Multiparameter exponentially-fitted Numerov methods applied to second-order boundary value problems

Marnix Van Daele, Davy Hollevoet, Guido Vanden Berghe
Vakgroep Toegepaste Wiskunde en Informatica, Universiteit Gent

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The Numerov method is a well-known numerical linear multistep method to solve second-order boundary value problems of the form

$$y'' = f(t,y)\,, \qquad y(a) = \alpha\,, \qquad y(b) = \beta\,. \tag{1}$$

This method has order four, indicating that the method is exact when the solution is a polynomial of degree 5 at most.

In recent papers so-called exponentially-fitted (EF) modified versions of this Numerov method were constructed. These EF methods depend upon a parameter μ which can be tuned to solve the problem at hand adequately. The idea is quite simple : suppose one wants to solve $y'' - \mu^2\, y = g(t,y)$ where $|g(t,y)| \ll |\mu^2\, y|$. In that case, the solution $y(t)$ can be well approximated by a linear combination of $\exp(\mu\, t)$ and $\exp(-\mu\, t)$. To mimic this behaviour, new methods are constructed which exactly integrate $\exp(\pm\mu\, t)$.

In general, EF methods exactly integrate functions of the form 1, t, ..., t^K and $\exp(\pm\mu t)$, $t\,\exp(\pm\mu\, t)$, ..., $t^P\,\exp(\pm\mu\, t)$. For modified versions of the Numerov method we have $2\,(P+1)+K = 5$. The number P is called the level of tuning : for $(P,K) = (-1,5)$ the purely polynomial method is obtained, while the combinations $(P,K) = (0,3)$, $(1,1)$ and $(2,-1)$ yield three new methods.

In practice, the value of μ of an EF rule can be computed by locally annihilating the leading term of local truncation error of the method : at the point t_j the value μ_j for μ is computed from the equation $D^{K+1}\,(D^2 - \mu_j^2)^{P+1} y(t_j) = 0$. Since this expression is a polynomial of degree $P+1$ in μ_j^2, it is not guaranteed in case P is odd that real μ_j^2 exist and that the numerical solution of (1) will be real.

In order to avoid this problem, we propose a new type of EF methods which exactly integrate 1, t, ..., t^K and $P+1$ couples of exponential functions $\exp(\pm\mu_i t)$, $i = 0, \ldots, P$. In this way, multiparameter EF methods arise.