Collocation Schemes for Nonlinear Index 1 DAEs with a Singular Point

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We discuss the convergence behavior of collocation schemes applied to approximate solutions of BVPs in nonlinear DAEs given in the form

$$f((D(t)x(t))', x(t), t) = 0, \quad t \in (0,1], \quad B_0 D(0)x(0) + B_1 D(1)x(1) = \beta,$$

where $f(y,x,t) \in \mathbb{R}^m$, $D(t) \in \mathbb{R}^{n\times m}$, $y \in \mathbb{R}^n$, $x \in \mathcal{D}$, with $\mathcal{D} \subseteq \mathbb{R}^m$ open, $t \in [0,1]$, $n \le m$. The data f, f_y, f_x, D are assumed to be at least continuous on their definition domains. Moreover, we require

$$\ker f_y(y,x,t) = 0, \quad (y,x,t) \in \mathbb{R}^n \times \mathcal{D} \times (0,1], \quad \mathcal{R}(D(t)) = \mathbb{R}^n, \quad t \in [0,1].$$

These conditions guarantee that the matrix $D(t)$ has constant full row rank n on the closed interval while $f_y(y,x,t)$ has full column rank n only on $\mathbb{R}^n \times \mathcal{D} \times (0,1]$; at $t = 0$ the matrix $f_y(y,x,t)$ may undergo a rank drop. Let us denote by Q_0 a continuous pointwise projector function onto $\ker D$, $Q_0(t)^2 = Q_0(t)$, $\mathcal{R}(Q_0(t)) = \ker D(t)$, $t \in [0,1]$. Moreover, let us define two matrices from the matrix chain used in the decoupling of the DAE system into the inherent ODE system and the algebraic constraints,

$$G_0(y,x,t) := f_y(y,x,t)D(t), \quad (y,x,t) \in \mathbb{R}^n \times \mathcal{D} \times [0,1],$$
$$G_1(y,x,t) := G_0(y,x,t) + f_x(y,x,t)Q_0(t), \quad (y,x,t) \in \mathbb{R}^n \times \mathcal{D} \times [0,1].$$

We discuss DAE systems which are regular with tractability index 1 on $\mathbb{R}^n \times \mathcal{D} \times (0,1]$. Consequently, $G_1(y,x,t)$ is nonsingular on $\mathbb{R}^n \times \mathcal{D} \times (0,1]$. We permit a singular behavior of $G_1(y,x,t)$ for $t \to 0$, causing a singularity in the associated inherent ODE. To this end, we assume that $tG_1(y,x,t)^{-1}$ has a continuous extension on $\mathbb{R}^n \times \mathcal{D} \times [0,1]$. The above assumptions result in the following form of the associated inherent ODE system:

$$u'(t) = \frac{1}{t}M(t)u(t) + h(u(t),t), \quad t \in (0,1], \quad B_0 u(0) + B_1 u(1) = \beta, \quad u \in C[0,1],$$

exhibiting a *singularity of the first kind*. We apply standard polynomial collocation at k interior collocation points to solve the *DAE system written in its original form*. We specify conditions under which for a certain class of well-posed boundary value problems in DAEs having sufficiently smooth solutions, the global error at the *collocation points* shows the convergence behavior $O(h^k)$ known from the ODE case. Here, the boundedness of the so-called canonical projector function plays a crucial role. The superconvergence order in the mesh points known for special choices of collocation points (Gaussian) for regular ODEs, does not hold even for singular ODEs, and thus neither for the DAEs, in general. The theoretical results are supported by numerical experiments.