Numerical solution of obstacle and parabolic obstacle problems based on Piecewise Linear Systems[1]

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Abstract. Piecewise Linear Systems (PLSs) are linear systems whose coefficient matrix is a piecewise constant function of the solution itself. Their general formulation has been introduced in [1] and their application to flows in porous media has already been studied in [2]. Here we consider another important application of such kind of systems, that is the numerical solution of obstacle and parabolic obstacle problems. The discrete formulation of such problems is expressed as a linear complementarity problem and it is then formulated as a specific PLS for the elliptic case and as a finite sequence of such systems for the parabolic case. A semi–iterative Newton–type method is proposed for the solution of the obtained PLSs and it is possible to prove that monotonic convergence in a finite number of steps is guaranteed. Some numerical results are presented to show the effectiveness of the proposed approach.

Keywords: Obstacle problems, Parabolic obstacle problems, Coincidence set, Piecewise linear systems, *M*–matrices, Picard iteration.
PACS: 65K10, 90C33, 90C53.

INTRODUCTION

Here we are interested in the numerical solution of obstacle and parabolic obstacle problems which have important applications for example in the elasticity theory [8]. In the literature, the classical approach used for this aim is based on a finite–element discretization combined with the use of projected relaxation methods [4] which however have a convergence rate heavily depending on the mesh refinement. In order to improve the efficiency, it has been later proposed the use of multigrid (e.g. see [5, 11]) and/or of active set (e.g. see [9, 10]) strategies. Two approaches recently introduced in the literature, respectively for the elliptic and the parabolic case, are presented in [9] and in [6], where the Lagrange multiplier strategy is used in order to express the problem as a higher dimension standard equality problem; in particular in [9] such strategy is combined with a semi-iterative procedure based on a suitable successive update of the *coincidence set* (that is the area where the solution touches the obstacle) while in [6] the solution of the parabolic variational inequality is obtained as the limit of the solutions of a family of appropriately regularized nonlinear parabolic equations.

In this paper, at the moment dealing with the linear case, we introduce a new approach for the numerical effective treatment of obstacle elliptic and parabolic problems which has a very compact formulation and does not require the use of Lagrange multipliers. The new method is obtained as an application of Piecewise Linear Systems to the obstacle problems. For the sake of simplicity and of clarity, it is here assumed to deal with hyper–rectangular domains because this allows us to use a standard finite difference discretization. However, the presented theory can be generalized to more complicated geometries and also to different discretization schemes. In fact the requirement for its safe application is that an *M*–matrix (a less restrictive condition on the matrix structure is also possible, see [3] for further details) characterize the discrete inequality modeling the given differential one. The method is at its initial formulation, for example mesh adaptation (see e.g. [7]) is an important non trivial aspect which has not yet been investigated.

[1] Work developed within the project "Numerical methods and software for differential equations".

APPLICATION OF PIECEWISE LINEAR SYSTEMS TO THE OBSTACLE PROBLEMS

The classical obstacle problem, often expressed as a variational inequality, can be also formulated as the following complementarity problem [8],

$$-\Delta u \geq f, \qquad u \geq \psi, \qquad (u-\psi)(\Delta u + f) = 0, \qquad \text{in } \Omega, \tag{1}$$

with suitable prescribed boundary conditions, where f and ψ are given functions, Ω is a domain in $\mathbb{R}^d, d \geq 1$, and ψ is the *obstacle* function. Assuming for simplicity a hyper–rectangular shape for Ω, a standard finite difference discretization of the Laplacian on a rectangular mesh can be used, which leads us to consider the following discrete complementarity problem,

$$T\,\mathbf{u} \geq \mathbf{f}, \qquad \mathbf{u} \geq \mathbf{p} \qquad (\mathbf{u}-\mathbf{p})^T\,(T\mathbf{u}-\mathbf{f}) = 0, \tag{2}$$

where $\mathbf{u}$ represents the unknown discrete solution and T is a square matrix. Observe that $\mathbf{u}, \mathbf{f}$ and $\mathbf{p}$ are vectors with a number n of entries equal to the number of inner mesh points if Dirichelet boundary conditions are considered (or, more generally, to the number of mesh points where the solution is unknown) and that the vector $\mathbf{f}$ depends on the function f and on the boundary conditions. The matrix T has a useful special structure because it turns out to be an M–matrix (i.e. it can be written as $\alpha I - B$ with $B \geq 0$ and $\rho(B) < \alpha$) if the solution is prescribed in at least one point on the boundary (conversely, it has a relaxed similar structure which anyway guarantees the robustness of the presented approach, see [3] for further details). Thus, using a suitable known vector $\mathbf{b} \in \mathbb{R}^n$, problem (2) can be transformed into the following standard complementarity problem,

$$T\,\mathbf{y} \geq \mathbf{b}, \qquad \mathbf{y} \geq \mathbf{0}, \qquad \mathbf{y}^T\,(T\mathbf{y}-\mathbf{b}) = 0, \tag{3}$$

whose solution $\mathbf{y}$ can be proved to be $\max\{\mathbf{0}\,,\mathbf{x}\}$, where $\mathbf{x} = (x_i)_{i=1}^n$ is the solution of the following PLS,

$$[I - P(\mathbf{x}) + TP(\mathbf{x})]\,\mathbf{x} = \mathbf{b}, \tag{4}$$

where $P(\mathbf{x}) = diag(p(x_1), \ldots, p(x_n))$, with $p(x)$ denoting the step function

$$p(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The following Picard iteration is used for iteratively solving system (4),

$$P^{(0)} = O, \qquad (I - P^{(k)} + TP^{(k)})\,\mathbf{x}^{k+1} = \mathbf{b}, \qquad P^{(k+1)} = P(\mathbf{x}^{k+1}), \qquad k = 0, 1, \ldots. \tag{5}$$

In fact, thanks to the structure of T, it is possible to prove that the sequence of matrices $P^{(k)}$ is not decreasing and that, if $P^{(k+1)} = P^{(k)}$, then $\mathbf{x}^{k+1} = \mathbf{x}$ [3]. Clearly, considering how $P(\mathbf{x})$ is defined, this implies that the iteration (5) converges in at most n steps.

With analogous considerations, the discrete formulation of the following parabolic obstacle problem,

$$u_t - \Delta u \geq f, \qquad u \geq \psi, \qquad (u-\psi)(u_t - \Delta u - f) = 0, \qquad \text{in } \Omega, \quad t > 0, \tag{6}$$

with suitable initial and boundary conditions can be reduced to consider, at each time step, a PLS in the form,

$$[I + TP(\mathbf{x})]\,\mathbf{x} = \mathbf{b}, \tag{7}$$

where $\mathbf{b}$ now changes at each time step and T is a matrix with the same structure obtained for the stationary problem. Using the proofs produced for the elliptic case, it can be easily deduced that even in this case the associated Picard iteration has a finite and monotonic convergence behavior.

TABLE 1. Example 1. Number of iterations (5) for various couples (C,N).

$N\backslash C$	25	50	75	100
-5	9	17	25	32
-10	5	10	13	16
-15	4	7	9	11
-20	1	5	7	9

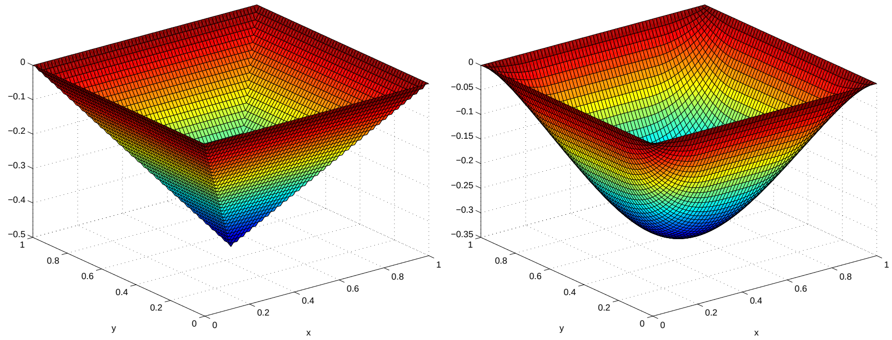

FIGURE 1. Example 1. On the left the obstacle function and on the right the corresponding numerical solution computed with $N = 127, C = -5$.

EXAMPLES

The torsion problem of an elastic–plastic isotropic and homogeneous cylindrical bar with square cross section $\Omega = (0\,,1)^2$ is here considered as a test example for the elliptic case. Its formulation as a complementarity problem is the following [8],

$$\begin{array}{lll} -\Delta u & \geq C, & u(x,y) \geq -\min\{x, 1-x, y, 1-y\} =: \psi(x,y), \qquad (x,y) \in \Omega = (0\,,1)^2, \\ (u-\psi)(\Delta u + C) & = 0, & u|_{\delta\Omega} = \psi|_{\delta\Omega}, \end{array} \tag{8}$$

where $u(x,y)$ is the unknown stress function and C is a negative constant depending on both the rigidity of the material and the angular rotation applied at the end cross sections of the cylinder. Observe that the obstacle ψ which is shown in the left plot in Figure 1 is actually the opposite of the distance of a domain point from the boundary. For the discretization, a uniform mesh is used in the reported experiments, with spatial steps $\Delta x = \Delta y = \frac{1}{N+1}$, which implies that the associated PLS has dimension $n = N^2$. In the right plot in Figure 1 the associated numerical solution when $N = 127$ and $C = -5$ is shown. In Table 1 the number of iterations (5) to get convergence is reported and it can be observed that it is slowly increasing with $N = \sqrt{n}$. In addition, the two left plots in Figure 2 show that, the higher is $|C|$, the larger is the extension of the associated active set. This behavior explains why the required number of iterations decreases when $|C|$ increases, as shown in Table 1. In fact, for $|C|$ increasing, the initial approximation used for u becomes progressively more suited because in our implementation it is chosen equal to ψ. The inner right plot in Figure 2 shows the nicely shaped coincidence set of the PLS numerical solution of a variant of Problem (8), where the case of a non constant right–hand side f in the differential inequality is considered. The domain, the obstacle function and the boundary condition are unchanged and f is defined as follows (see Example 5.2 in [9]),

$$f(x,y) = -45(x - x^2)\,[1 + \sin(11\pi x)].$$

We observe that even in this case the required number of Picard iterations is slowly depending on $N = \sqrt{n}$ (for $N = 25$ and $N = 100$ it is respectively equal to 7 and to 26.)

As a test example for the parabolic case we consider now the deformation of a thin homogeneous membrane loaded by a normal uniformly distributed force f, constrained to lie above a body (represented by the obstacle function) where it is initially positioned and fixed to the body on the boundary of the domain [8],

$$\begin{array}{llll} u_t - \Delta u & \geq f(x,y), & u \geq \psi(x,y), & (x,y) \in \Omega, \quad 0 < t \leq T_{max}, \\ (u-\psi)(u_t - \Delta u - f) & = 0, & u|_{\delta\Omega} = \psi|_{\delta\Omega}, & \forall t \leq T_{max}, \qquad u(0,x,y) = \psi(x,y) \quad \forall (x,y) \in \Omega, \end{array} \tag{9}$$

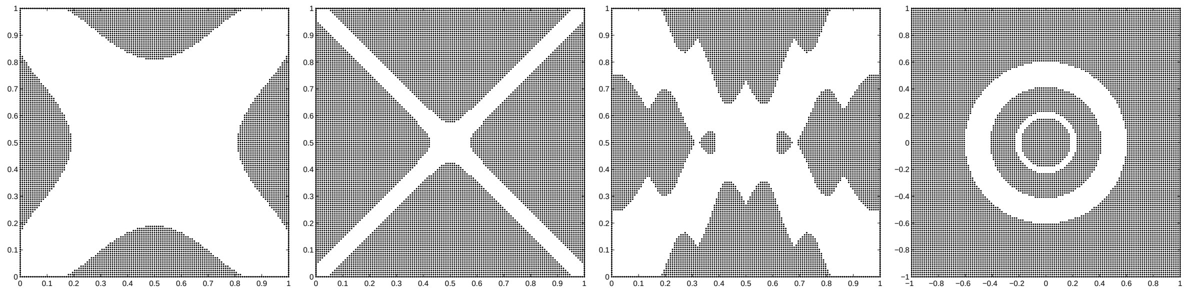

FIGURE 2. The coincidence sets (black area) related to the numerical solutions of Example 1 with $C = -5$ (outer–left) and $C = -20$ (inner–left), of its variant case (inner–right) and of the numerical solution at time $t = T_{max}$ of Example 2 (outer–right). Uniform discretizations used, with $N = 127$ and $\Delta t = 20$ for the parabolic case.

where in particular, as in the example considered in Section 5.2 in [7], we assume $\Omega = (-1\,,\,1)^2$, $T_{max} = 0.1$, $f(x,y) \equiv -4$ and the obstacle function is chosen equal to the following radial symmetric function:

$$\psi(x,y) = \max\{0, -0.1 + 0.6 * \exp(-10 * r^2), 0.5 - r\}, \qquad \text{with} \quad r = \sqrt{x^2 + y^2}.$$

As for the previous examples, we report here some results obtained by using a uniform grid with $\Delta x = \Delta y = \frac{2}{N+1}$ and $\Delta t = \frac{T_{max}}{Nt}$. In particular, the coincidence set of the numerical solution at the final time T_{max} obtained with $N = 127$ and $Nt = 20$ is depicted in the outer right plot in Figure 2. The number of Picard iterations required at each time step ranges from 9 to 11.

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