PGSCM: A FAMILY OF P-STABLE BOUNDARY VALUE METHODS FOR SECOND-ORDER INITIAL VALUE PROBLEMS[§]

LIDIA ACETO*, PAOLO GHELARDONI*, AND CECILIA MAGHERINI*

Abstract. In this paper, we introduce a family of Linear Multistep Methods used as Boundary Value Methods, that we call PGSCMs, for the numerical solution of initial value problems for second order ordinary differential equations of special type. We prove rigorously that it is composed by P-stable schemes, in a generalized sense, of arbitrarily high order. This overcome the barrier that Lambert and Watson established in [16] on Linear Multistep Methods used in the classic way; that is as Initial Value Methods. A numerical illustration which confirms the theoretical results of the paper is finally given.

Key words. Second order ordinary differential equations; P-stability; Boundary Value Methods

AMS subject classifications. 65L05, 65L20, 65L04

1. Introduction. The numerical solution of initial value problems for second order ordinary differential equations of special type given by

$$y''(x) = f(x,y)\,, \quad y(x_0) = y_0\,, \quad y'(x_0) = y'_0, \qquad x \in [x_0, X]\,, \tag{1.1}$$

having periodic and oscillatory solution $y(x) \in \mathbb{R}^r$, has attracted much interest in recent decades. It is well-known that these problems can be easily reformulated as systems of first order ODEs of size $2r$ so that one of the several schemes currently available in the literature for the latter type of problems can be applied for their solution. It is evident, however, that the use of numerical schemes designed for solving (1.1) in its original formulation is more competitive from the point of view of the computational complexity.

In this context, the application of Linear Multistep Methods (LMMs) is one of the classical approach. If the interval of integration is discretized with a uniform partition with stepsize $h = (X - x_0)/N$, then a k-step LMM with coefficients α_j's and β_j's replace the equation in (1.1) with the following difference equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j y''_{n+j}\,, \tag{1.2}$$

where $y_n \approx y(x_n)$, $y''_n = f(x_n, y_n)$, with $x_n = x_0 + nh$, for all $n = 0, 1, \ldots, N$.

When the problem to be solved is *stiff,* namely when its solution is a combination of components with dominant short frequencies and components with large frequencies and small amplitudes, the use of schemes satisfying "good" stability properties is mandatory. Following the idea of Dahlquist, a rigorous definition of them was given by Lambert and Watson in [16]. In such paper, the authors applied a linear stability analysis of (1.2) based on the following test equation

$$y'' = -\lambda^2 y\,, \quad \lambda \in \mathbb{R}\,, \tag{1.3}$$

whose general exact solution, given by $y(x) = A\cos(\lambda x) + B\sin(\lambda x)$, is periodic with period $2\pi/\lambda$ (actually with the only exception of the cases $\lambda = 0$ or $A = B = 0$). The

[§]Work developed within the project "Numerical methods and software for differential equations"

*Dipartimento di Matematica Applicata, Università di Pisa, Via Buonarroti 1/c, I-56127 Pisa, Italy (`l.aceto@dma.unipi.it`, `ghelardoni@dma.unipi.it`, `cecilia.magherini@dma.unipi.it`)

aim of the analysis is then that of finding the conditions for which the corresponding numerical solution has (essentially) the same qualitative behaviour.
This led to the definition of interval of periodicity and of P-stability of a method which ensures that the numerical solution has the desired behaviour independently of the used stepsize. In the same paper [16], however, the authors established that the order of a P-stable LMM, used as Initial Value Method (IVM), cannot exceed two which is exactly the analogous of the famous second Dahlquist barrier.
In order to overcome this undeniable negative result, a number of approaches has been adopted across the years. Among them, we mention the hybrid methods proposed in [9, 10, 11] and the class of symmetric two-step Obrechkoff methods recently studied by Van Daele and Vanden Berghe in [19]. In particular, the latter ones are P-stable schemes of order $p = 2m$, with $m \in \mathbb{N}$, which make use of the derivatives of the unknown solution up to order $2m$. In addition, in the last years, particular attention has been devoted to exponential-fitting methods (see, for example, [12, 14, 18, 20]). This field of research is surely interesting even though the derived methods require the a priori knowledge of good approximations of the involved frequencies.
In this article, we shall investigate if the use of LMMs as Boundary Value Methods (BVMs) is successful in overcoming the barrier of Lambert and Watson. The main idea on which such schemes rely is that of completing the discrete problem generated by a LMM with a set of boundary conditions instead of just initial ones as classically done. This approach was introduced in the nineties for the definition of schemes for solving first order ODEs and the principal reference for them is [8]. Their linear stability properties have been studied in details in several papers where it is proved rigorously that they are able to overcome the second Dahlquist barrier [1, 2, 3, 5, 7, 17].

The article is organized as follows. In Section 2 we recall the definitions of interval of periodicity and of P-stability for IVMs and we give their generalization for the case where the LMMs are used as BVMs. In Section 3 we introduce a family of BVMs, that we call PGSCMs, and we prove some properties of their coefficients. The linear stability analysis of the new methods is carried out in Section 4 where it is proved that they are P-stable formulae, in the sense corresponding to BVMs, of arbitrarily high order. Finally, in Section 5 we propose additional formulae to be coupled with the main LMM in order to recover the boundary values required by the discrete problem. The results of a numerical experiment conducted with the new schemes are also reported which confirm the theory of the previous sections.

2. P-stability for Initial and Boundary Value Methods. When the method (1.2) is applied for solving (1.3) the discrete problem reduces to the following linear difference equation

$$\sum_{j=0}^{k} \alpha_j y_{n+j} + q^2 \sum_{j=0}^{k} \beta_j y_{n+j} = 0\,, \qquad q = h\lambda.$$

The corresponding stability polynomial is

$$\pi(z, q^2) = \rho(z) + q^2 \sigma(z),$$

where, as usual,

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j, \qquad \sigma(z) = \sum_{j=0}^{k} \beta_j z^j$$

are the characteristic polynomials of the method. We recall that such method is consistent if

$$\rho(1) = \rho'(1) = 0, \qquad \rho''(1) = 2\sigma(1). \tag{2.1}$$

Let, from now on, $z_1(q^2), z_2(q^2), \ldots, z_k(q^2)$ be the roots of $\pi(z, q^2)$ ordered with increasing modulus. In particular, when the LMM is used as IVM, namely when the discrete problem (1.2) is completed by fixing the values of $y_0, y_1, \ldots y_{k-1}$, let $z_{k-1}(q^2) = \overline{z_k(q^2)}$ be the principal roots of the method; that is, $z_{k-1}(0) = z_k(0) = 1$. It is well-known that, if $|z_{k-2}(q^2)| < |z_{k-1}(q^2)|$ then the solution provided by an IVM is essentially given by a linear combination of $z_{k-1}^n(q^2)$ and $z_k^n(q^2)$ and this led Lambert and Watson to give the following definitions. Before them, we recall that a polynomial is said to be of type (m_1, m_2, m_3) if it has m_1, m_2 and m_3 roots inside, on the boundary, and outside the unit circle in the complex plane, respectively. This notation will be used extensively in the sequel.

Definition 2.1. *A k-step IVM has interval of periodicity $I = (0, q_0^2)$, if $q^2 \in I$ implies that its stability polynomial $\pi(z, q^2)$ is of type $(k - m, m, 0)$ where $m = m(q^2)$ with $2 \le m(q^2) \le k$.*

Definition 2.2. *An IVM is P-stable if $I = (0, \infty)$ being I its interval of periodicity.*

The use of a LMM with a non empty interval of periodicity means that the numerical solution has the desired qualitative behaviour provided the stepsize is chosen sufficiently small in such a way that $q^2 \in I$. This is the case, for example, of the famous Numerov method,

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{12}\left(y''_{n+2} + 10y''_{n+1} + y''_n\right),$$

which has interval of periodicity $(0, 6)$. A similar restriction on the stepsize does not occur if the method used is P-stable and this is surely mandatory if the problem to be solved is stiff. For example, the following methods introduced in [16]

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{2 - 2\cos\phi}\left(y''_{n+2} - 2\cos\phi\, y''_{n+1} + y''_n\right) \tag{2.2}$$

have order two and are P-stable for all $\phi \in (0, 2\pi)$. The important negative result stated in the same paper, however, establishes that the order of accuracy of a P-stable LMM used as IVM cannot exceed two.

In this paper we shall investigate if the use of the BVM approach allows to overcome such barrier. In this case a set of boundary conditions is associated to the difference equation (1.2). More precisely, when applied for solving (1.1), the discrete problem generated by a k-step BVM used with (k_1, k_2)-boundary conditions, $k_1 + k_2 = k$, is given by (1.2) coupled with

$$y_0, y_1, \ldots, y_{k_1-1}, \quad y_{N-k_2+1}, \ldots, y_N \qquad \text{fixed.} \tag{2.3}$$

We shall talk in Section 5 about a possible strategy that can be used for getting an approximation of the boundary values. The important advantage that arises from this approach is that the principal roots of the method are no longer restricted to be the ones of largest modulus. This is a consequence of the following result.

THEOREM 2.3. *Suppose that a linear difference equation of order k with constant coefficients has characteristic roots z_i satisfying*

$$|z_1| \le \ldots \le |z_{k_1-2}| < |z_{k_1-1}| = |z_{k_1}| < |z_{k_1+1}| \le \ldots \le |z_k|, \qquad 1 < |z_{k_1+1}|,$$

with $z_{k_1-1} \neq z_{k_1}$. Then, the solution of an associated boundary value problem with k_1 initial values and $k_2 = k - k_1$ final ones as in (2.3) behaves as

$$y_n = |z_{k_1}|^n \left[\hat{\gamma}_1 \left(\frac{z_{k_1-1}}{|z_{k_1-1}|} \right)^n + \hat{\gamma}_2 \left(\frac{z_{k_1}}{|z_{k_1}|} \right)^n + O\left(\left| \frac{z_{k_1-2}}{z_{k_1}} \right|^n \right) \right.$$
$$\left. + O\left(\left| \frac{z_{k_1}}{z_{k_1+1}} \right|^{N-n} \right) + O\left(|z_{k_1+1}|^{-N} \right) \right] + O\left(|z_{k_1+1}|^{-(N-n)} \right),$$

when n and $N - n$ are sufficiently large. In the previous asymptotic estimate, the coefficients $\hat{\gamma}_1$ and $\hat{\gamma}_2$ depend only on the initial values $y_0, y_1, \ldots, y_{k_1-1}$.

Proof. The statement can be proved by using arguments similar to the ones considered in the proof of Theorem 2.6.1 in [8]. □

Clearly, from the previous theorem one gets that, for a fixed $q^2 > 0$, the numerical solution provided by a k-step BVM with (k_1, k_2)-boundary conditions is (essentially) periodic if $\pi(z, q^2)$ is of type $(k_1 - 2, 2, k_2)$. In this regard, in [16] it was proved that this may happen only if the method is symmetric, i.e.

$$\alpha_j = \alpha_{k-j}, \quad \beta_j = \beta_{k-j}, \qquad j = 0, 1, \ldots, k.$$

In the same paper, it was also proved that a symmetric irreducible LMM has step-number and order even. In the sequel, we shall therefore assume $k = 2\nu$ with $\nu \ge 1$. We can now give the following definitions which extend the ones given for an IVM.

DEFINITION 2.4. *A (2ν)-step BVM with $(\nu+1, \nu-1)$-boundary conditions is said to have interval of ν-**periodicity** $I_\nu = (0, q_0^2)$, if $\pi(z, q^2)$ is of type $(\nu - 1, 2, \nu - 1)$ for all $q^2 \in I_\nu$.*

DEFINITION 2.5. *A (2ν)-step BVM with $(\nu+1, \nu-1)$-boundary conditions is said* **P_ν-stable** *if $I_\nu = (0, \infty)$.*

The main target of this article is to determine a family of P_ν-stable BVMs of order greater than two, i.e. methods that overcome the barrier established by Lambert and Watson in [16]. The tool that we are going to use for the linear stability analysis is the boundary locus of the method defined by

$$\Gamma = \left\{ q^2 \in \mathbb{C} : q^2 \equiv \psi(\theta) = -\frac{\rho(\mathrm{e}^{\mathrm{i}\theta})}{\sigma(\mathrm{e}^{\mathrm{i}\theta})}, \theta \in [0, 2\pi) \right\}. \tag{2.4}$$

It is not difficult to verify that

- the elements of Γ are the values of q^2 such that $\pi(z, q^2)$ has at least one root on the unit circle;
- if the method is symmetric then $\Gamma \subset \mathbb{R}$, $\psi(\theta) = \psi(2\pi - \theta) = \psi(-\theta)$;
- $I_\nu \subseteq \Gamma$ so that a (2ν)-step BVM can be P_ν-stable only if Γ is unbounded, i.e. if there exists $\theta \in (0, 2\pi)$ such that $\sigma(\mathrm{e}^{\mathrm{i}\theta}) = 0$.

3. PGSCMs for second order ODEs. In this section, we shall derive a family of BVMs obtained as a generalization of the popular Störmer-Cowell methods. They verify the necessary conditions to be P_ν-stable, namely their boundary locus is unbounded and they are symmetric. The first property is verified by construction while the second one will be proved after their derivation. We name these schemes PGSCMs, acronym for *P_ν-stable Generalized Störmer-Cowell Methods*.
When applied for solving (1.1), the difference equation generated by the (2ν)-step PGSCM reads

$$y_{n+1} - 2y_n + y_{n-1} = h^2 \sum_{j=-\nu}^{\nu} \beta_{j+\nu}^{(2\nu)} y''_{n+j}, \qquad n = \nu, \nu+1, \dots, N-\nu, \tag{3.1}$$

with $\nu \in \mathbb{N}$. Observe that we have introduced an upper index on the coefficients β_j's to denote the stepnumber of the corresponding method. Like the Störmer-Cowell methods, these formulae have the first characteristic polynomial

$$\rho_{2\nu}(z) = \sum_{j=0}^{2\nu} \alpha_j^{(2\nu)} z^j = z^{\nu-1}(z-1)^2 \tag{3.2}$$

fixed a priori which verifies the first two consistency conditions $\rho_{2\nu}(1) = \rho'_{2\nu}(1) = 0$, see (2.1). The second characteristic polynomial

$$\sigma_{2\nu}(z) = \sum_{j=0}^{2\nu} \beta_j^{(2\nu)} z^j \tag{3.3}$$

is determined by imposing the formula to have order $p = 2\nu$ and

$$\sigma_{2\nu}(-1) = 0, \tag{3.4}$$

so that the associated boundary locus (2.4) is unbounded. The method has order $p = 2\nu$ if the following order conditions, obtained by considering the Taylor series expansion of the exact solution at $x = x_\nu$, are verified

$$\sum_{j=-\nu}^{\nu} \beta_{j+\nu}^{(2\nu)} j^{s-2} = \frac{(-1)^s + 1}{s(s-1)}, \qquad s = 2, 3, \dots, 2\nu+1. \tag{3.5}$$

It is important to observe that the so-obtained 2-step method coincides with the one in (2.2) corresponding to $\phi = \pi$ so that the family of PGSCMs represents a generalization of it. In addition, the 4-step method has been already derived in [6] even though its stability properties were not proved in such paper.
With the aim of writing (3.4)-(3.5) in matrix form, we introduce the following notation. For each $\ell \geq 1$ and $x \in \mathbb{R}$, let

$$\xi_\ell(x) = \left(x^0, x^1, \dots, x^{\ell-1}\right)^T. \tag{3.6}$$

In addition, let

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ -\nu & -\nu+1 & \cdots & \nu \\ \vdots & \vdots & \vdots & \vdots \\ (-\nu)^{2\nu} & (-\nu+1)^{2\nu} & \cdots & \nu^{2\nu} \end{pmatrix}, \tag{3.7}$$

$$\mathbf{v}_{2\nu} = \left(\frac{2}{2\cdot 1}, 0, \frac{2}{4\cdot 3}, 0, \ldots, \frac{2}{2\nu\cdot(2\nu-1)}, 0 \right)^T, \tag{3.8}$$

$$\tilde{I} = \begin{pmatrix} I_{2\nu} & \mathbf{0}_{2\nu} \\ \mathbf{0}_{2\nu}^T & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} O_{2\nu} & \mathbf{0}_{2\nu} \\ \xi_{2\nu}^T(-1) & 1 \end{pmatrix}, \tag{3.9}$$

where $I_{2\nu}$, $O_{2\nu}$ and $\mathbf{0}_{2\nu}$ are the identity matrix, the zero matrix and the zero vector of size 2ν, respectively. Then, one verifies that (3.4)-(3.5) can be reformulated in matrix form as

$$(\tilde{I}V + E)\boldsymbol{\beta}^{(2\nu)} = \begin{pmatrix} \mathbf{v}_{2\nu} \\ 0 \end{pmatrix} \tag{3.10}$$

where $\boldsymbol{\beta}^{(2\nu)} = (\beta_0^{(2\nu)}, \beta_1^{(2\nu)}, \ldots, \beta_{2\nu}^{(2\nu)})^T$. The methods obtained as just described satisfy the following proposition.

PROPOSITION 3.1. *For each $\nu \geq 1$, the coefficient vector $\boldsymbol{\beta}^{(2\nu)}$ of the (2ν)-step PGSCM (3.1) satisfying (3.4)-(3.5) is unique. Moreover, the method is symmetric, namely, by denoting with J the anti-identity matrix of size $2\nu+1$, its coefficient vectors satisfy*

$$\boldsymbol{\alpha}^{(2\nu)} = J\boldsymbol{\alpha}^{(2\nu)}, \qquad \boldsymbol{\beta}^{(2\nu)} = J\boldsymbol{\beta}^{(2\nu)}, \tag{3.11}$$

where $\boldsymbol{\alpha}^{(2\nu)} = (\alpha_0^{(2\nu)}, \alpha_1^{(2\nu)}, \ldots, \alpha_{2\nu}^{(2\nu)})^T$ has all zero entries with the exception of $\alpha_{\nu-1}^{(2\nu)} = \alpha_{\nu+1}^{(2\nu)} = 1$ and $\alpha_\nu^{(2\nu)} = -2$.

Proof. By applying the Laplace expansion along the last row and using the fact that the determinant of a Vandermonde matrix with increasing abscissae is positive, it is not difficult to verify that the coefficient matrix $\tilde{I}V + E$ of system (3.10) has a positive determinant so that $\boldsymbol{\beta}^{(2\nu)}$ is uniquely determined.
Concerning the symmetry of the method, the first relation in (3.11) is trivially verified by construction while, in view of the uniqueness of the method, the second relation holds true if $\boldsymbol{\beta}^{(2\nu)}$ and $J\boldsymbol{\beta}^{(2\nu)}$ are both solution of (3.10). We observe that, see (3.6)–(3.9), $\tilde{I}VJ = \mathrm{diag}\,(\xi_{2\nu+1}(-1))\,\tilde{I}V$ and $E\,J = \mathrm{diag}\,(\xi_{2\nu+1}(-1))\,E$. This implies

$$\begin{aligned} (\tilde{I}V + E)J\boldsymbol{\beta}^{(2\nu)} &= \mathrm{diag}\,(\xi_{2\nu+1}(-1))\,(\tilde{I}V + E)\boldsymbol{\beta}^{(2\nu)} \\ &= \mathrm{diag}\,(\xi_{2\nu+1}(-1)) \begin{pmatrix} \mathbf{v}_{2\nu} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_{2\nu} \\ 0 \end{pmatrix}, \end{aligned}$$

where, see (3.8), the last equality is due to the fact that the entries with even index in $\mathbf{v}_{2\nu}$ are all zero. The vector $J\boldsymbol{\beta}^{(2\nu)}$ is therefore solution of (3.10) and this completes the proof. □

In Table 3.1 the normalized coefficients $\hat{\beta}_j^{(2\nu)} = \eta_{2\nu}\beta_j^{(2\nu)}$, $j = 0, 1, \ldots, \nu$ have been reported for $\nu = 1, 2, 3, 4$.

3.1. Properties of the second characteristic polynomial. The first result we are going to prove is that the second characteristic polynomials of PGSCMs are related by a recurrence relation and to this aim we need the following lemma.

Table 3.1
Normalized coefficients of PGSCMs

ν	$\eta_{2\nu}$	$\hat\beta_0^{(2\nu)}$	$\hat\beta_1^{(2\nu)}$	$\hat\beta_2^{(2\nu)}$	$\hat\beta_3^{(2\nu)}$	$\hat\beta_4^{(2\nu)}$
1	4	1	2			
2	24	-1	6	14		
3	960	9	-58	231	596	
4	60480	-134	1103	-4190	14017	38888

Lemma 3.2. *For each integer m, let $P = (p_{ij})_{i,j=1,\dots,m}$ be the lower triangular Pascal matrix whose nonzero entries are*

$$p_{ij} = \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}, \qquad 1 \le j \le i \le m,$$

and let

$$H = \begin{pmatrix} \mathbf{0}^T & 0 \\ I_{m-1} & \mathbf{0} \end{pmatrix}. \tag{3.12}$$

Then, for each $\ell = 1, \dots, m-1$,

$$P^T H^\ell = (I_m + H)^\ell P^T + R_\ell, \tag{3.13}$$

where R_ℓ has the first $m-\ell$ columns with all zero entries.

Proof. We proceed by induction on ℓ. If $\ell = 1$ we verify the statement by direct inspection. In fact,

$$\left(P^T H\right)_{ij} = \begin{pmatrix} j \\ i-1 \end{pmatrix} = \begin{pmatrix} j-1 \\ i-1 \end{pmatrix} + \begin{pmatrix} j-1 \\ i-2 \end{pmatrix} = \left(P^T\right)_{ij} + \left(HP^T\right)_{ij},$$
$$j = 1, 2, \dots, m-1, \qquad i = j, j+1, \dots, m.$$

This implies that, when $\ell = 1$, (3.13) is verified with R_1 a suitable matrix having the first $m-1$ columns with all zero entries.
Next, by induction, if it holds true for ℓ it holds true also for $\ell+1$. In fact, from the induction hypothesis and by taking into account that $P^T H = (I_m + H)P^T + R_1$, as just proved, we obtain

$$\begin{aligned} P^T H^{\ell+1} &= (I_m + H)^\ell P^T H + R_\ell H \\ &= (I_m + H)^{\ell+1} P^T + (I_m + H)^\ell R_1 + R_\ell H \\ &\equiv (I_m + H)^{\ell+1} P^T + R_{\ell+1}, \end{aligned}$$

where $R_{\ell+1}$ has the first $m - \ell - 1$ columns with all zero entries. $\square$

We can now state the following theorem.

Theorem 3.3. *The second characteristic polynomials of PGSCMs verify the recurrence relation*

$$\sigma_2(z) = \gamma_1 (z+1)^2 \equiv \frac{1}{4}(z+1)^2, \tag{3.14}$$

$$\sigma_{2\nu}(z) = z\ \sigma_{2\nu-2}(z) + \gamma_\nu (z-1)^{2\nu-2}(z+1)^2, \quad \nu = 2, 3, \dots, \tag{3.15}$$

for suitable coefficients γ_ν, $\nu \geq 2$.

Proof. Concerning (3.14) nothing has to be proved (see Table 3.1). With reference to (3.15), the relation holds true if $\boldsymbol{\beta}^{(2\nu)}$ can be written in the form

$$\boldsymbol{\beta}^{(2\nu)} = \begin{pmatrix} 0 \\ \boldsymbol{\beta}^{(2\nu-2)} \\ 0 \end{pmatrix} + \gamma_\nu \mathbf{c}^{(\nu)} \tag{3.16}$$

where γ_ν is a suitable coefficient and $\mathbf{c}^{(\nu)} = \left(c_0^{(\nu)}, c_1^{(\nu)}, \ldots, c_{2\nu}^{(\nu)}\right)^T$ satisfies, see (3.6),

$$(z-1)^{2\nu-2}(z+1)^2 = \sum_{i=0}^{2\nu} c_i^{(\nu)} z^i = \xi_{2\nu+1}^T(z)\mathbf{c}^{(\nu)} \,. \tag{3.17}$$

¿From (3.10) one gets that (3.16) is equivalent to

$$(\tilde{I}V + E)\begin{pmatrix} 0 \\ \boldsymbol{\beta}^{(2\nu-2)} \\ 0 \end{pmatrix} - \begin{pmatrix} \mathbf{v}_{2\nu} \\ 0 \end{pmatrix} = -\gamma_\nu\, (\tilde{I}V + E)\mathbf{c}^{(\nu)} \,. \tag{3.18}$$

Now, it results

$$(\tilde{I}V + E)\begin{pmatrix} 0 \\ \boldsymbol{\beta}^{(2\nu-2)} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{v}_{2\nu-2} \\ \chi \\ 0 \\ 0 \end{pmatrix},$$

for a suitable $\chi \in \mathbb{R}$. In fact, the first $2\nu-2$ of the previous equalities and the last one are the conditions (3.10), with $\nu-1$ in place of ν, which uniquely determine $\boldsymbol{\beta}^{(2\nu-2)}$. The second last equality, instead, is due to the symmetry of the $(2\nu-2)$-step method. In addition, see (3.8),

$$\mathbf{v}_{2\nu} = \begin{pmatrix} \mathbf{v}_{2\nu-2} \\ \frac{2}{2\nu\cdot(2\nu-1)} \\ 0 \end{pmatrix}.$$

This implies that the vector on the left hand-side in (3.18) belongs to $\mathrm{span}\{\mathbf{e}_{2\nu-1}\}$ where, from now on, $\mathbf{e}_\ell$ will denote the ℓ-th unit vector of size $2\nu+1$. It follows that equation (3.18) holds true if $(\tilde{I}V + E)\mathbf{c}^{(\nu)} \in \mathrm{span}\{\mathbf{e}_{2\nu-1}\}$. ¿From (3.9) and (3.17), one gets $E\mathbf{c}^{(\nu)} = \mathbf{0}_{2\nu+1}$ so that it remains to verify that $\tilde{I}V\mathbf{c}^{(\nu)} \in \mathrm{span}\{\mathbf{e}_{2\nu-1}\}$ or, equivalently, that $V\mathbf{c}^{(\nu)} \in \mathrm{span}\{\mathbf{e}_{2\nu-1}, \mathbf{e}_{2\nu+1}\}$.
It is known that the Vandermonde matrix V can be decomposed as [4]

$$V = P^{-\nu} S D_f P^T,$$

where P is the Pascal matrix of size $2\nu+1$ given in Lemma 3.2, S is the unit lower triangular matrix of order $2\nu+1$ whose nonzero entries are the Stirling numbers of the second kind and $D_f = \mathrm{diag}(0!, 1!, \ldots, (2\nu)!)$. Therefore, if we let $\mathbf{w}_\nu = P^T\mathbf{c}^{(\nu)}$, then we need to verify that

$$P^{-\nu} S D_f \mathbf{w}_\nu \in \mathrm{span}\{\mathbf{e}_{2\nu-1}, \mathbf{e}_{2\nu+1}\} \,.$$

It is known that the nonzero entries of P^{-T} are given by [4]

$$\left(P^{-T}\right)_{ij} = \begin{pmatrix} j-1 \\ i-1 \end{pmatrix} (-1)^{i-j}, \qquad 1 \le i \le j \le 2\nu+1$$

from which, see (3.6) and (3.12), one gets

$$z^\ell (z-1)^{2\nu-2} = \left(\xi_{2\nu+1}^T(z) H^\ell\right) P^{-T}\mathbf{e}_{2\nu-1}, \qquad \ell = 0,1,2.$$

The coefficient vector $\mathbf{c}^{(\nu)}$ in (3.17) can be therefore written as

$$\mathbf{c}^{(\nu)} = (I_{2\nu+1} + 2H + H^2)P^{-T}\mathbf{e}_{2\nu-1}$$

so that, from (3.13) and considering that the last two entries of $P^{-T}\mathbf{e}_{2\nu-1}$ are zero, we get

$$\begin{aligned} \mathbf{w}_\nu = P^T\mathbf{c}^{(\nu)} &= \left(I + 2(I+H) + (I+H)^2 + 2R_1P^{-T} + R_2P^{-T}\right)\mathbf{e}_{2\nu-1} \\ &= 4\mathbf{e}_{2\nu-1} + 4\mathbf{e}_{2\nu} + \mathbf{e}_{2\nu+1}. \end{aligned}$$

By virtue of the fact that $P^{-\nu}SD_f$ is lower triangular, we then obtain

$$P^{-\nu}SD_f\mathbf{w}_\nu \in \mathrm{span}\{\mathbf{e}_{2\nu-1}, \mathbf{e}_{2\nu}, \mathbf{e}_{2\nu+1}\}.$$

The result is therefore proved if $\mathbf{e}_{2\nu}^T P^{-\nu}SD_f\mathbf{w}_\nu = 0$, i.e. if

$$\begin{aligned} \mathbf{e}_{2\nu}^T P^{-\nu}SD_f\mathbf{w}_\nu &= 4\mathbf{e}_{2\nu}^T P^{-\nu}SD_f\left(\mathbf{e}_{2\nu-1} + \mathbf{e}_{2\nu}\right) \\ &= 4(2\nu-2)!\,\mathbf{e}_{2\nu}^T P^{-\nu}S\left(\mathbf{e}_{2\nu-1} + (2\nu-1)\mathbf{e}_{2\nu}\right) \\ &= 4(2\nu-2)!\left(\mathbf{e}_{2\nu}^T P^{-\nu}S\mathbf{e}_{2\nu-1} + 2\nu - 1\right) = 0, \end{aligned}$$

but the latter equality holds true since $P^{-\nu}$ and S are both unit lower triangular and [4, 13]

$$\left(P^{-\nu}\right)_{2\nu,2\nu-1} = -\nu\begin{pmatrix} 2\nu-1 \\ 2\nu-2 \end{pmatrix}, \qquad (S)_{2\nu,2\nu-1} = \begin{pmatrix} 2\nu-1 \\ 2 \end{pmatrix}$$

so that

$$\mathbf{e}_{2\nu}^T P^{-\nu}S\mathbf{e}_{2\nu-1} = \left(P^{-\nu}\right)_{2\nu,2\nu-1} + (S)_{2\nu,2\nu-1} = 1 - 2\nu.$$

$\square$

Remark 3.4. *For each $\nu \ge 1$, the coefficient γ_ν in (3.14)-(3.15) is the leading coefficient of $\sigma_{2\nu}(z)$. This clearly implies $\gamma_\nu = \beta_{2\nu}^{(2\nu)}$.*

We are now going to establish some properties of the coefficient $\beta_{2\nu}^{(2\nu)}$. If we apply the Cramer method to (3.10), we get

$$\beta_{2\nu}^{(2\nu)} = \frac{\det(W)}{\det(\tilde{I}V + E)}$$

where W is obtained from $\tilde{I}V + E$ by replacing its last column with the vector of constant terms. It can be verified by direct inspection that $\tilde{I}V + E$ can be factorized as

$$\tilde{I}V + E \equiv \begin{pmatrix} \hat{V} & \xi_{2\nu}(\nu) \\ \xi_{2\nu}^T(-1) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} I_{2\nu} & \mathbf{0}_{2\nu} \\ \xi_{2\nu}^T(-1)\hat{V}^{-1} & 1 \end{pmatrix} \begin{pmatrix} \hat{V} & \xi_{2\nu}(\nu) \\ \mathbf{0}_{2\nu}^T & 1 - \xi_{2\nu}^T(-1)\hat{V}^{-1}\xi_{2\nu}(\nu) \end{pmatrix},$$

where, see (3.7), $\hat{V} \in \mathbb{R}^{(2\nu)\times(2\nu)}$ is obtained from V by removing its last row and column. It follows that $\det(\tilde{I}V + E) = \det(\hat{V})(1 - \xi_{2\nu}^T(-1)\hat{V}^{-1}\xi_{2\nu}(\nu))$. With a similar factorization for W one gets $\det(W) = -\det(\hat{V})(\xi_{2\nu}^T(-1)\hat{V}^{-1}\mathbf{v}_{2\nu})$. Therefore,

$$\beta_{2\nu}^{(2\nu)} = \frac{\xi_{2\nu}^T(-1)\hat{V}^{-1}\mathbf{v}_{2\nu}}{\xi_{2\nu}^T(-1)\hat{V}^{-1}\xi_{2\nu}(\nu) - 1}. \tag{3.19}$$

We observe that the entries of $\hat{V}^{-T}\xi_{2\nu}(-1)$ can be read as the coefficients with respect to the monomial basis of the polynomial $p_\nu(t) = \xi_{2\nu}^T(-1)\hat{V}^{-1}\xi_{2\nu}(t) \in \Pi_{2\nu-1}$ that interpolates the following data set

$$p_\nu(j) = (-1)^{j+\nu}, \qquad j = -\nu, 1-\nu, \ldots, \nu-1. \tag{3.20}$$

This clearly implies that the denominator in (3.19) is equal to $p_\nu(\nu) - 1$. Concerning the numerator, one may verify that, see (3.6)-(3.8)

$$\xi_{2\nu}^T(-1)\hat{V}^{-1}\mathbf{v}_{2\nu} = \xi_{2\nu}^T(-1)\hat{V}^{-1}\int_0^1\int_0^x (\xi_{2\nu}(t) + \xi_{2\nu}(-t))\,dt\,dx$$
$$= \int_0^1\int_0^x (p_\nu(t) + p_\nu(-t))\,dt\,dx.$$

¿From all these considerations, we obtain that (3.19) can be rewritten as

$$\beta_{2\nu}^{(2\nu)} = \frac{\int_0^1\int_{-x}^x p_\nu(t)\,dt\,dx}{p_\nu(\nu) - 1}. \tag{3.21}$$

In order to prove some properties of $\beta_{2\nu}^{(2\nu)}$ we need the results concerning the polynomial $p_\nu(t)$ stated in the following lemma.

LEMMA 3.5. *For each $\nu \geq 1$, the polynomial $p_\nu(t) \in \Pi_{2\nu-1}$ that interpolates the data set (3.20) satisfies the following properties:*

- **P1.** $p_\nu(\nu) - 1 = -4^\nu$;
- **P2.** *the leading coefficient of $p_\nu(t)$, say ω_ν, is negative;*
- **P3.** *$p_\nu(t)$ is symmetric with respect to $t = -\frac{1}{2}$, i.e. $p_\nu(-1/2+t) + p_\nu(-1/2-t) = 0$ for all $t \in \mathbb{R}$;*
- **P4.** $(-1)^\nu \int_{-x}^x p_\nu(t)\,dt > 0$, *for all $x \in (0,1)$;*
- **P5.** $(-1)^\nu \int_{-x}^x (p_\nu(t) + p_{\nu+1}(t))\,dt \geq 0$, *for all $x \in [0,1]$.*

Proof. Concerning property **P1**, by using the Lagrange basis for the interpolating polynomial, we get

$$p_\nu(t) = \sum_{j=-\nu}^{\nu-1} (-1)^{j+\nu}\ell_j(t), \qquad \ell_j(t) = \prod_{i=-\nu, i\neq j}^{\nu-1} \frac{t-i}{j-i}.$$

Now, one may verify that

$$\ell_j(\nu) = \prod_{i=-\nu, i\neq j}^{\nu-1} \frac{\nu-i}{j-i} = \frac{(2\nu)!}{(\nu-j)} \frac{(-1)^{\nu-j-1}}{(\nu+j)!(\nu-j-1)!} = (-1)^{\nu-j-1} \begin{pmatrix} 2\nu \\ \nu+j \end{pmatrix}.$$

and, consequently,

$$p_\nu(\nu) - 1 = -\left(\sum_{j=-\nu}^{\nu-1} \begin{pmatrix} 2\nu \\ \nu+j \end{pmatrix} \right) - 1 = -\sum_{j=0}^{2\nu} \begin{pmatrix} 2\nu \\ j \end{pmatrix} = -4^\nu.$$

The property **P2** is a consequence of the fact that $p_\nu \in \Pi_{2\nu-1}$, $p_\nu(-\nu) = 1$ and the zeros of $p_\nu(t)$ are all real and belong to $[-\nu, \nu-1]$ since in such interval $p_\nu(t)$ changes sign $2\nu - 1$ times. This implies $\lim_{t\to-\infty} p_\nu(t) = +\infty$, i.e. $\omega_\nu < 0$.
In order to prove **P3**, it suffices to observe that $p_\nu(-1/2+t_j)+p_\nu(-1/2-t_j) = 0$, for $t_j = -\nu - \frac{1}{2} + j$ with $j = 1, 2, \ldots, 2\nu$ which implies that $p_\nu(-1/2+t)+p_\nu(-1/2-t)$ is the zero polynomial.
Concerning **P4**, notice that $p_\nu(t) \in \Pi_{2\nu-1}$, $\omega_\nu < 0$ and, see (3.20), $p_\nu(j)-p_\nu(-j) = 0$, for each $j = 1-\nu, \ldots, \nu-1$. Consequently, $p_\nu(t) - p_\nu(-t) = 2\omega_\nu \prod_{j=1-\nu}^{\nu-1}(t-j)$ and therefore

$$(-1)^\nu \left(p_\nu(t) - p_\nu(-t)\right) > 0, \qquad \text{for all } t \in (0,1). \tag{3.22}$$

This implies that if $x \in (0,1)$ then

$$(-1)^\nu \int_{-x}^{x} p_\nu(t)\,dt > (-1)^\nu \left(\int_0^x p_\nu(-t)\,dt + \int_{-x}^0 p_\nu(t)\,dt \right) = (-1)^\nu 2 \int_{-x}^0 p_\nu(t)\,dt \geq 0,$$

where the last inequality is due to property **P3** and to the facts that $p_\nu(0) = (-1)^\nu$ and, when $t \in [-1, 0]$, $p_\nu(t) = 0$ only for $t = -\frac{1}{2}$.
Finally, in order to obtain property **P5** we proceed by applying arguments similar to the ones used for proving the inequality in (3.22). In fact, by letting $q_\nu(t) = p_\nu(t) + p_{\nu+1}(t) \in \Pi_{2\nu+1}$, from (3.20) and property **P3** we get $q_\nu(j) = 0$, for each $j = -\nu, \ldots, \nu-1, -\frac{1}{2}$, i.e.

$$q_\nu(t) = \omega_{\nu+1}(t+\frac{1}{2}) \prod_{j=-\nu}^{\nu-1}(t-j) = \omega_{\nu+1}\, t(t+\frac{1}{2})(t+\nu) \prod_{j=1}^{\nu-1}(t^2-j^2),$$

where we recall that $\omega_{\nu+1} < 0$ represents the leading coefficient of $p_{\nu+1}(t)$. This implies

$$\begin{aligned} q_\nu(t) + q_\nu(-t) &= \omega_{\nu+1}\, t \left((t+\frac{1}{2})(t+\nu) - (-t+\frac{1}{2})(-t+\nu) \right) \prod_{j=1}^{\nu-1}(t^2-j^2) \\ &= \omega_{\nu+1}(1+2\nu)\, t^2 \prod_{j=1}^{\nu-1}(t^2-j^2), \end{aligned}$$

so that $(-1)^\nu \left(q_\nu(t) + q_\nu(-t)\right) \geq 0$, for all $t \in [0,1]$, from which property **P5** immediately follows. ∎

We now have all the instruments for proving the following result.

PROPOSITION 3.6. *For all $\nu \geq 1$, the following inequalities hold true*

$$(-1)^{\nu+1}\beta_{2\nu}^{(2\nu)} > 0, \qquad (-1)^{\nu+1}\left(4\beta_{2\nu+2}^{(2\nu+2)} + \beta_{2\nu}^{(2\nu)}\right) \geq 0. \tag{3.23}$$

Proof. The first inequality follows immediately from (3.21) and properties **P1**, **P4** in the previous lemma.
Concerning the second inequality, again from (3.21) and property **P1**, one gets that it is verified if

$$(-1)^{\nu}\int_0^1 \int_{-x}^{x} (p_\nu(t) + p_{\nu+1}(t))\, dt\, dx \geq 0,$$

and this holds true because of property **P5.** □

We conclude this section with the following result which establishes the type of the second characteristic polynomial $\sigma_{2\nu}(z)$.

THEOREM 3.7. *For each $\nu \geq 1$ and $\theta \in [0, 2\pi)$ it results*

$$\sigma_{2\nu}(\mathrm{e}^{\mathrm{i}\theta}) = \left(\mathrm{e}^{\mathrm{i}\theta} + 1\right)^2 \mathrm{e}^{\mathrm{i}(\nu-1)\theta} g_{\nu-1}(\theta), \tag{3.24}$$

where

$$g_{\nu-1}(\theta) = \sum_{j=0}^{\nu-1} (-1)^j \beta_{2j+2}^{(2j+2)} \left(2\sin\frac{\theta}{2}\right)^{2j} > 0. \tag{3.25}$$

It follows that $\sigma_{2\nu}(z)$ is of type $(\nu-1, 2, \nu-1)$.

Proof. In order to obtain (3.24)-(3.25), it is sufficient to consider Remark 3.4 and to apply Theorem 5.1 in [1] to the sequence of polynomials $(z+1)^{-2}\sigma_{2\nu}(z)$, $\nu \geq 1$. In addition, from the first inequality in (3.23), we obtain $g_{\nu-1}(\theta) > 0$. Consequently $\sigma_{2\nu}(z)$ has exactly two roots, namely $z = -1$ with multiplicity 2, of unit modulus. In view of the symmetry of the same polynomial, see (3.11), we therefore deduce that it is of the indicated type. □

4. P_ν-stability of PGSCMs. This section is devoted to the proof of the main result of this paper consisting of the P_ν-stability of the family of PGSCMs. As mentioned in Section 2, the main tool we are going to use is the boundary locus (2.4). We will in fact establish that, for $\theta \in [0, \pi)$, the map $\theta \to \psi(\theta)$ is one-to-one and onto with respect to the positive semireal axis (origin included). By using this result, we will then prove that the stability polynomial $\pi(z, q^2)$ is of type $(\nu-1, 2, \nu-1)$ for all $q^2 \in (0, \infty)$, i.e. that the method is P_ν-stable.

THEOREM 4.1. *For each $\nu \geq 1$, let $\rho_{2\nu}(z)$ and $\sigma_{2\nu}(z)$ be the characteristic polynomials of the (2ν)-step PGSCM defined in (3.2)-(3.3) with coefficients $\beta_j^{(2\nu)}$'s uniquely determined from (3.10). Then, the map $\psi : [0, \pi) \to [0, \infty)$ given by*

$$\psi(\theta) = -\frac{\rho_{2\nu}(\mathrm{e}^{\mathrm{i}\theta})}{\sigma_{2\nu}(\mathrm{e}^{\mathrm{i}\theta})}$$

is one-to-one and onto.

Proof. ¿From (3.2) and (3.24), one immediately gets

$$\psi(\theta) = -\frac{\mathrm{e}^{\mathrm{i}(\nu-1)\theta}(\mathrm{e}^{\mathrm{i}\theta}-1)^2}{(\mathrm{e}^{\mathrm{i}\theta}+1)^2\mathrm{e}^{\mathrm{i}(\nu-1)\theta}g_{\nu-1}(\theta)} = -\frac{(\mathrm{e}^{\mathrm{i}\theta/2}-\mathrm{e}^{-\mathrm{i}\theta/2})^2}{(\mathrm{e}^{\mathrm{i}\theta/2}+\mathrm{e}^{-\mathrm{i}\theta/2})^2}\,\frac{1}{g_{\nu-1}(\theta)}$$
$$= \left(\tan\frac{\theta}{2}\right)^2 \frac{1}{g_{\nu-1}(\theta)}$$

so that the map is onto (recall that, see (3.25), $g_{\nu-1}(\theta) > 0$). With the aim of proving that it is also one-to-one, we need to verify that $\psi(\theta)$ is an increasing function for $\theta \in (0,\pi)$. If we let $s(\theta) \equiv \sin^2\frac{\theta}{2}$ then, see (3.25),

$$\psi(\theta) = \phi(s(\theta)) \equiv \frac{s(\theta)}{1-s(\theta)}\frac{1}{g_{\nu-1}(s(\theta))}, \qquad g_{\nu-1}(s) = \sum_{j=0}^{\nu-1}(-4)^j\beta_{2j+2}^{(2j+2)}s^j. \tag{4.1}$$

Clearly, $s(\theta)$ is increasing for $\theta \in (0,\pi)$ so that it is sufficient to prove that

$$\phi'(s) = \frac{g_{\nu-1}(s) - s(1-s)g'_{\nu-1}(s)}{\left((1-s)g_{\nu-1}(s)\right)^2} > 0$$

or, equivalently, that its numerator is positive. From (4.1), with some computations, one gets

$$\begin{aligned}
&\big(g_{\nu-1}(s) - sg'_{\nu-1}(s)\big) + s^2g'_{\nu-1}(s) = \\
&\quad= \sum_{j=0}^{\nu-1}(-4)^j\beta_{2j+2}^{(2j+2)}(1-j)s^j + \sum_{j=2}^{\nu}(-1)^j\beta_{2j}^{(2j)}4^{j-1}(1-j)s^j \\
&\quad= \beta_2^{(2)} + (-4)^{\nu-1}\beta_{2\nu}^{(2\nu)}(\nu-1)s^\nu + \sum_{j=2}^{\nu-1}(-4)^{j-1}(j-1)\left(4\beta_{2j+2}^{(2j+2)} + \beta_{2j}^{(2j)}\right)s^j
\end{aligned}$$

which is strictly positive since $\beta_2^{(2)} = 1/4$ and, in view of (3.23), all the other addends are nonnegative. □

THEOREM 4.2. *For each $\nu \geq 1$, let $\pi(z,q^2) = \rho_{2\nu}(z) + q^2\sigma_{2\nu}(z)$ be the stability polynomial associated to the (2ν)-step PGSCM. Then, for all $q^2 \in (0,\infty)$ the type of $\pi(z,q^2)$ is $(\nu-1,2,\nu-1)$ and the method is P_ν-stable when used with $(\nu+1,\nu-1)$-boundary conditions.*

Proof. By virtue of the previous theorem and considering that $\pi(z,q^2)$ has real coefficients it is sufficient to observe that, see (2.4), for all $q^2 \in (0,\infty)$ there exists a unique $\theta \in (0,\pi)$ such that $\pi(\mathrm{e}^{\mathrm{i}\theta},q^2) = \pi(\mathrm{e}^{-\mathrm{i}\theta},q^2) = 0$. ¿From the symmetry of the method, one therefore gets that the type of $\pi(z,q^2)$ is $(\nu-1,2,\nu-1)$ for all $q^2 > 0$ so that, when used with $(\nu+1,\nu-1)$-boundary conditions, the method is P_ν-stable according to Definitions 2.1,2.2. □

5. Additional methods and a numerical illustration. The effective use of PGSCMs requires the definition of a suitable strategy for recovering the boundary values in (2.3). Clearly, the initial value y_0 is provided by the continuous problem. Concerning the remaining ones, we have applied the usual technique for BVMs of getting them implicitly through the application of a set of $2\nu-2$ additional formulae together with a discretization of the first order derivative $y'(x_0) = y'_0$ at the initial

point.
In more details, if the interval of integration is $[x_0, X]$ and $h = (X - x_0)/N$ then the following set of $\nu - 1$ initial and $\nu - 1$ final additional methods

$$y_{i-1} - 2y_i + y_{i+1} = h^2 \sum_{j=0}^{2\nu-1} \beta_j^{(i,2\nu)} y_j'' , \qquad i = 1, 2, \ldots, \nu - 1, \tag{5.1}$$

$$y_{m-1} - 2y_m + y_{m+1} = h^2 \sum_{j=0}^{2\nu-1} \beta_j^{(i,2\nu)} y_{m-i+j+1}'', \qquad i = \nu + 1, \ldots, 2\nu - 1, \quad m = N + i - 2\nu. \tag{5.2}$$

are coupled with the main formula in (3.1). Here, for each $i = 1, 2, \ldots, \nu - 1, \nu + 1, \ldots, 2\nu - 1$, the coefficients $\beta_j^{(i,2\nu)}$'s of the ith additional formula are uniquely determined by imposing it to be of order 2ν, i.e. of the same order as that of the main method.
With reference to the discretization of $y'(x_0)$ we have used a formula analogous to the one considered in [6] for the 4-step method which is given by

$$-y_0 + y_1 - hy_0' = h^2 \sum_{j=0}^{2\nu-1} \beta_j^{(0,2\nu)} y_j'' \tag{5.3}$$

where again the coefficients are computed in order to keep the same order of the other formulae.

We have applied PGSCMs coupled with (5.1)–(5.3) for solving the initial value problem

$$y''(x) = \begin{pmatrix} \mu - 2 & 2\mu - 2 \\ 1 - \mu & 1 - 2\mu \end{pmatrix} y(x), \qquad y(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

whose exact solution is $y(x) = (2\cos(x), -\cos(x))^T$ independently of $\mu > 0$. When μ is large, this is a typical example of stiff problem for second order ODEs which is frequently used for testing the performance of P-stable schemes. The eigenvalues of the Jacobian matrix are in fact $-\mu$ and -1. With the chosen initial value, however, the continuous solution is smooth, i.e. it does not contain modes corresponding to the high frequency. In spite of this the application of methods with not appropriate stability properties determines a severe restriction on the choice of the stepsize in order to maintain the stiff mode under control.
We have solved the problem with $\mu = 2500$, known in the literature as Kramarz's system [15], over the interval $[0, 20\pi]$, by using the PGSCMs of orders 2, 4, 6, and 8 with stepsize $h = \pi/32$. In Figure 5.1, the obtained maximum error over each semi-period for the first component of the solution has been reported. The graphics corresponding to the second component are similar.

As one can see, the figure confirms that the property of P_ν-stability of PGSCMs allows to get good approximations of oscillatory solutions of IVPs for second order ODEs even when stiff modes are present. Moreover, it is clear that the accuracy of the approximations increases together with the order of the method.

REFERENCES

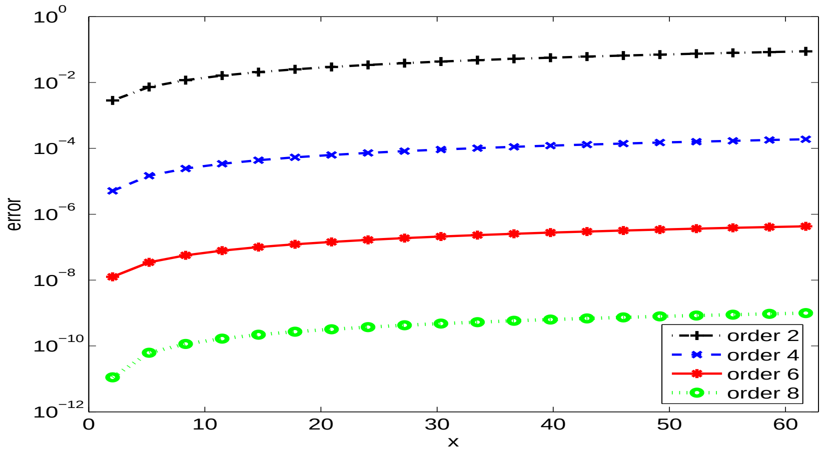

Fig. 5.1. *Error in the approximation of the first component of the solution of Kramarz's system*

[1] L. ACETO AND R. PANDOLFI, *Theoretical analysis of the stability for Extended Trapezoidal Rules*, Nonlinear Analysis, 71 (2009), pp. 2521-2534.

[2] L. ACETO, R. PANDOLFI, AND D. TRIGIANTE, *One parameter family of linear difference equations and the stability problem for the numerical solution of ODEs,* Adv. Difference Equ., (2006), Art. 19276, pp. 1-14.

[3] L. ACETO, R. PANDOLFI, AND D. TRIGIANTE, *Stability analysis of linear multistep methods via polynomial type variation,* J. Numer. Anal. Ind. Appl. Math., 2 (2007), pp. 1-9.

[4] L. ACETO AND D. TRIGIANTE, *The matrices of Pascal and other greats,* Amer. Math. Monthly, 108 (2001), pp. 232-245.

[5] L. ACETO AND D. TRIGIANTE, *On the A-stable methods in the GBDF class,* Nonlinear Anal. Real World Appl., 3 (2002), pp. 9-23.

[6] P. AMODIO AND F. IAVERNARO, *Symmetric boundary value methods for second order initial and boundary value problems*, Mediterr. J. Math., 3 (2006), pp. 383-398.

[7] L. BRUGNANO AND D. TRIGIANTE,*Convergence and stability of Boundary Value Methods for Ordinary Differential Equations*, J. Comp. Appl. Math., 66 (1996), pp. 97-109.

[8] L. BRUGNANO AND D. TRIGIANTE, *Solving ODEs by Linear Multistep Initial and Boundary Value Methods*, Gordon & Breach, Amsterdam, 1998.

[9] M.V. BULATOV AND G.VANDEN BERGHE, *Two-step fourth order methods for linear ODEs of the second order,* Numer. Algor., 51 (2009), pp. 449-460.

[10] M.M. CHAWLA, *Two-step fourth order P-stable methods for second order differential equations,* BIT, 21 (1981), pp. 190-193.

[11] M.M. CHAWLA AND B. NETA, *Families of two-step fourth order P-stable methods for second order differential equations,* J. Comp. Appl. Math., 15 (1986), pp. 213-223.

[12] J.P. COLEMAN AND L.GR. IXARU, *P-stability and exponential-fitting methods for $y'' = f(x, y)$*, IMA Journal of Numerical Analysis, 16 (1996), pp. 179-199.

[13] R.L. GRAHAM, D.E. KNUTH, AND O. PATASHNIK, *Concrete mathematics. A foundation for computer science,* Second edition. Addison-Wesley Publishing Company, Reading, MA, 1994.

[14] L.GR. IXARU AND B. PATERNOSTER, *A conditionally P-stable fourth-order exponential-fitting method for $y'' = f(x, y)$,* J. Comput. Appl. Math., 106 (1999), pp. 87-98.

[15] L. KRAMARZ, *Stability of collocation methods for the numerical solution of $y''(t) = f(t, y)$*, BIT, 20 (1980), pp. 215-222.

[16] J.D. LAMBERT AND I.A. WATSON, *Symmetric multistep methods for periodic initial value problems*, J. Inst. Math. Appl., 18 (1976), pp.189-202.

[17] F. MAZZIA, A. SESTINI, AND D. TRIGIANTE, *B-spline linear multistep methods and their continuous extensions,* SIAM J. Numer. Anal., 44 (2006), pp. 1954-1973.

[18] M. VAN DAELE, G. VANDEN BERGHE, H. DE MEYER, AND L.GR.IXARU, *Exponential-fitted four-step methods for $y'' = f(x, y)$,* Int. J. Comput. Math., 66 (1998), pp. 299-309.

[19] M. VAN DAELE AND G. VANDEN BERGHE, *P-stable Obrechkoff methods of arbitrary order for second-order differential equations*, Numer. Algor., 44 (2007), pp. 115-131.

[20] M. VAN DAELE AND G. VANDEN BERGHE, *P-stable exponentially-fitted Obrechkoff methods of arbitrary order for second-order differential equations,* Numer. Algor., 46 (2007), pp. 333-350.