Chapter 3

Infinity HBVMs

From the previous arguments, it is clear that the orthogonality conditions (1.13), i.e., the fulfillment of the *Master Functional Equation* (1.15), is in principle only a sufficient condition for the conservation property (1.12) to hold, when a generic polynomial basis $\{P_j\}$ is considered. Such a condition becomes also necessary, when such basis is orthonormal.

Theorem 3. *Let* $\{P_j\}$ *be an orthonormal basis on the interval* $[0, 1]$*. Then, assuming* $H(y)$ *to be analytical, (1.12) implies that each term in the sum has to vanish.*

Proof Let us consider the expansion

$$g(\tau) \equiv \nabla H(\sigma(t_0 + \tau h)) = \sum_{\ell \geq 1} \rho_\ell P_\ell(\tau), \qquad \rho_\ell = (P_\ell, g), \qquad \ell \geq 1,$$

where, in general,

$$(f, g) = \int_0^1 f(\tau) g(\tau) \mathrm{d}\tau.$$

Substituting into (1.12), yields

$$\sum_{j=1}^{s} \gamma_j^T (P_j, g) = \sum_{j=1}^{s} \gamma_j^T \left(P_j, \sum_{\ell \geq 1} \rho_\ell P_\ell \right) = \sum_{j=1}^{s} \gamma_j^T \rho_j = 0.$$

Since this has to hold whatever the choice of the function $H(y)$, one concludes that

$$\gamma_j^T \rho_j = 0, \qquad j = 1, \dots, s. \ \square \tag{3.1}$$

Remark 8. *In the case where* $\{P_j\}$ *is an orthonormal basis, from (3.1) one then derives that*

$$\gamma_j = S\rho_j, \qquad i = 1, \ldots, s,$$

where S *is any nonsingular skew-symmetric matrix. The natural choice* $S = J$ *then leads to (1.13).*

Moreover, we observe that, if the Hamiltonian $H(y)$ is a polynomial, the integral appearing at the right-hand side in (1.18) is exactly computed by a quadrature formula, thus resulting into a HBVM(k,s) method with a sufficient number of silent stages. As already stressed in the Chapter 1, in the non-polynomial case such formulae represent the limit of the sequence HBVM(k,s), as $k \to \infty$.

Definition 3. *For general Hamiltonians, we call the limit formula (1.18)* Infinity Hamiltonian Boundary Value Method of degree s *(in short,* ∞-HBVM of degree s *or* HBVM(∞, s)*) [7].*

More precisely, due to the choice of the orthonormal basis (1.8),

$$\text{HBVM}(\infty, s) = \lim_{k \to \infty} \text{HBVM}(k, s),$$

whatever is the choice of the fundamental abscissae $\{c_i\}$.

A worthwhile consequence of Theorems 1 and 2 is that one can transfer to HBVM(∞, s) all those properties of HBVM(k,s) which are satisfied starting from a given $k \geq k_0$ on: for example, the order and stability properties.

Corollary 1. *Whatever the choice of the abscissae* $c_1, \ldots, c_s$*, HBVM*(∞, s) (1.18) *has order* $2s$ *and is perfectly* A*-stable.*