] & & (1,117,312) \arrow[dl] \\
& (15,1747,4659) \arrow[dl] \arrow[dr] & \\
(224,26088,69573) & & (8730,1016751,2711530)
\end{tikzcd}
$$

Finally, we summarize the results of this section in the following theorem:

Theorem 4.3.5. *Let E be an exceptional bundle on $\mathbb{P}^2$ with slope $\mu \in [-\frac{1}{2}, 0]$. Let $\widehat{E}_k$ denote the normalized exceptional Steiner bundles with resolution*

$$0 \to \mathcal{O}(-2)^{a_{k-1}} \to \mathcal{O}(-1)^{a_k} \to \widehat{E}_k \to 0.$$

Then, E has one of the following descriptions:

- *E comes from a right mutation of the triple $(\widehat{E}_{k-1}, \widehat{E}_k, \mathcal{O})$, for $k \geq 3$, and $E = \widehat{E}_{k+1}$;*
- *E comes from a left mutation of the triple $(\widehat{E}_{k-1}, \widehat{E}_k, \mathcal{O})$, for $k \geq 3$, and $E = D_k$, where*
$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{3r_k a_{k-2}} \to \mathcal{O}(-1)^{3r_k a_{k-1}} \to D_k \to 0;$$
- *E comes from a right mutation of the triple (A, F, B) of exceptional bundles such that $r = \mathrm{rk}(B)$,*
$$0 \to \mathcal{O}(-3)^d \oplus \mathcal{O}(-2)^e \to \mathcal{O}(-1)^f \to F \to 0,$$
$$0 \to \mathcal{O}(-3)^a \oplus \mathcal{O}(-2)^b \to \mathcal{O}(-1)^c \to A \to 0,$$
and E admit the following resolution:
$$0 \to \mathcal{O}(-3)^{3rd-a} \oplus \mathcal{O}(-2)^{3re-b} \to \mathcal{O}(-1)^{3rf-c} \to E \to 0;$$

- *E comes from a left mutation of the triple of exceptional bundles (A, F, B), where $r = \mathrm{rk}(A)$,*

$$0 \to \mathcal{O}(-3)^d \oplus \mathcal{O}(-2)^e \to \mathcal{O}(-1)^f \to F \to 0,$$

and

$$0 \to \mathcal{O}(-3)^g \oplus \mathcal{O}(-2)^h \to \mathcal{O}(-1)^i \to B \to 0,$$

and E admits the following resolution:

$$0 \to \mathcal{O}(-3)^{3rd-g} \oplus \mathcal{O}(-2)^{3re-h} \to \mathcal{O}(-1)^{3rf-i} \to E \to 0.$$

Remark 4.3.6. All the exceptional bundles with slope in $[-\frac{1}{2}, 0]$ are described by Lemmas 4.3.3 and 4.3.4, except the bundles D_k, studied in Lemma 4.2.2. In fact D_k comes from a left mutation of the triple $(\widehat{E}_{k-1}, \widehat{E}_k, \mathcal{O})$ and $\mathcal{O}$ is not of the form

$$0 \to \mathcal{O}(-3)^g \oplus \mathcal{O}(-2)^h \to \mathcal{O}(-1)^i \to B \to 0,$$

as requested in Lemma 4.3.3. This is a reason for which the bundles D_k have a very particular resolution.

4.4 Dual case

Up to now, we have only considered exceptional bundles with slope in $[-\frac{1}{2}, 0]$. Since $\mu(E(i)) = \mu(E) + i$ and $\mu(E^*) = -\mu(E)$, it is obvious that the exceptional bundles with slope in $[-1, -\frac{1}{2}]$ are obtained by dualizing and twisting the bundles that we have already classified. We know that when we dualize, the left and right mutations interchange (see Section 2.5). We can simply apply the same arguments used in the case of $\mu \in [-\frac{1}{2}, 0]$ and the classification follows.

Remark 4.4.1. It is easy to check that when we apply a left mutation to the exceptional triple $(\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1))$, we get the triple $(\mathcal{O}(-1), T, \mathcal{O})$, where $T = T_{\mathbb{P}^2}(-2)$ has resolution

$$0 \to \mathcal{O}(-2) \to \mathcal{O}(-1)^3 \to T \to 0.$$

Then, we can apply another left mutation and obtain the triple $(\mathcal{O}(-1), H_2, T)$, where H_2 has resolution

$$0 \to \mathcal{O}(-3) \to \mathcal{O}(-1)^6 \to H_2 \to 0,$$

Thus, by applying recursively right mutations to the triple $(\mathcal{O}(-1), H_2, T)$, we obtain bundles, with slope $\mu \in [-\frac{1}{2}, 0]$, of the form

$$0 \to \mathcal{O}(-3)^{b_{k-1}} \to \mathcal{O}(-1)^{b_k} \to H_k \to 0,$$

where

$$b_k = \frac{\left(3+2\sqrt{2}\right)^k - \left(3-2\sqrt{2}\right)^k}{4\sqrt{2}}.$$

These bundles are exactly a particular case (for h=2 and L=6) of the bundles studied in Section 3.5, arising as the natural generalization of Steiner bundles. The sequence $\{b_k\}$ is also defined recursively as follows:

$$\begin{cases} b_0 = 0, \\ b_1 = 1, \\ b_{k+1} = 6b_k - b_{k-1}. \end{cases}$$

Notice that $H_1 = \mathcal{O}(-1)$ and $H_2 = E_3^*(-1)$.

Theorem 4.4.2. *Let E be an exceptional bundle on $\mathbb{P}^2$ with slope $\mu \in [-1, -\frac{1}{2}]$. Let H_k denote the exceptional bundles with resolution*

$$0 \to \mathcal{O}(-3)^{b_{k-1}} \to \mathcal{O}(-1)^{b_k} \to H_k \to 0,$$

for all $k \geq 1$, and let T denote the first non-trivial exceptional Steiner bundle $T = E_2 = T_{\mathbb{P}^2}(-2)$. Then E has one of the following description:

- *E comes from a right mutation of the triple (H_{k-1}, H_k, T), for $k \geq 2$, and $E = H_{k+1}$;*

- *E comes from a left mutation of the triple (H_{k-1}, H_k, T), for $k \geq 2$, and $E = C_k$ is defined by the following exact sequence*

$$0 \to \mathcal{O}(-3)^{3s_{k-1}b_{k-1}} \to \mathcal{O}(-2) \oplus \mathcal{O}(-1)^{3s_{k-1}b_k - 3} \to C_k \to 0,$$

  where $s_k = b_k - b_{k-1}$;

- *E comes from a right mutation of the triple (A, F, B) of exceptional bundles such that $r = \mathrm{rk}(B)$,*

$$0 \to \mathcal{O}(-3)^d \to \mathcal{O}(-2)^e \oplus \mathcal{O}(-1)^f \to F \to 0,$$

$$0 \to \mathcal{O}(-3)^a \to \mathcal{O}(-2)^b \oplus \mathcal{O}(-1)^c \to A \to 0,$$

and E admit the following resolution:

$$0 \to \mathcal{O}(-3)^{3rd-a} \to \mathcal{O}(-2)^{3re-b} \oplus \mathcal{O}(-1)^{3rf-c} \to E \to 0;$$

- *E comes from a left mutation of the triple of exceptional bundles (A, F, B), where $r = \mathrm{rk}(A)$,*

  $$0 \to \mathcal{O}(-3)^d \to \mathcal{O}(-2)^e \oplus \mathcal{O}(-1)^f \to F \to 0,$$

  and

  $$0 \to \mathcal{O}(-3)^g \to \mathcal{O}(-2)^h \oplus \mathcal{O}(-1)^i \to B \to 0,$$

  and E admits the following resolution:

  $$0 \to \mathcal{O}(-3)^{3rd-g} \to \mathcal{O}(-2)^{3re-h} \oplus \mathcal{O}(-1)^{3rf-i} \to E \to 0.$$

Remark 4.4.3. Notice that the resolutions of the exceptional bundles with slope in $[-1, -\frac{1}{2}]$ are all of the type (4.1.4) of Section 4.1. This allows us to complete the proof of Proposition 4.1.6.

Chapter 5

Bundles with only one quadratic form on $\mathbb{P}^2$

In this chapter, we will undertake the study of the bundles on $\mathbb{P}^2$ of the form

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0. \tag{5.0.1}$$

As in the case of Steiner bundles, we want to give a criterion for simplicity and stability. By Proposition 2.6.7, we know that, on $\mathbb{P}^2$, proving stability is equivalent to proving generic simplicity. Then we will use both approaches: on one hand, we will apply the Drézet-Le Potier criterion to check stability, on the other hand, we will prove generic simplicity in a direct way.

We can parameterize the bundles with resolution (5.0.1) by means of the pair of natural numbers (s, t) and we can represent them graphically in a plane with coordinates (s, t). First of all, let us observe that in this plane there exists no line separating stable from non-stable bundles, even if we exclude exceptional ones. In fact, let F_1, F_2, F_3 be defined by the following sequences:

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{14} \to \mathcal{O}(-1)^{42} \to F_1 \to 0,$$

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{16} \to \mathcal{O}(-1)^{47} \to F_2 \to 0,$$

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^{21} \to \mathcal{O}(-1)^{60} \to F_3 \to 0.$$

It is possible to compute that F_1 and F_3 are simple, stable and not exceptional, whereas F_2 is not simple and not stable. Now let us suppose that there exists a line

$l = \{t = ms + q\}$, separating F_1 from F_2. Therefore, we would have

$$\begin{cases} 42 \leq 14m + q, \\ 47 > 16m + q, \end{cases}$$

and, for q to exist, we need to suppose that $m \leq 2.5$. On the other hand, the line l separates F_2 from F_3, then

$$\begin{cases} 60 \leq 21m + q, \\ 47 > 16m + q, \end{cases}$$

and so we get $m \geq 2.6$. These two conditions are clearly in contradiction, hence the line does not exist.

Furthermore, we immediately observe that if F has resolution

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0,$$

the property $\chi(\operatorname{End} F) \leq 1$ does not imply that F is simple. For example, if F has resolution

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^4 \to \mathcal{O}(-1)^{16} \to F \to 0,$$

we get $\chi(\operatorname{End} F) = -3$, but F is not simple because it can be shown that $\mathrm{h}^0(\operatorname{End} F) = 5$ (see the appendix for more details). Another example is F_2, for which $\chi(\operatorname{End} F_2) = -24$, but $\mathrm{h}^0(\operatorname{End} F_2) = 2$. In other words, the hyperbola $\chi(\operatorname{End} F) = 1$ is not a boundary between simple and non-simple bundles. The condition, that we are searching for, needs to be more complicated and we will find it in the next section.

5.1 Stability

In this section we want to apply the Drézet-Le Potier criterion in the case of bundles with resolution of the form (5.0.1), i.e.

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0.$$

We will see that in this case the condition for the stability of the normalized bundle can be expressed by a condition on the variables s, t. Let F be a bundle on $\mathbb{P}^2$ with resolution (5.0.1). Recall that F is exceptional if and only if $s = 3r_k a_{k-2}$ and $t = 3r_k a_{k-1}$ (see Corollary 4.2.8). In the plane with coordinates (s, t), the points corresponding to the exceptional bundles D_k are $(3r_k a_{k-2}, 3r_k a_{k-1})$, for all $k \geq 3$,

and we denote them again by D_k, by abuse of notation.
Recall that in Chapter 2 we have defined the height of a bundle on $\mathbb{P}^2$ as

$$h(r, c_1, c_2) = \Delta(r, c_1, c_2) - \delta(\frac{c_1}{r})$$

where, for every $\mu \in I_E$,

$$\delta(\mu) = P(-|\mu - \mu_E|) - \Delta_E.$$

and $P(x) = \frac{x^2+3x+2}{2}$. If the pair (s, t) corresponds to a bundle with rank r and Chern classes c_1 and c_2, we will denote $\delta(s, t) = \delta(\frac{c_1}{r})$ and $h(s, t) = h(r, c_1, c_2)$. Let Q_k be the points $(r_{k+1}a_{k-2}, r_{k+1}a_{k-1})$, for any $k \geq 2$. We will see (in Remark 5.1.5) that they correspond to bundles with height 0 and slope $\mu(Q_k) = \mu(\widehat{E}_k)$, where $\widehat{E}_k$ are the normalized exceptional Steiner bundles. Let p denote the polygonal

$$p = \{Q_2, D_3, Q_3, D_4, \ldots, Q_{k-1}, D_k, Q_k, \ldots\},$$

i.e. the union of the segments (Q_{k-1}, D_k) and (D_k, Q_k), for any $k \geq 3$. We can give now a characterization of bundles with positive height.

Theorem 5.1.1. *Let F be a normalized bundle on $\mathbb{P}^2$ with resolution* (5.0.1) *and let p be the polygonal*

$$p = \{Q_2, D_3, Q_3, D_4, \ldots, Q_{k-1}, D_k, Q_k, \ldots\},$$

where $D_k = (3r_k a_{k-2}, 3r_k a_{k-1})$ and $Q_k = (r_{k+1}a_{k-2}, r_{k+1}a_{k-1})$. Then

(i) if (s, t) is above the polygonal p, then $h(s, t) < 0$,

(ii) if (s, t) lies on the polygonal p, then either $h(s, t) = 0$ or $(s, t) = D_k$,

(iii) if (s, t) lies between the polygonal p and the line $t = 2s + 3$, then $h(s, t) > 0$.

Remark 5.1.2. It is easy to check that F is normalized if and only if $t \geq 2s + 3$.

By using Drézet-Le Potier criterion, it immediately follows from this theorem the following interesting corollary.

Corollary 5.1.3. *Let F be a normalized bundle on $\mathbb{P}^2$ with resolution* (5.0.1) *and let p be the polygonal defined in the statement of the previous theorem. Then F is stable if and only if (s, t) lies on the polygonal p or between the polygonal p and the line $t = 2s + 3$; F is not stable if and only if (s, t) is above the polygonal p.*

The first points in the polygonal are

$$\{(0,5),(15,45),(13,39),(117,312),\ldots\}.$$

We can see the situation in Figure 5.1 and in the corresponding enlargement in Figure

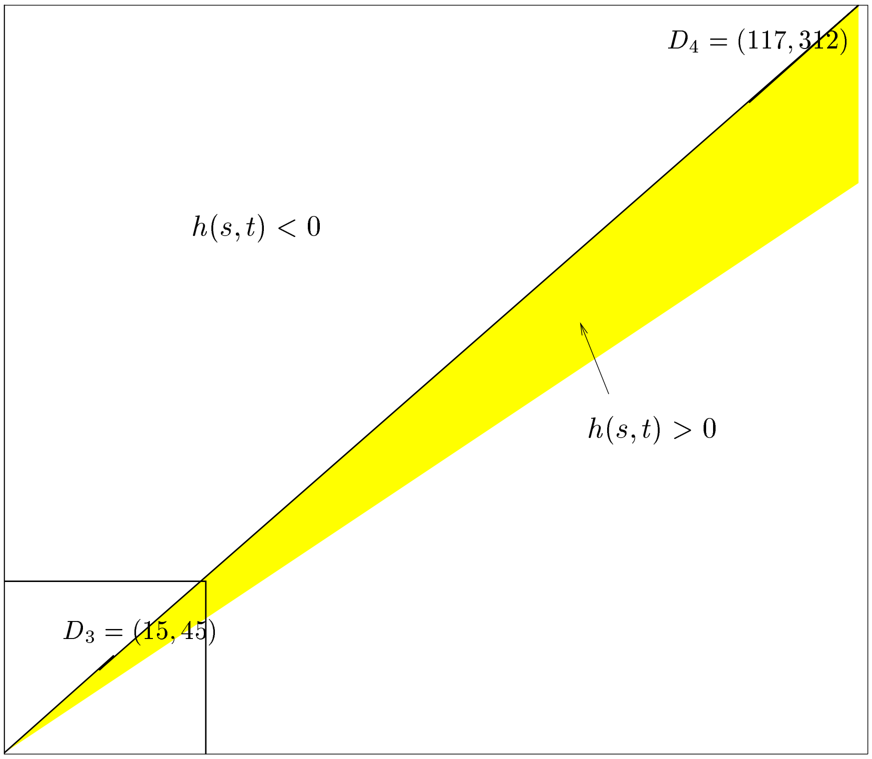

Figure 5.1: Bundles on $\mathbb{P}^2$ with resolution (5.0.1).

5.2: the shaded region corresponds to stable normalized bundles.

In order to prove Theorem 5.1.1, we have to compute the height of a bundle F corresponding to the point (s,t). We have that

$$\Delta(s,t) = \frac{st - s + 4t}{2(t-s-1)^2} \qquad \text{and} \qquad \mu(s,t) = \frac{-t+2s+3}{t-s-1}.$$

Now, we define a new function, depending only on the normalized exceptional Steiner bundles $\widehat{E}_k$, in the following way. First of all, recall that a left mutation of the triple of exceptional bundles $(\widehat{E}_{k-1}, \widehat{E}_k, \mathcal{O})$ gives the triple $(\widehat{E}_{k-1}, D_k, \widehat{E}_k)$, for any $k \geq 3$ (see Theorem 4.3.5). By Remark 2.5.5, it follows that $\mu(\widehat{E}_{k-1}) < \mu(D_k) < \mu(\widehat{E}_k) < \mu(D_{k+1})$, for any $k \geq 3$. Therefore, if the point (s,t) corresponds to a bundle with

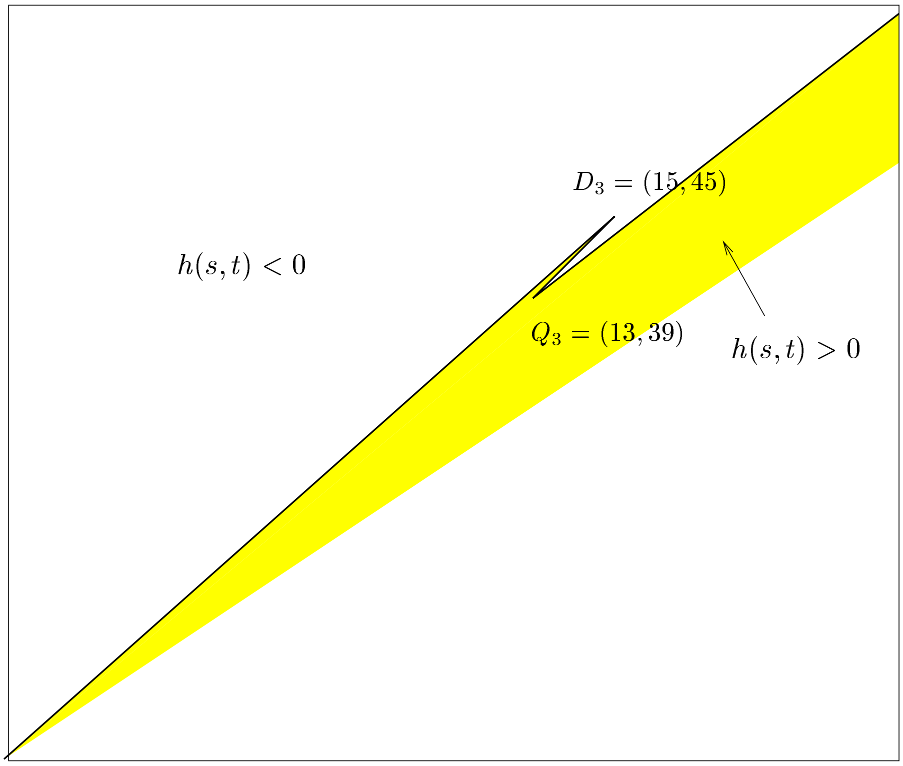

Figure 5.2: Enlargement of Figure 5.1: the polygonal p is here visible.

slope μ such that $\mu(D_k) \leq \mu \leq \mu(D_{k+1})$, we define $\delta_E(s,t) = P(-|\mu - \mu(\widehat{E}_k)|) - \Delta(\widehat{E}_k)$, and consequently $h_E(s,t) = \Delta(s,t) - \delta_E(s,t)$. Notice that $\delta_E(s,t) = \delta(s,t)$ if and only if the slope of the bundle corresponding to (s,t) lies in an interval $I_{\widehat{E}_k}$, for an exceptional Steiner bundle $\widehat{E}_k$.

Now, we prove the following lemma.

Lemma 5.1.4. *If $t \geq 2s+3$, we have that $h_E(s,t) > 0$ if and only if (s,t) is below the polygonal p. Moreover, $h_E(s,t) = 0$ exactly on the polygonal p.*

Proof. Recall that $\mu(\widehat{E}_k) = \frac{2a_{k-1} - a_k}{r_k}$ and $\Delta(\widehat{E}_k) = \frac{a_k a_{k-1}}{2r_k^2}$, where $r_k = a_k - a_{k-1}$. Then

$$h_E(s,t) = \frac{st - s + 4t}{2(t-s-1)^2} + \frac{a_k a_{k-1}}{2r_k^2} - P(-|\frac{-t+2s+3}{t-s-1} - \frac{2a_{k-1} - a_k}{r_k}|)$$

By solving $h_E(s,t) \geq 0$, we get

$$t \leq \tfrac{a_{k+1}}{a_k} s + \tfrac{3r_{k+1}}{a_k}, \quad \text{when} \quad \mu > \mu(\widehat{E}_k)$$

and

$$t \geq \tfrac{a_{k-1}}{a_{k-2}} s, \quad \text{when} \quad \mu \leq \mu(\widehat{E}_k).$$

We check that these segments meet exactly in the points D_k and Q_k. In fact

$$\{t = \frac{a_k}{a_{k-1}} s + \frac{3r_k}{a_{k-1}}\} \cap \{t = \frac{a_{k-1}}{a_{k-2}} s\} = (3r_k a_{k-2}, 3r_k a_{k-1}) = D_k$$

and

$$\{t = \frac{a_{k-1}}{a_{k-2}} s\} \cap \{t = \frac{a_{k+1}}{a_k} s + \frac{3r_{k+1}}{a_k}\} = (r_{k+1} a_{k-2}, r_{k+1} a_{k-1}) = Q_k.$$

This means that $h_E(s,t) < 0$ above p, $h_E(s,t) = 0$ exactly on p and $h_E(s,t) > 0$ below p. □

Remark 5.1.5. It is clear from the previous proof that the points Q_k arise from the intersection of the segments corresponding to the vanishing of the function $h_E(s,t)$. Hence it is easily seen that $h(Q_k) = 0$ and $\mu(Q_k) = \mu(\widehat{E}_k)$. Furthermore, it is possible to prove that these are the unique points satisfying these two properties.

In order to prove Theorem 5.1.1 we have to show that if $\mu(F) \in I_G$ and $G \neq \widehat{E}_k$ is an exceptional non-Steiner bundle, then F is not stable. First of all, we introduce some preliminary notations and remarks. Given two bundles A, B, we will say that

$$A \leq B \qquad \text{iff} \qquad \mu(A) \leq \mu(B).$$

Remark 5.1.6. We recall that (A, C, B) is called a triple of adjacent exceptional bundles when it comes from the helix $(\mathcal{O}(-1), \mathcal{O}, \mathcal{O}(1))$ by left and right mutations (see Section 2.5). It is obvious that $(A, L_C B, C)$ and $(C, R_C A, B)$ are triples of adjacent exceptional bundles too. Therefore, by Remark 2.5.5, we get

$$A < L_C B < C < B \quad \text{and} \quad A < C < R_C A < B.$$

Lemma 5.1.7. *If* $F \in I_E$, *where* E *is exceptional, then* $\operatorname{rk} F \leq \operatorname{rk} E$.

Proof. We prove that every fraction p/q in $I_E = (\mu(E) - x_E, \mu(E) + x_E)$ has denominator $q \geq r = \operatorname{rk} E$. Observe that in any interval centered in s/r, there are no fractions with denominator less than r if the radius of the interval is less than $\frac{1}{r(r-1)}$. Then, it suffices to show that $x_E < \frac{1}{r(r-1)}$, i.e. $\frac{3r - \sqrt{9r^2-4}}{2r} < \frac{1}{r(r-1)}$, and this is true by simple computations. □

Lemma 5.1.8. *If* G *is an exceptional bundle such that* $\widehat{E}_{k-1} < G < \widehat{E}_k$ *and* $F \in I_G$, *then* $\operatorname{rk}(F) \geq \operatorname{rk}(D_k) = 3r_k r_{k-1} - 1$.

Proof. By the previous lemma we know that $\operatorname{rk} F \geq \operatorname{rk} G$. Now, it is easy to check that when an exceptional bundle G comes from a mutation of an exceptional pair (A, B), the rank of G is greater than the rank of A and of B. By Remark 5.1.6, we know that any exceptional bundle such that $\widehat{E}_{k-1} < G < \widehat{E}_k$ is given by some mutations starting from $\widehat{E}_{k-1}$ and $\widehat{E}_k$. It follows that $\operatorname{rk} G \geq \operatorname{rk} D_k = 3r_k r_{k-1}$, therefore $\operatorname{rk} F \geq \operatorname{rk} G \geq 3r_k r_{k-1}$. □

Remark 5.1.9. If F has resolution (5.0.1), then if $\mu(F) < \mu_0$, solving the following inequality

$$\frac{-t+2s+3}{t-s-1} < \mu_0,$$

we get

$$t > \frac{2+\mu_0}{1+\mu_0}s + \frac{3+\mu_0}{1+\mu_0}$$

(respectively if $\mu(F) > \mu_0$, the latter inequality holds with $<$).

Now we can give the proof of the main theorem.

Proof of Theorem 5.1.1. In Lemma 5.1.4 we describe the behavior of the function $h_E(s,t)$. We know that $h(s,t) = h_E(s,t)$ holds if and only if the pair (s,t) corresponds to a bundle F such that $\mu(F) \in I_{\widehat{E}_k}$, for an exceptional Steiner bundle $\widehat{E}_k$.

On the contrary, suppose $\mu(F) \in I_G$ and G exceptional, non-Steiner. Since $\{I_A : A \text{ exceptional}\}$ is a partition of $\mathbb{Q}$, we get $\mu(F) \notin I_{\widehat{E}_k}$, for all the exceptional Steiner bundles $\widehat{E}_k$. Therefore, by Remark 2.6.5, we know that

$$\delta(s,t) \geq \frac{1}{2} > \delta_E(s,t),$$

hence $h(s,t) < h_E(s,t)$. Then, by Lemma 5.1.4, it follows that if the point (s,t) is above the polygonal p, then

$$h(s,t) < h_E(s,t) < 0,$$

as statement *(i)* of the theorem claims.

Now, in order to prove statements *(ii)* and *(iii)*, it suffices to prove that the only bundles F which lie either below or on p and such that $\mu(F) \in I_G$, with G exceptional and non-Steiner, are the exceptional bundles D_k.

Suppose $\mu(F) \in I_G$ and $\widehat{E}_{k-1} < G < \widehat{E}_k$. It follows that $\widehat{E}_{k-1} < F < \widehat{E}_k$, i.e.

$$\mu_1 = \frac{2a_{k-2} - a_{k-1}}{r_{k-1}} < \mu(F) < \frac{2a_{k-1} - a_k}{r_k} = \mu_2.$$

By Remark 5.1.9, this gives the following conditions on s, t:

$$t < \frac{2+\mu_1}{1+\mu_1}s + \frac{3+\mu_1}{1+\mu_1} \qquad \text{and} \qquad t > \frac{2+\mu_2}{1+\mu_2}s + \frac{3+\mu_2}{1+\mu_2},$$

i.e.

$$t > \frac{a_{k-1}}{a_{k-2}}s + \frac{r_k}{a_{k-2}} \qquad \text{and} \quad t < \frac{a_k}{a_{k-1}}s + \frac{r_{k+1}}{a_{k-1}}.$$

Moreover, by Lemma 5.1.8, we know that $\mathrm{rk}(F) \geq 3r_k r_{k-1} - 1$, i.e.

$$t - s \geq 3r_{k-1}r_k.$$

Then, any bundle F such that $\mu(F) \in I_G$ satisfies the following inequalities

$$\begin{cases} t > \frac{a_{k-1}}{a_{k-2}}s + \frac{r_k}{a_{k-2}}, \\ t < \frac{a_k}{a_{k-1}}s + \frac{r_{k+1}}{a_{k-1}}, \\ t \geq s + 3r_{k-1}r_k. \end{cases} \tag{5.1.1}$$

Now, the region

$$C = \{(s,t) : t > \frac{a_{k-1}}{a_{k-2}}s + \frac{r_k}{a_{k-2}}, t < \frac{a_k}{a_{k-1}}s + \frac{r_{k+1}}{a_{k-1}}\} \cap \{(s,t) \text{ below or on } p\}$$

is a convex figure bounded by the segments $Q_{k-1}D_k$ and D_kQ_k. Therefore, by simple computations, we see that the line $t = s + 3r_{k-1}r_k$ intersects the figure C exactly in the point D_k. Hence, it follows that if the point (s,t) satisfies the conditions in (5.1.1), then either $(s,t) = D_k$, or (s,t) is above the polygonal p, as claimed. $\square$

5.1.1 Graphic description

Recall that the Drézet-Le Potier criterion characterizes stable bundles on $\mathbb{P}^2$ by means of the function $h(s,t)$. Notice that the polygonal p represents the locus $\{h_E(s,t) = 0\}$, where $h_E(s,t)$ is the auxiliary function introduced in the previous section. On the other hand, the locus $\{h(s,t) = 0\}$ is much more complicate than the polygonal p. For example, in Figure 5.3, which is an enlargement and an affine transformation of Figure 5.2, we can see that below the polygonal p, there is a region R in which we have $h(s,t) < 0$. In fact this region is bounded by the polygonal p and by the segments corresponding to $\{h_{D_3}(s,t) = 0\}$ (where h_{D_3} is the height associated to the exceptional bundle D_3). By the previous proof, we know that in this region R there are no integer points except $D_3 = (15, 45)$. When we consider the heights associated

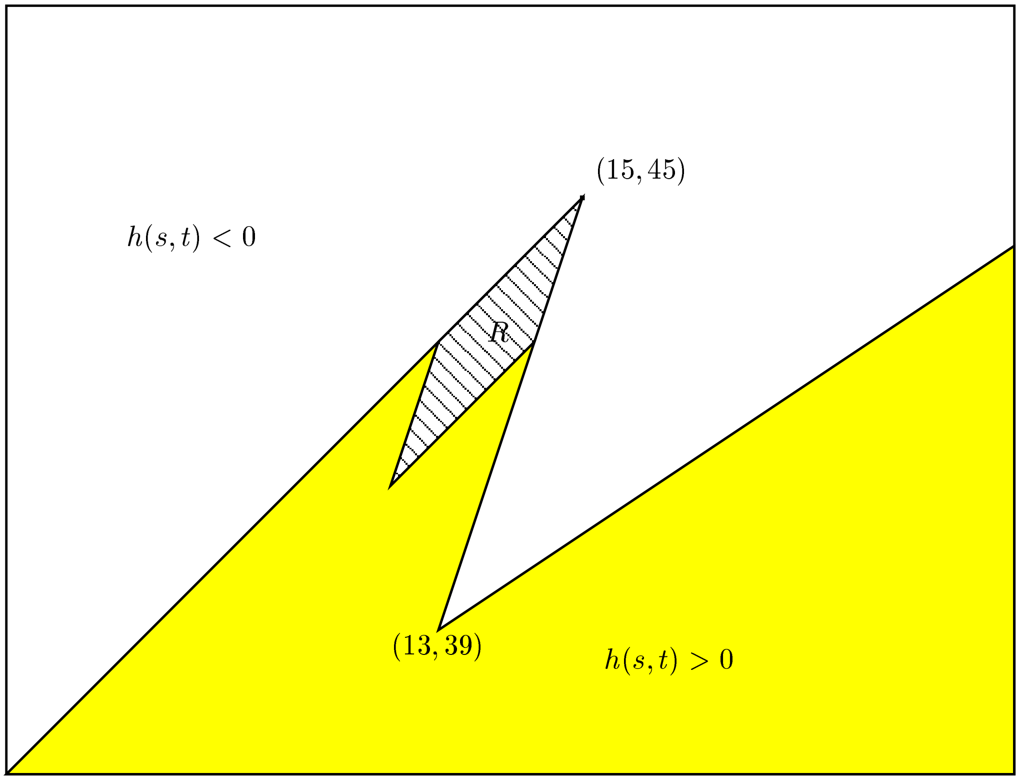

Figure 5.3: If there were any bundle inside the region R, it would not be stable.

to the other exceptional bundles, we can draw other new regions in which $h(s,t) < 0$, but, by the same reason, we know that they contain no integer points. Then, we can say that the behavior of the function $h(s,t)$ is "fractal", nevertheless in the case of bundles with resolution (5.0.1), the characterization of stable bundles can be strongly simplified by our Theorem 5.1.1.

Unfortunately it seems impossible to extend this method to bundles on $\mathbb{P}^2$ with resolution

$$0 \to \mathcal{O}(-3)^q \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0 \tag{5.1.2}$$

for any $q \geq 2$. In fact, for any $q \geq 2$, we can consider the locus $\{h_E(s,t) = 0\}$, but this function is not sufficient to characterize the stable bundles. For example, let F be the generic bundle with resolution

$$0 \to \mathcal{O}(-3)^7 \oplus \mathcal{O}(-2)^{102} \to \mathcal{O}(-1)^{307} \to F \to 0.$$

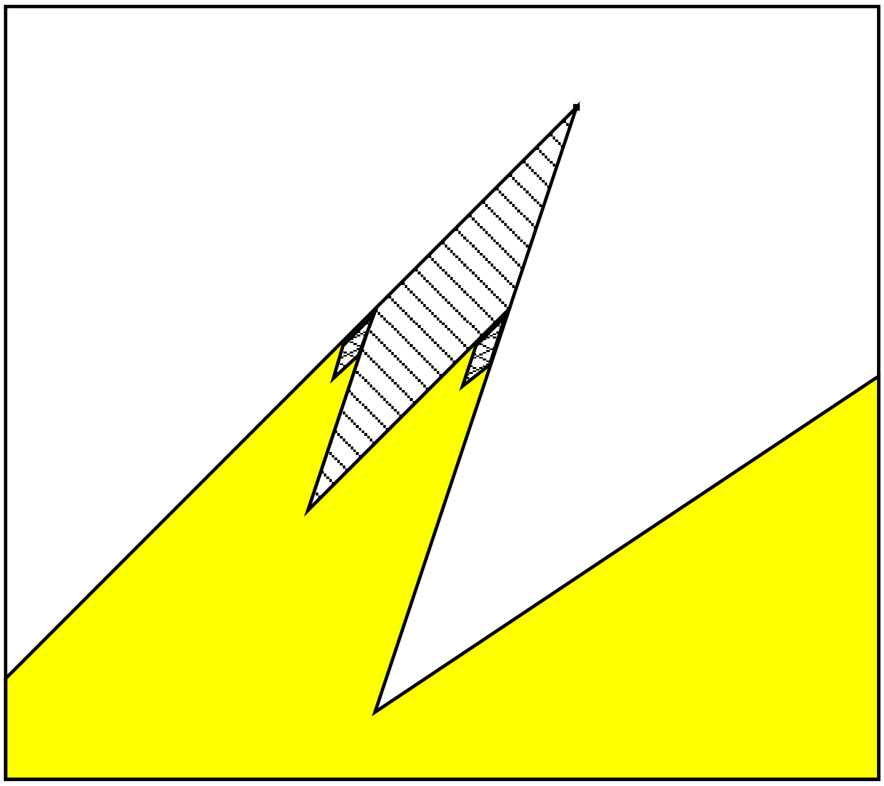

Figure 5.4: We can see the first steps of the constructions of the graphic $h(s,t)=0$

Then F is not stable because its height is negative, whereas the function h_E of F is positive. In other words, if for any $q \geq 1$, we consider the zero locus of $h_E(s,t)$ we get a polygonal p' and the bundle F in the example is below this polygonal p', but is not stable. It follows that Theorem 5.1.1 is not true in general for bundles with resolution (5.1.2). Notice that some vertices of the polygonal p' correspond exactly to the semi-exceptional bundles given by q copies of the exceptional bundles D_k, defined above.

Remark 5.1.10. It is interesting to notice that the intervals associated to the exceptional Steiner bundles cover a "very large" part of $\mathbb{Q}$. This implies that for "almost all" the bundles the two functions h and h_E coincide.

More precisely, we denote

$$T = \bigcup_{A \in \mathcal{E}} I_A \cap [-1, 0],$$

where $\mathcal{E}$ is the set of exceptional bundles on $\mathbb{P}^2$. Therefore, by Theorem 2.6.1, we know that $\mathbb{Q} \subset T$ and, since T is a countable union of intervals containing $\mathbb{Q}$, then the measure of $T \cap [-1, 0]$ is 1. Denoting by $\mathcal{S} \subset \mathcal{E}$ the set of exceptional Steiner bundles (we are not assuming them normalized), we define

$$S = \bigcup_{A \in \mathcal{S}} I_A \cap [-1, 0].$$

We can compute the measure l of S as follows

$$l(S) = \sum_{k=1}^{\infty} \left(3 - \frac{\sqrt{9r_k^2 - 4}}{r_k} \right),$$

where

$$r_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2k-1}}{\sqrt{5}}.$$

By using Maple, we can compute this value and it turns out to be $l(S) \approx 0.9669138184$. This means that for a normalized bundle the functions h_E and h coincide with a probability of 96.69%.

In particular, since all the normalized exceptional Steiner bundles have slope in $[-\frac{1}{2}, 0]$, it is interesting to compute

$$l(S \cap [-\frac{1}{2}, 0]) \approx 0.4991613707.$$

It follows that for a bundle with slope in $[-\frac{1}{2}, 0]$ the functions h_E and h coincide with a probability of 99.83%.

5.2 Simplicity

In the previous section, we were able to give a condition for stability only for normalized bundles. This implies that we can only consider bundles with $t \geq 2s + 3$. Now, we study the remaining cases, i.e. $s + 3 \leq t \leq 2s + 3$. The following theorem proves that all these bundles are generically simple. First, let us observe that if F is a vector bundle on $\mathbb{P}^{N-1}$, then $\operatorname{rk} F = t - s - 1 \geq 2$. It follows that $t \geq s + 3$ is a necessary condition.

Now, we give a lemma, which is the generalization of Lemma 3.2.7 to bundles with resolution

$$0 \longrightarrow \mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s \longrightarrow \mathcal{O}^t \longrightarrow F \longrightarrow 0,$$

for any $q, s, t \in \mathbb{N}$, such that $t - s - q \geq N - 1$.

Lemma 5.2.1. *If F is defined by the sequence*

$$0 \longrightarrow \mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s \xrightarrow{M} \mathcal{O}^t \longrightarrow F \longrightarrow 0 \tag{5.2.1}$$

and $\dim \mathrm{Stab}(M) = 1$, *then F is simple.*

Proof. If, by contradiction, F is not simple, then there exists $\phi : F \to F$ non-trivial.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s \arrow[r] \arrow[d, "A"] & \mathcal{O}^t \arrow[r] \arrow[d, "B"] & F \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s \arrow[r] & \mathcal{O}^t \arrow[r] & F \arrow[r] & 0
\end{tikzcd} \tag{5.2.2}$$

Applying the functor $\mathrm{Hom}(-, F)$ to the sequence (5.2.1), we get that ϕ induces $\widetilde{\phi}$ non-trivial in $\mathrm{Hom}(\mathcal{O}^t, F)$. Now, applying the functor $\mathrm{Hom}(\mathcal{O}^t, -)$ again to the same sequence, we get $\mathrm{Hom}(\mathcal{O}^t, \mathcal{O}^t) \cong \mathrm{Hom}(\mathcal{O}^t, F)$ because $\mathrm{Hom}(\mathcal{O}^t, \mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s) = 0$ and $\mathrm{Ext}^1(\mathcal{O}^t, \mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s) = 0$. It follows that there exists $\widetilde{\phi}$ non-trivial in $\mathrm{End}(\mathcal{O}^t)$, i.e. a matrix $B \neq \mathrm{Id}$ in $\mathrm{GL}(t)$. Restricting $\widetilde{\phi}$ to $\mathcal{O}(-2)^q \oplus \mathcal{O}(-1)^s$ and calling A the corresponding matrix, we get the commutative diagram (5.2.2). Therefore, $(A, B) \neq (\lambda \mathrm{Id}, \lambda \mathrm{Id})$ belongs to $\mathrm{Stab}(M)$ and hence $\dim \mathrm{Stab}(M) > 1$. $\square$

Theorem 5.2.2. *Let F be a generic bundle on $\mathbb{P}^2$ with resolution*

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0.$$

If $s + 3 \leq t \leq 2s + 3$, then F is simple.

Proof. We only need to prove that the condition $t \leq 2s + 3$ implies the generic simplicity.

We consider first the case $t = 2s + 3$. Let $M = M(x_0, x_1, x_2)$ be the following matrix of size $(s + 1) \times ((s + 1) + 1 + (s + 1))$ of forms:

$$M = \left[\begin{array}{ccccc|c|cccc}
x_0^2 & x_1^2 & & & & & x_2^2 & & & \\
& x_0 & x_1 & & & & & x_2 & & \\
& & x_0 & x_1 & & & & & x_2 & \\
& & & \ddots & \ddots & & & & \ddots & \\
& & & & x_0 & x_1 & & & & x_2
\end{array}\right]$$

We claim that $\dim \mathrm{Stab}(M) = 1$. Indeed it is obvious that $(\lambda \mathrm{Id}, \lambda \mathrm{Id}) \in \mathrm{Stab}(M)$. If (A, B) is a pair of matrices in $\mathrm{Stab}(M)$, it follows that

$$A^{-1} M(x_0, x_1, x_2) B = M(x_0, x_1, x_2),$$

i.e.

$$M(x_0, x_1, x_2) B = A M(x_0, x_1, x_2)$$

for all $(x_0, x_1, x_2) \in \mathbb{P}^2$. In particular we can consider the three points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and we get the following matrices

$$M_0 = M(1,0,0) = \left[\begin{array}{c|c|c} I_{s+1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & 0_{s+1} \end{array} \right],$$

$$M_1 = M(0,1,0) = \left[\begin{array}{c|c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_{s+1} & 0_{s+1} \end{array} \right],$$

$$M_2 = M(0,0,1) = \left[\begin{array}{c|c|c} 0_{s+1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_{s+1} \end{array} \right],$$

where I_n denote the identity matrix of size $(n \times n)$ and 0_n the zero matrix of size $(n \times n)$. By simple computations, we see that from the following three conditions

$$\begin{cases} M_0 B = A M_0 \\ M_1 B = A M_1 \\ M_2 B = A M_2 \end{cases}$$

it follows that $A = \lambda \mathrm{Id}$ and $B = \lambda \mathrm{Id}$, i.e. $\mathrm{Stab}(M)$ has dimension 1. Therefore, by Lemma 5.2.1, we deduce that the bundle $F = \mathrm{Coker}\, M$ is simple. Since the simplicity is an open condition, the statement of the theorem follows.

If $t < 2s + 3$, we consider the matrix given by the first t columns of M and the same proof holds. $\square$

By joining Theorem 5.1.1 and Theorem 5.2.2 we get the final result

Theorem 5.2.3. *Let F be a generic bundle on $\mathbb{P}^2$ with resolution*

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to F \to 0.$$

and let p be the polygonal

$$p = \{Q_2, D_3, Q_3, D_4, \dots, Q_{k-1}, D_k, Q_k, \dots\},$$

where $D_k = (3r_k a_{k-2}, 3r_k a_{k-1})$ and $Q_k = (r_{k+1}a_{k-2}, r_{k+1}a_{k-1})$. Then

(i) F is stable if and only if F is simple if and only if either

- *(s,t) lies on the polygonal p, or*
- *(s,t) lies between the polygonal p and the line $t = s+2$;*

(ii) F is not stable if and only if F is not simple if and only if (s,t) is above the polygonal p.

Furthermore, we can generalize Theorem 5.2.2 as follows

Theorem 5.2.4. *Let G be a generic bundle on $\mathbb{P}^2$ with resolution*

$$0 \to \mathcal{O}(-3)^q \oplus \mathcal{O}(-2)^s \to \mathcal{O}(-1)^t \to G \to 0.$$

If $t \leq 2(q+s)+1$, then G is simple.

Proof. We can repeat the proof of Theorem 5.2.2 with the following $(q+s)\times t$-matrix

$$M(x_0,x_1,x_2) = \left[\begin{array}{cccccc|c|cccccc}
x_0^2 & x_1^2 & & & & & & x_2^2 & & & & & \\
& \ddots & \ddots & & & & & & \ddots & & & & \\
& & x_0^2 & x_1^2 & & & & & & x_2^2 & & & \\
& & & x_0 & x_1 & & & & & & x_2 & & \\
& & & & \ddots & \ddots & & & & & & \ddots & \\
& & & & & x_0 & x_1 & & & & & & x_2
\end{array}\right],$$

where the first q rows contain quadratic forms and the other s rows contain linear forms.

$\square$

Appendix

Macaulay 2 is a software system devoted to supporting research in algebraic geometry and commutative algebra. Now, we present a very simple program written in Macaulay 2.

If E is a generic bundle with resolution

$$0 \to \mathcal{O}(-3) \oplus \mathcal{O}(-2)^4 \xrightarrow{M} \mathcal{O}(-1)^{16} \to E \to 0,$$

then the following program computes the dimension of $\mathrm{H}^0(\operatorname{End} E)$.

```
--ring:
R=ZZ/32003[x_0..x_2]
--generic matrix M:
M=random(R^16,R^{-2,-1,-1,-1,-1})
E=coker(M)
--dual matrix:
N=transpose M;
F = ker N;
--resolution of End E:
S=res (E**F)
out=S.dd_1;
tout=transpose out;
k=kernel tout;
--compute H^0(End E):
hilbertFunction(0,k)
```

When we want to consider bundles with other resolutions, it suffices to change the definition of the matrix M and the program should work. Nevertheless, when the dimension of the matrix M is very big, it is possible that the program aborts. In this case, we can compute $\mathrm{H}^0(\operatorname{End} E)$ in a less direct way, by substituting the last five lines of the program with the following

```
P=presentation(E**F);
Q=prune transpose P;
betti res coker Q
```

and by searching the dimension of H^0 among the Betti numbers obtained.

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