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APPLIED NUMERICAL MATHEMATICS

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Recent advances in linear analysis of convergence for splittings for solving ODE problems ☆

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Abstract

In the nineties, Van der Houwen et al. (see, e.g., [P.J. van der Houwen, B.P. Sommeijer, J.J. de Swart, Parallel predictor–corrector methods, J. Comput. Appl. Math. 66 (1996) 53–71; P.J. van der Houwen, J.J.B. de Swart, Triangularly implicit iteration methods for ODE-IVP solvers, SIAM J. Sci. Comput. 18 (1997) 41–55; P.J. van der Houwen, J.J.B. de Swart, Parallel linear system solvers for Runge–Kutta methods, Adv. Comput. Math. 7 (1–2) (1997) 157–181]) introduced a linear analysis of convergence for studying the properties of the iterative solution of the discrete problems generated by implicit methods for ODEs. This linear convergence analysis is here recalled and completed, in order to provide a useful quantitative tool for the analysis of splittings for solving such discrete problems. Indeed, this tool, in its complete form, has been actively used when developing the computational codes `BiM` and `BiMD` [L. Brugnano, C. Magherini, The BiM code for the numerical solution of ODEs, J. Comput. Appl. Math. 164–165 (2004) 145–158. Code available at: http://www.math.unifi.it/~brugnano/BiM/index.html; L. Brugnano, C. Magherini, F. Mugnai, Blended implicit methods for the numerical solution of DAE problems, J. Comput. Appl. Math. 189 (2006) 34–50]. Moreover, the framework is extended for the case of special second order problems. Examples of application, aimed to compare different iterative procedures, are also presented.
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PACS: 65L05; 65L06; 65L80; 65H10

Keywords: Ordinary differential equations; Initial value problems; Stiff problems; Second order problems; Implicit numerical methods; Iterative solution of algebraic systems; Blended implicit methods

1. Introduction

The study of numerical methods for ordinary differential equation has been, and continues to be, a very active field of investigation. Across the years, many topics in this field have been the subject of systematic studies and usually each topic has greatly benefited from the use of specific tools of analysis. As an example, the linear stability analysis introduced by Dahlquist [13], and its successive generalizations, has been the main tool for both studying and devising most of the currently available numerical methods. The subject that we shall study here concerns the

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doi:10.1016/j.apnum.2008.03.008

Please cite this article in press as: L. Brugnano, C. Magherini, Recent advances in linear analysis of convergence for splittings for solving ODE problems, Applied Numerical Mathematics (2008), doi:10.1016/j.apnum.2008.03.008

efficient implementation of implicit methods, which are usually recommended, when solving stiff initial value ODE problems. Their use, in turn, requires the solution of a (generally nonlinear) system of equations at each integration step: its efficient solution greatly affects the performance of computational codes designed for solving stiff ODE-IVPs. In particular, in this paper we shall refer to the discrete problems generated by *block implicit methods* [29], namely methods that, when used for solving the problem

$$y' = f(t, y), \quad t \in [t_0, T], \qquad y(t_0) = y_0 \in \mathbb{R}^m, \tag{1}$$

generate, at the nth integration step, a discrete problem in the form

$$F(\mathbf{y}_n) \equiv A \otimes I_m \, \mathbf{y}_n - h_n B \otimes I_m \, \mathbf{f}_n - \boldsymbol{\eta}_n = \mathbf{0}. \tag{2}$$

Here, the matrices $A, B \in \mathbb{R}^{r\times r}$ (which we assume nonsingular hereafter) define the method, h_n is the current stepsize, the block vectors

$$\begin{aligned} &\mathbf{y}_n = (y_{n1}, \dots, y_{nr})^T, \qquad \mathbf{f}_n = (f_{n1}, \dots, f_{nr})^T, \\ &f_{nj} = f(t_{nj}, y_{nj}), \quad t_{nj} = t_n + c_j h_n, \quad j = 1, \dots, r, \end{aligned} \tag{3}$$

contain the discrete solution and, finally, $\boldsymbol{\eta}_n$ only depends on already known quantities. This is a rather general framework, including the majority of implicit Runge–Kutta methods, for which $A = I$, the identity matrix, a number of General Linear Methods (see, for example [11,17,18]), and, as a more recent instance, block Boundary Value Methods [7]. In the following, for the sake of simplicity, we shall omit the index n of the integration step, since we shall always refer to a generic step of integration.

The most straightforward approach for solving (2) is the use of the simplified Newton iteration, which amounts to carry out the iteration

$$\begin{aligned} &(A \otimes I_m - hB \otimes J)\boldsymbol{\delta}^{(i)} = F(\mathbf{y}^{(i)}), \\ &\mathbf{y}^{(i+1)} = \mathbf{y}^{(i)} - \boldsymbol{\delta}^{(i)}, \quad i = 0, 1, \dots, \end{aligned} \tag{4}$$

where J is the "frozen" Jacobian of $f(t, y)$, evaluated at the last known point. This procedure usually converges fastly; however, the involved linear systems require the factorization of a matrix of dimension $rm \times rm$ which, therefore, may be very costly. The first successful attempt to get rid of this fact is the use of the Butcher procedure [10] which, in this setting, essentially amounts to have the two matrices A and B diagonalized by the same similarity transformation. In such a case, after a variable transformation, one has only to solve $m \times m$ (possibly complex) linear systems, thus reducing the overall computational cost. This strategy has been implemented, for example, in the codes RADAU and RADAU5 [18,23]. Nevertheless, this procedure may still be quite costly, when the matrix pencil $A - sB$ has many distinct eigenvalues. Even though additional different approaches have been pursued across the years (see, e.g., [9,12,22]), another general way for iteratively solving (2) relies on the definition of a suitable nonlinear *splitting*; i.e., having chosen two (suitably simple) matrices A^* and B^*, one then solves the iteration

$$A^* \otimes I_m \mathbf{y}^{(i+1)} - hB^* \otimes I_m \mathbf{f}^{(i+1)} = (A^* - A) \otimes I_m \mathbf{y}^{(i)} - h(B^* - B) \otimes I_m \mathbf{f}^{(i)} + \boldsymbol{\eta}, \quad i = 0, 1, \dots, \tag{5}$$

with an obvious meaning of the involved vectors. Depending on the structure of the matrices A^* and B^* (which we shall always assume to be nonsingular), the cost for carrying out the iteration may be much lower than that required for the simplified Newton iteration: for example, if A^* and B^* are both lower triangular with constant diagonal entries, only the factorization of an $m \times m$ real matrix is required. This procedure may be very efficient and has been implemented in the codes GAM [19] and GAMD [23].

This latter procedure is exactly the one we are interested in. In fact, it is not an easy task, in general, to define "efficient" splitting matrices A^* and B^*: as a matter of fact, usually the simpler the matrices, the less efficient the iteration (5), due to its poor convergence properties. With the aim of providing a tool of analysis for studying the properties of this iteration, van der Houwen et al. [28,24,25] proposed a linear analysis of convergence. This analysis studies the convergence properties of the iterative procedure when problem (1) is the test equation,

$$y' = \mu y, \quad \Re(\mu) < 0, \tag{6}$$

usually considered to carry out the linear stability analysis of the methods. This idea is quite interesting because of many reasons:

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- the resulting analysis is relatively simple;
- it straightforwardly extends to autonomous linear systems of equations;
- by considering that one can also use the splitting for solving the linearized equation in (4),

$$(A^* \otimes I_m - hB^* \otimes J)\boldsymbol{\delta}^{(i,j+1)} = \big((A^* - A) \otimes I_m - h(B^* - B) \otimes J\big)\boldsymbol{\delta}^{(i,j)} + F\big(\mathbf{y}^{(i)}\big), \quad j = 0, 1, \ldots,$$

  it provides a first order tool of analysis in the case of nonlinear and/or non-autonomous systems;
- it allows to compare possible stepsize limitations due to stability reasons with those due to the convergence of the iterative procedure.

Even though this analysis has been used in various settings (see, e.g., [1–6,8,15,26,27,19,21]), nevertheless, to the best of our knowledge, it is not explicitly presented in a corresponding paper. For this reason, we here recall the basic facts concerning it and, in addition, we provide additional useful extensions which allow to better quantify the convergence properties of a given splitting procedure. With this premise, the paper is then organized as follows: in Section 2 we provide the basic linear convergence analysis (mostly, a reformulation of the material in [24,25], along with a suitable set of new results and related concepts); in Section 3 the extensions of the theory for handling a modified form of the discrete problem and the case of special second order problems are presented; at last, Section 4 contains a practical application of the analysis, aimed to compare some general procedures for getting efficient splittings for relevant classes of methods.

2. Linear convergence analysis

When applying the block method (2)–(3) to the test equation (6) the discrete problem reduces to a linear system of dimension r:

$$(A - qB)\mathbf{y} = \boldsymbol{\eta}, \quad q = h\mu. \tag{7}$$

Similarly, the splitting procedure (5) reduces to a linear iteration,

$$(A^* - qB^*)\mathbf{y}^{(i+1)} = \big((A^* - A) - q(B^* - B)\big)\mathbf{y}^{(i)} + \boldsymbol{\eta}, \quad i = 0, 1, \ldots. \tag{8}$$

Moreover, let

$$\mathbf{e}^{(i)} = \mathbf{y} - \mathbf{y}^{(i)}$$

be the convergence error at the ith iteration. Consequently, from (7)–(8) one obtains that the error equation for the iteration (8) is given by

$$(A^* - qB^*)\mathbf{e}^{(i+1)} = \big((A^* - A) - q(B^* - B)\big)\mathbf{e}^{(i)}, \quad i = 0, 1, \ldots.$$

Therefore, the iteration will converge if and only if the spectral radius, say $\rho(q)$, of the iteration matrix

$$Z(q) = (A^* - qB^*)^{-1}\big((A^* - A) - q(B^* - B)\big), \tag{9}$$

is less than 1. Moreover, the set

$$\Gamma = \big\{q \in \mathbb{C}\colon\ \rho(q) < 1\big\}$$

is the *region of convergence* of the iteration (8). Clearly, the larger Γ the better the properties of the iteration. Indeed, ideally it should contain the absolute stability region of the method, in order to avoid stepsize restrictions depending only on the convergence of the iteration (8). However, this is not enough to characterize an efficient iteration: additional evaluation parameters for this purpose will be considered in the sequel. A minimal requirement for the iteration is the following one.

Definition 1. The iteration (8) is said to be 0-*convergent* if $\rho(0) = 0$.

It is evident that the property of 0-convergence is a minimal requirement for the iteration, since it means that the procedure converges quickly, if a suitably small stepsize is used. Moreover, it implies that the matrix

$$Z(0) \equiv I - (A^*)^{-1}A \tag{10}$$

is nilpotent. Concerning this point, hereafter we shall make the following assumption.

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Assumption 1. $A^* = A$.

Consequently, one has that $Z(0) \equiv O$, the zero matrix, so that a constant solution is found in at most one iteration. This assumption is not restrictive, in practice, since it is evident that (2) can be cast in many equivalent forms, due to the fact that both A and B are assumed to be nonsingular. As a consequence, we can always assume that A is a suitably "simple structured" matrix and, therefore, the previous assumption can be easily fulfilled. Moreover, in this case one easily realizes that the iteration (8) is well defined in a neighborhood of the origin, and

$$Z(q) = qZ'(0) + \mathrm{O}\big(q^2\big), \quad q \approx 0,\ Z'(0) = A^{-1}(B - B^*). \tag{11}$$

This naturally leads to the following definition.

Definition 2. The *nonstiff amplification factor* of the iteration (8) is given by $\tilde{\rho} \equiv \rho(Z'(0))$, where, as usual, $\rho(Z'(0))$ denotes the spectral radius of that matrix.

Consequently, one obtains that

$$\rho(q) \approx \tilde{\rho}|q|, \quad q \approx 0. \tag{12}$$

Remark 3. Obviously, the smaller $\tilde{\rho}$, the better the convergence properties of the iteration (8) when $q \approx 0$. Therefore, the nonstiff amplification factor $\tilde{\rho}$ is an evaluation parameter for the iteration.

The previous arguments apply to the properties of the iteration when q is small. Anyway, when dealing with stiff problems, we are interested in having the convergence region Γ as large as possible. For this reason, the following definitions are given.

Definition 4. The iteration (8) is said to be A-*convergent* if $\mathbb{C}^- \subseteq \Gamma$. If, in addition,

$$\rho(q) \to 0, \quad \text{as } q \to \infty, \tag{13}$$

then the iteration is said to be L-*convergent*.

Remark 5. Evidently, an A-convergent iteration is highly desirable, if the underlying method is A-stable. Conversely, one could have stepsize restrictions not due to stability reasons but to the convergence of the iterative procedure. Similarly, an L-convergent iteration is highly desirable, when the underlying method is L-stable since, in this way, the damped stiff components don't affect at all the convergence of the iteration (8).

In a similar way, we can define the following (weaker) notions, which are appropriate for $A(\alpha)$-stable (respectively, $L(\alpha)$-stable) methods.

Definition 6. The iteration (8) is said to be $A(\alpha)$-*convergent*, for a given $\alpha \in [0, \frac{\pi}{2}]$, if

$$\mathbb{C}^-(\alpha) \equiv \big\{q \in \mathbb{C}\colon \big|\arg(q) - \pi\big| \leqslant \alpha\big\} \subseteq \overline{\Gamma}.$$

If, in addition, (13) holds true, then the iteration is said to be $L(\alpha)$-*convergent*.

Let us now consider the problem of defining efficient criteria for:

- characterizing an A-convergent/L-convergent iteration;
- ranking an A-convergent/L-convergent iteration.

Concerning the first issue, we shall make, hereafter, the following additional assumption.

Assumption 2. All the eigenvalues of the matrix $A^{-1}B^*$ have positive real part.

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Indeed, by taking into account both Assumptions 1 and 2, it then follows that the iteration (8) is well defined for all $q \in \mathbb{C}^-$; conversely, the iteration cannot be A-convergent. As a consequence, we can give the following definition.

Definition 7. By setting, as usual, i the imaginary unit, the *maximum amplification factor* of the iteration (8) is defined as

$$\rho^* = \sup_{x\in\mathbb{R}} \rho(\mathrm{i}x). \tag{14}$$

From the maximum-modulus principle, the following simple characterization of A-convergence easily follows.

Theorem 8. *If the iteration* (8) *is* 0*-convergent, then it is A-convergent if and only if* $\rho^* \leqslant 1$.

Let now consider the additional characterization for L-convergence. The following result then holds true.

Theorem 9. *If the iteration* (8) *is A-convergent, then L-convergence is equivalent to require that the matrix*

$$Z_\infty \equiv \lim_{q\to\infty} Z(q) = I - (B^*)^{-1}B \tag{15}$$

is nilpotent.

Proof. Since the iteration is A-convergent, we have only to prove that (13) holds true. From (9) and (15), and by taking into account Assumption 1, we obtain that, for $|q| \gg 1$,

$$\begin{aligned} Z(q) &= q(A - qB^*)^{-1}(B - B^*) = \sum_{n\geqslant 0} q^{-n}\big((B^*)^{-1}A\big)^n Z_\infty \\ &= Z_\infty + q^{-1}(B^*)^{-1}AZ_\infty + \mathrm{O}\big(q^{-2}\big) \equiv Z_\infty + q^{-1}Z'_\infty + \mathrm{O}\big(q^{-2}\big). \end{aligned} \tag{16}$$

Therefore, the iteration will be L-convergent if and only if the matrix Z_∞ is nilpotent. □

Because of the above arguments, we state the following definitions.

Definition 10. The *stiff amplification factor* of the iteration (8) is defined (see (15)) as $\rho_\infty = \rho(Z_\infty)$.

Definition 11. We say that an L-convergent iteration has *index* ν_∞, if ν_∞ is the index of nilpotency of the corresponding matrix Z_∞.

Clearly, the smaller ν_∞, the better the convergence properties of the iteration. However, the following straightforward result provides a lower bound for the index of an L-convergent iteration.

Theorem 12. *Under Assumption* 1*, and assuming $B^* \neq B$* (*otherwise, the splitting would coincide with the original problem*)*, an L-convergent iteration has always index $\nu_\infty \geqslant 2$* (*i.e., $Z_\infty \neq O$*).

For an L-convergent iteration, we now want to study how the spectral radius $\rho(q)$ decays to zero as $q \to \infty$. For this purpose, we need the following preliminary result.

Lemma 13. *If $|q| \gg 1$ then*

$$Z(q)^\ell = \sum_{n\geqslant 0} q^{-n} T^{(\ell,n)}, \quad \ell \geqslant 1, \tag{17}$$

where, for each $n \geqslant 0$,

$$T^{(\ell,n)} = \sum_{(s_0,\ldots,s_n)\in S^{(\ell,n)}} Z_\infty^{s_0}\left(\prod_{j=1}^{n} F Z_\infty^{s_j}\right), \qquad F = (B^*)^{-1}A, \tag{18}$$

being

$$S^{(\ell,n)} = \left\{ (s_0,\dots,s_n) \in \mathbb{N}^{n+1}\colon \sum_{i=0}^{n} s_i = \ell,\ s_n \geqslant 1 \right\}, \tag{19}$$

and assuming, as usual, $\prod_{j=1}^{0}(\cdot) = I$*, the identity matrix.*

Proof. For each $n \geqslant 0$, the thesis can be proved by induction on ℓ. If $\ell = 1$ then $S^{(1,n)} = \{(0,\dots,0,1) \in \mathbb{N}^{n+1}\}$ and, consequently, (18) reduces to $T^{(1,n)} = F^n Z_\infty$, which follows immediately from (15)–(16). Let us now assume that the result holds true for ℓ and prove it for $\ell+1$. From (17), by considering that $Z(q)^{\ell+1} = Z(q)Z(q)^{\ell}$, one obtains

$$T^{(\ell+1,n)} = \sum_{k=0}^{n} T^{(1,k)} T^{(\ell,n-k)} = \sum_{k=0}^{n} F^k Z_\infty T^{(\ell,n-k)}. \tag{20}$$

Moreover, by the induction hypothesis, for each $k = 0,\dots,n$,

$$\begin{aligned} F^k Z_\infty T^{(\ell,n-k)} &= \left(Z_\infty^0 F\right)^k Z_\infty \sum_{(s_0,\dots,s_{n-k})\in S^{(\ell,n-k)}} Z_\infty^{s_0}\left(\prod_{j=1}^{n-k} F Z_\infty^{s_j}\right) \\ &= \sum_{(s_0,\dots,s_{n-k})\in S^{(\ell,n-k)}} \left(Z_\infty^0 F\right)^k Z_\infty^{s_0+1}\left(\prod_{j=1}^{n-k} F Z_\infty^{s_j}\right) \\ &= \sum_{(s_0,\dots,s_n)\in S^{(\ell+1,n,k)}} Z_\infty^{s_0}\left(\prod_{j=1}^{n} F Z_\infty^{s_j}\right), \end{aligned}$$

where $S^{(\ell+1,n,k)} = \{(s_0,\dots,s_n) \in S^{(\ell+1,n)}\colon\ s_0 = \dots = s_{k-1} = 0,\ s_k \geqslant 1\}$. Therefore, see (18)–(20), the thesis follows by considering that

$$k_1 \neq k_2 \quad\Longrightarrow\quad S^{(\ell+1,n,k_1)} \cap S^{(\ell+1,n,k_2)} = \emptyset \quad\text{and}\quad S^{(\ell+1,n)} = \bigcup_{k=0}^{n} S^{(\ell+1,n,k)}. \qquad \square$$

The following result then holds true.

Theorem 14. *If* $T^{(\nu_\infty-1,1)}$ *is not nilpotent then*

$$\rho(q)^{\nu_\infty-1} \approx |q|^{-1}\rho\left(T^{(\nu_\infty-1,1)}\right), \quad |q| \gg 1. \tag{21}$$

Proof. From (17) we get, for $|q| \gg 1$,

$$Z(q)^{\ell} = Z_\infty^{\ell} + q^{-1}T^{(\ell,1)} + \mathrm{O}\left(|q|^{-2}\right) \equiv W^{(\ell)}(q) + \mathrm{O}\left(|q|^{-2}\right), \tag{22}$$

where, according to (18),

$$T^{(\ell,1)} = \sum_{s=0}^{\ell-1} Z_\infty^{s} F Z_\infty^{\ell-s}. \tag{23}$$

Let now consider the Jordan canonical form of Z_∞,

$$Z_\infty = VJV^{-1}, \quad J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}, \tag{24}$$

where

$$J_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}_{\nu_i\times\nu_i}, \quad i = 1,\dots,k, \quad\text{and}\quad \nu_\infty = \max_{i=1,\dots,k} \nu_i. \tag{25}$$

From (22)–(23) it follows then that

$$T^{(\ell,1)} \sim \sum_{s=0}^{\ell-1} J^s X J^{\ell-s} \equiv \hat{T}^{(\ell,1)}, \quad X = V^{-1}FV, \tag{26}$$

and, consequently,

$$W^{(\ell)}(q) \sim J^\ell + q^{-1}\hat{T}^{(\ell,1)} \equiv \hat{W}^{(\ell)}(q). \tag{27}$$

Let us first prove the result in the simpler case where $\nu_\infty = r$. In this case, the matrix $\hat{T}^{(r-1,1)}$, see (26), is upper triangular. In fact, for each $s = 0, \ldots, r-2$,

$$1 \leqslant j < i \leqslant r \quad \Rightarrow \quad \mathbf{e}_i^T J^s X J^{r-s-1}\mathbf{e}_j = 0,$$

being $\mathbf{e}_i$ and $\mathbf{e}_j$ the ith and jth unit vector in $\mathbb{R}^r$, respectively, since:

- $i > r - s \Rightarrow \mathbf{e}_i^T J^s = \mathbf{0}^T$;
- $i \leqslant r - s \Rightarrow j < r - s \Rightarrow J^{r-s-1}\mathbf{e}_j = \mathbf{0}$.

Therefore, by considering that $J^{r-1} = \mathbf{e}_r\mathbf{e}_r^T$, from (26)–(27) it follows that the spectrum of $\hat{W}^{(r-1)}(q)$ coincides with that of $q^{-1}\,\hat{T}^{(r-1,1)}$ which, in turn, coincides with that of $q^{-1}\,T^{(r-1,1)}$. Consequently,

$$\rho(q)^{r-1} \approx \rho\big(\hat{W}^{(r-1)}\big) = |q|^{-1}\rho\big(T^{(r-1,1)}\big), \quad |q| \gg 1, \tag{28}$$

which proves the thesis when $\nu_\infty = r$.

If $\nu_\infty < r$ then the canonical form (24)–(25) contains more than one Jordan block. Then, the matrices X and $\hat{T}^{(\nu_\infty-1,1)}$ (see (26)) can be decomposed as

$$X = \begin{pmatrix} X_{11} & \ldots & X_{1k} \\ \vdots & & \vdots \\ X_{k1} & \ldots & X_{kk} \end{pmatrix}, \qquad \hat{T}^{(\nu_\infty-1,1)} = \begin{pmatrix} \hat{T}_{11}^{(\nu_\infty-1,1)} & \ldots & \hat{T}_{1k}^{(\nu_\infty-1,1)} \\ \vdots & & \vdots \\ \hat{T}_{k1}^{(\nu_\infty-1,1)} & \ldots & \hat{T}_{kk}^{(\nu_\infty-1,1)} \end{pmatrix},$$

$$X_{ij} \in \mathbb{R}^{\nu_i \times \nu_j}, \quad \hat{T}_{ij}^{(\nu_\infty-1,1)} = \sum_{s=0}^{\nu_\infty-2} J_i^s X_{ij} J_j^{\nu_\infty-s-1}, \quad i,j = 1,\ldots,k.$$

By using the same arguments used in the case $\nu_\infty = r$, it is possible to prove that the diagonal blocks $\hat{T}_{ii}^{(\nu_\infty-1,1)}$, $i = 1,\ldots,k$, are upper triangular. Moreover: the first columns of the off-diagonal blocks $\hat{T}_{ij}^{(\nu_\infty-1,1)}$, $i \neq j$, have all zero entries; from (24)–(25) and (27) it follows that the entries of $J_i^{\nu_\infty-1}$ at most affect the entry $(1, \nu_i)$ of the diagonal block $q^{-1}\hat{T}_{ii}^{(\nu_\infty-1,1)}$, $i = 1,\ldots,k$. Therefore, when computing the characteristic polynomial of $\hat{W}^{(\nu_\infty-1)}$, one obtains:

$$\begin{aligned}\det\big(\hat{W}^{(\nu_\infty-1)} - \lambda I\big) &\equiv \det\big(J^{\nu_\infty-1} + q^{-1}\hat{T}^{(\nu_\infty-1,1)} - \lambda I\big) \\ &= \det\big(q^{-1}\hat{T}^{(\nu_\infty-1,1)} - \lambda I\big).\end{aligned}$$

Indeed, by developing the second determinant on the first column, one eliminates the contribution of $J_1^{\nu_\infty-1}$. Then, to compute the determinant of the submatrix made up of rows and columns from 2 to r, one develops it on its ν_1th column, thus eliminating the contribution of $J_2^{\nu_\infty-1}$, and so forth. Therefore, the matrices $\hat{W}^{(\nu_\infty-1)}(q)$ and $q^{-1}\,\hat{T}^{(\nu_\infty-1,1)}$ do have the same characteristic polynomial and, consequently, the same spectral radius. In view of (28), with r replaced by ν_∞, this proves the statement of the theorem. □

Definition 15. If $T^{(\nu_\infty-1,1)}$ is not nilpotent, the *stiff convergence factor* of the iteration (8) is defined as

$$\tilde{\rho}_\infty = \rho\big(T^{(\nu_\infty-1,1)}\big)^{1/(\nu_\infty-1)}, \quad \nu_\infty \geqslant 2. \tag{29}$$

From (21) it then follows that,

$$\rho(q) \approx \tilde{\rho}_\infty\,|q|^{-1/(\nu_\infty-1)}, \quad |q| \gg 1. \tag{30}$$

Consequently, the larger ν_∞ the slower $\rho(q)$ converges towards zero as $q \to \infty$. In turn, this implies that, when comparing the convergence properties of different L-convergent iterations, one has first to look at the values of ν_∞ and, only when they are equal, one compares the corresponding stiff convergence factors.

Remark 16. By using the above results, we are able to easily rank the convergence properties of A-convergent iterations, as well as L-convergent ones. In fact:

- for an A-convergent iteration, the parameters ρ^*, $\tilde{\rho}$, and ρ_∞ completely characterize the convergence properties of the iteration, usually in increasing order of importance;
- for an L-convergent iteration, the reference parameters become, in (usually) increasing order of relevance, ρ^*, $\tilde{\rho}$, ν_∞, and $\tilde{\rho}_\infty$.

Remark 17. A possible way to improve on the (partially negative) result of Theorem 12, would be that of using a splitting for which Assumption 1 does not hold: in fact, in this case one could set $B^* = B$, thus obtaining $Z_\infty = O$. Nevertheless, this would be made at the "price" of having, for a 0-convergent iteration, a nilpotent matrix $Z(0)$ (see (10)) with index $\nu > 1$. This implies that, for a constant solution, convergence would be generally obtained in more than 1 iteration. Moreover, by using arguments similar to those used in the proof of Theorem 14, one would have $\rho(q) = \mathrm{O}(|q|^{1/(\nu-1)})$, when $q \approx 0$, instead of (12).

2.1. Averaged amplification factors

The previous amplification factors measure the asymptotic speed of convergence when an infinite number of iterations are performed. It is also customary to define corresponding "averaged" amplification factors, which measure the "average" convergence when a prescribed number of iterations is performed [28,24,25]. In particular, by considering a suitable matrix norm $\|\cdot\|$, and with reference to what previously has been set out, we define the following *averaged amplification factors* when ℓ ($\ell \geqslant 1$) iterations of (8) are carried out:

$$\rho^{*(\ell)} = \sup_{x\in\mathbb{R}} \sqrt[\ell]{\big\|\big(Z(\mathrm{i}x)\big)^\ell\big\|}, \quad \tilde{\rho}^{(\ell)} = \sqrt[\ell]{\big\|\big(Z'(0)\big)^\ell\big\|}, \quad {\rho_\infty}^{(\ell)} = \sqrt[\ell]{\big\|(Z_\infty)^\ell\big\|}.$$

Concerning the averaged convergence factor corresponding to $\tilde{\rho}_\infty$, we have the following result.

Theorem 18. *For an L-convergent iteration of index $\nu_\infty \geqslant 2$, one has that, for $|q| \gg 1$:*

$$\rho^{(\ell)}(q) \equiv \sqrt[\ell]{\big\|Z(q)^\ell\big\|} = \mathrm{O}\big(|q|^{-n_\ell/\ell}\big), \qquad n_\ell = \left\lceil \frac{\ell}{\nu_\infty - 1} - 1 \right\rceil, \tag{31}$$

being $\lceil\cdot\rceil$ the ceil function.

Proof. In order for (31) to be satisfied, from (17)–(19), it follows that one must have

$$T^{(\ell,n)} \neq O \quad \Longrightarrow \quad n \geqslant n_\ell,$$

that is, by considering that $n \in \mathbb{N}$,

$$n \geqslant \frac{\ell}{\nu_\infty - 1} - 1. \tag{32}$$

Indeed, see (18)–(19), $T^{(\ell,n)} \neq O$ implies that there exists $(s_0, \ldots, s_n) \in S^{(\ell,n)}$ such that

$$\ell = \sum_{i=0}^{n} s_i \quad \text{and} \quad \max_{i=0,\ldots,n} s_i \leqslant \nu_\infty - 1.$$

Consequently, $\ell \leqslant (n+1)(\nu_\infty - 1)$, i.e. (32). □

Remark 19. We observe that Theorem 18 generalizes the specific result of Theorem 3.4 in [25], which holds true for the splitting defined in that paper.

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Because of (31) and (17)–(18), if $T^{(\ell,n_\ell)} \neq O$, we define the *averaged stiff convergence factor* for the iteration (8) as

$$\tilde{\rho}_\infty^{(\ell)} \equiv \lim_{q\to\infty} |q|^{n_\ell/\ell}\rho^{(\ell)}(q) = \sqrt[\ell]{\left\|T^{(\ell,n_\ell)}\right\|}.$$

Clearly, as $\ell \to \infty$, the averaged amplification factors converge towards the corresponding amplification factors. Moreover, for an L-convergent iteration one has $\rho_\infty^{(\ell)} = 0$, for $\ell \geqslant \nu_\infty$.

3. Extensions

In this section we consider quite straightforward extensions of the previous linear convergence analysis, which are aimed to handle the case in which:

- the matrices A and B in (7), defining the discrete problem, do depend on the parameter q;
- the discrete problem is generated by the numerical solution of special second order problems.

3.1. Blended implicit methods

An interesting extension of the above analysis concerns the case of *blended implicit methods* [2–6,8]. For such methods, the discrete problem (7) assumes the form

$$\big(A(q) - qB(q)\big)\mathbf{y} = \boldsymbol{\eta}(q), \tag{33}$$

where

$$A(0) = A^*, \qquad \lim_{q\to\infty} B(q) = B^*. \tag{34}$$

In such a case, the *blended iteration*

$$(A^* - qB^*)\mathbf{y}^{(i+1)} = \big(\big(A^* - A(q)\big) - q\big(B^* - B(q)\big)\big)\mathbf{y}^{(i)} + \boldsymbol{\eta}(q), \quad i = 0, 1, \ldots, \tag{35}$$

is naturally induced. The iteration (35) will be convergent if and only if the spectral radius of the corresponding iteration matrix,

$$Z(q) = I - (A^* - qB^*)^{-1}\big(A(q) - qB(q)\big), \tag{36}$$

is less than 1. All the arguments seen in Section 2 are then generalized by taking into account (36). In particular, a remarkable property of the blended iteration (35) is that, because of (34), one obtains that

$$Z_\infty \equiv \lim_{q\to\infty} Z(q) = O \quad \Longrightarrow \quad \rho_\infty = 0, \qquad \nu_\infty = 1, \tag{37}$$

whatever is the considered value of the blocksize r. This allows to nicely improve on the negative result, given by Theorem 12, which holds for usual splittings. In addition, if we assume $A(q)$ and $B(q)$ analytical in $\mathbb{C}^-$, one can prove that, for a suitable matrix Z'_∞,

$$Z(q) = q^{-1}Z'_\infty + \mathrm{O}\big(|q|^{-2}\big), \quad |q| \gg 1, \tag{38}$$

thus leading to the following definition.

Definition 20. The *stiff convergence factor* for the blended iteration (35) is defined as $\tilde{\rho}_\infty = \rho(Z'_\infty)$.

It follows then that, for each value of r, (30) now becomes

$$\rho(q) \approx \tilde{\rho}_\infty |q|^{-1}, \quad |q| \gg 1, \tag{39}$$

with an obvious improvement of the convergence properties, as $q \to \infty$. Moreover, we mention that averaged convergence factors for the blended iteration can be defined according to what is stated in Section 2.1, with one main

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difference concerning the definition of the corresponding averaged stiff convergence factor. Indeed, in this case the result of Theorem 18 does not apply since $\nu_\infty = 1$, and (31) now becomes

$$\rho(q)^{(\ell)} \equiv \sqrt[\ell]{\|Z(q)^\ell\|} \approx |q|^{-1} \sqrt[\ell]{\|(Z'_\infty)^\ell\|}, \quad |q| \gg 1,$$

where Z'_∞ is the same matrix as in (38). Consequently, the averaged stiff convergence factor, corresponding to ℓ blended iterations, is defined as

$$\tilde{\rho}^{(\ell)}_\infty = \sqrt[\ell]{\|(Z'_\infty)^\ell\|}.$$

As a particular, relevant, instance, we consider the case of the *blended implementation* of a given block implicit method (2). This is obtained by setting (see, e.g., [5,6]):

$$C = A^{-1}B, \qquad \theta(q) = (1-\gamma q)^{-1}I, \quad \gamma > 0, \tag{40}$$

and

$$A(q) = \theta(q)I + \big(I - \theta(q)\big)\gamma C^{-1}, \qquad B(q) = \theta(q)C + \big(I - \theta(q)\big)\gamma I,$$
$$\boldsymbol{\eta}(q) = \theta(q)A^{-1}\boldsymbol{\eta} + \big(I - \theta(q)\big)\gamma B^{-1}\boldsymbol{\eta}. \tag{41}$$

It is an easy matter to verify that, in this way, the discrete problem (33) and (40)–(41) is obtained as the *"blending"*, with weights $\theta(q)$ and $I - \theta(q)$, of the following equivalent formulations of Eq. (7):

$$(I - qC)\mathbf{y} = A^{-1}\boldsymbol{\eta}, \qquad \gamma\big(C^{-1} - qI\big)\mathbf{y} = \gamma B^{-1}\boldsymbol{\eta}.$$

Consequently, from (34) and (36) one obtains:

$$A^* = I, \qquad B^* = \gamma I, \tag{42}$$

and

$$Z(q) = \frac{q}{(1-\gamma q)^2} C^{-1}(C - \gamma I)^2.$$

It then follows that:

$$Z(0) = O \quad \Longrightarrow \quad \rho(0) = 0, \tag{43}$$
$$Z'(0) = C^{-1}(C-\gamma I)^2 \quad \Longrightarrow \quad \tilde{\rho} = \rho\big(C^{-1}(C-\gamma I)^2\big), \tag{44}$$
$$Z'_\infty = \gamma^{-2}C^{-1}(C-\gamma I)^2 \quad \Longrightarrow \quad \tilde{\rho}_\infty = \gamma^{-2}\tilde{\rho}, \tag{45}$$
$$\max_{x\in\mathbb{R}} \frac{|\mathrm{i}x|}{|(1-\gamma \mathrm{i}x)^2|} = (2\gamma)^{-1} \quad \Longrightarrow \quad \rho^* = (2\gamma)^{-1}\tilde{\rho}. \tag{46}$$

The positive parameter γ in (40) is here chosen in order to minimize some amplification factor for the iteration (35). As an example, for many methods of interest (for example, Radau IIA and Gauss–Legendre Runge–Kutta methods, as well as the methods implemented in the codes `BiM` and `BiMD` [4,6]), the choice [3,5]

$$\gamma = |\lambda_1| \equiv \min_{\lambda\in\sigma(C)} |\lambda|, \tag{47}$$

minimizes the corresponding maximum amplification factor ρ^*, providing corresponding L-convergent blended iterations (see also Section 4.3).

Remark 21. The relations (12) and (39) proved to be extremely useful in the development of the computational codes `BiM` and `BiMD` [4,6]. Indeed, they have been actually used to derive the order variation strategy implemented in the two codes, which turns out to be very efficient and reliable, as confirmed by an extensive experimentation.

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3.2. Special second order problems

We now consider an extension of the linear analysis described in Section 2, when the continuous problem is a second order problem of special type,

$$y'' = f(t, y), \quad t \in [t_0, T], \qquad y(t_0) = y_0, \; y'(t_0) = y_0' \in \mathbb{R}^m.$$

When the problem is *stiff*, i.e., with $y(t)$ combining dominant components with short frequencies and components with high frequencies and small amplitudes, again implicit methods are conveniently used [16]. In this case, methods which are popular for solving such problems (for example, in the class of Runge–Kutta–Nyström methods), lead to a discrete problem, to be solved at each integration step, in the form

$$F(\mathbf{y}) \equiv \mathbf{y} - h^2 B \otimes I_m \mathbf{f} - \boldsymbol{\eta} = \mathbf{0}, \tag{48}$$

with the block vectors $\mathbf{y}$ and $\mathbf{f}$, which contain the discrete solution, similarly defined as in (3), h being the current stepsize, and $\boldsymbol{\eta}$ a known vector. Also in this case, by choosing a suitably simple matrix B^*, it is possible to define a nonlinear splitting for solving (48):

$$\mathbf{y}^{(i+1)} - h^2 B^* \otimes I_m \mathbf{f}^{(i+1)} = h^2 (B - B^*) \otimes I_m \mathbf{f}^{(i)} + \boldsymbol{\eta}, \quad i = 0, 1, \ldots. \tag{49}$$

For studying the linear stability analysis of the method, it is customary to consider the linear test equation (see, e.g., [20])

$$y'' = -\mu^2 y, \quad \mu \in \mathbb{R},$$

which will be also considered for the linear analysis of convergence for the iteration (49). In particular, the discrete problem (48) and the iteration itself now become, respectively,

$$\left(I - q^2 B\right)\mathbf{y} = \boldsymbol{\eta}, \quad q = \mathrm{i}h\mu \equiv \mathrm{i}x, \; x \in \mathbb{R}, \tag{50}$$

$$\left(I - q^2 B^*\right)\mathbf{y}^{(i+1)} = q^2 (B - B^*)\mathbf{y}^{(i)} + \boldsymbol{\eta}, \quad i = 0, 1, \ldots. \tag{51}$$

In this setting, the equivalent of Assumption 2 becomes that B^* has no real and negative eigenvalues. The corresponding iteration matrix is then given by

$$Z\left(q^2\right) = q^2 \left(I - q^2 B^*\right)^{-1} (B - B^*). \tag{52}$$

As usual, the iteration will be convergent if and only if the corresponding spectral radius, say $\rho(q^2)$, is less than 1. Clearly, Definition 1, Eq. (11) (with $A = I$), and Definition 2 straightforwardly extend to this setting, via the formal substitution

$$q \to q^2. \tag{53}$$

In particular, (12) now becomes

$$\rho\left(q^2\right) \approx \tilde{\rho}|q^2|, \quad q \approx 0,$$

so that (see also (52)) one easily obtains that the iteration (51) is 0-convergent. By considering the notion of P-stability, formerly introduced for LMFs [20] and subsequently extended to block one step methods (see, e.g., [14]), we provide the related notion of P-convergence as follows.

Definition 22. The iteration (51) is P*-convergent* if the corresponding *maximum amplification factor*, still given by (14) by considering the formal substitution $\mathrm{i}x \to (\mathrm{i}x)^2$, is less than 1.

The notion of L-convergence (see Definition 4) can be extended to this setting, by replacing the requirement of A-convergence with that of P-convergence:

Definition 23. A P-convergent iteration is L-convergent if, in addition, (13)–(53) hold true.

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Moreover, by considering that

$$Z(q^2) = \begin{cases} q^2(B - B^*) + O(q^4) \equiv q^2 Z'(0) + O(q^4), & q \approx 0, \\ I - (B^*)^{-1}B + q^{-2}(B^*)^{-1}(I - (B^*)^{-1}B) + O(q^{-4}), & \\ \quad \equiv Z_\infty + q^{-2}Z'_\infty + O(q^{-4}), & |q| \gg 1, \end{cases}$$

the following result generalizes that of Theorem 9.

Theorem 24. *A P-convergent iteration is L-convergent if and only if the matrix Z_∞, formally still given by* (15)*, is nilpotent.*

Also Definitions 10, 11, and 15 formally still hold, as well as the result of Theorem 12, whereas (30) is replaced by

$$\rho\big(q^2\big) \approx \tilde{\rho}_\infty |q|^{-2/(\nu_\infty - 1)}, \quad |q| \gg 1. \tag{54}$$

Moreover, corresponding average amplification factors can be also defined, according to that explained in Section 2.1. Finally, in a way similar to that explained in Section 3.1 (see, e.g. [5]), the arguments straightforwardly generalize to the case where the discrete problem (50) is replaced by a discrete problem in the form (compare with (33))

$$\big(A\big(q^2\big) - q^2 B\big(q^2\big)\big)\mathbf{y} = \boldsymbol{\eta}\big(q^2\big).$$

Indeed, what has been set out in Section 3.1 formally still holds, via the formal substitution (53), leading to the definition of a corresponding blended iteration. In particular, for the blended iteration the property (37) holds true and (54) becomes

$$\rho\big(q^2\big) \approx \tilde{\rho}_\infty |q|^{-2}, \quad |q| \gg 1,$$

where $\tilde{\rho}_\infty$ is defined according to Definition 20. Finally, we also mention that (see [5] for details) the blended iteration, induced by the blended implementation of Runge–Kutta-Nyström methods derived from Gauss–Legendre or Lobatto IIIA formulae, turns out to be L-convergent.

4. Examples of application

In this section, we use the previous analysis for comparing the convergence properties of different splittings defined for solving the same discrete problem (2). In particular, we shall refer to the following Runge–Kutta methods (and, therefore, $A = I$): Radau IIA and Gauss–Legendre. For such methods, we shall compare a few general procedures for getting corresponding splittings for solving the discrete problems, namely that defined in [24,25] and that proposed in [1]. Moreover, we shall compare the convergence properties of these splittings with those of the *blended iteration* (35) deriving from the corresponding *blended implementation* (40)–(41) of the methods [2,3,5,6]. In the following subsections, we briefly describe the three procedures that we are going to compare.

4.1. The triangular splitting

In [24,25], the authors study the efficient solution of the discrete problem

$$(I - qB)\mathbf{y} = \boldsymbol{\eta}, \tag{55}$$

where the matrix B is nonsingular and factorizable as

$$B = LU, \tag{56}$$

with L a lower triangular matrix with positive diagonal entries, and U an upper triangular matrix with unit diagonal entries. The methods we are considering (i.e., Radau IIA and Gauss–Legendre Runge–Kutta methods) do satisfy (56). In such a case, they consider the splitting (let us recall that Assumption 1 holds true)

$$A^* = I, \qquad B^* = L.$$

Consequently, since the diagonal entries of L are usually different from each other, when a system of equation is considered then one has to factor r real matrices of the same size as that of the continuous problem. Since this

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splitting has been proposed for implementation on parallel computers, this is not a severe drawback, even though the dimension of the problem is large. Anyway, we now want to study only the convergence properties of the splitting. The splitting is obviously 0-convergent. Moreover,

$$\tilde{\rho} = \rho\big(L(U - I)\big), \qquad \rho_\infty = \rho(I - U). \tag{57}$$

A remarkable property is that $\rho_\infty \equiv 0$, since the matrix $Z_\infty = I - U$ is nilpotent of index $\nu_\infty = r$ (the blocksize of the method), since U has unit diagonal. Therefore, from (23) and (29) one obtains

$$\tilde{\rho}_\infty = \rho\big(T^{(r-1,1)}\big)^{1/(r-1)}, \qquad T^{(r-1,1)} = \sum_{s=0}^{r-2} (I - U)^s L^{-1} (I - U)^{r-1-s}. \tag{58}$$

Finally, a few calculation allow to obtain that

$$\rho^* = \max_{x>0} x\rho\big(L(I - \mathrm{i}xL)^{-1}(U - I)\big). \tag{59}$$

Clearly, the parameters in (57)–(58) can be computed directly, whereas that in (59) must be computed numerically. We then conclude that, provided that $\rho^* \leqslant 1$, the iteration is L-convergent with index $\nu_\infty = r$.

4.2. The modified triangular splitting

When a sequential computer is used, the previous procedure may become too expensive, due to the fact that the diagonal entries of L are, in general, distinct. For this reason, in [1] a modification of the approach described in Section 4.1 has been proposed, which is aimed to get rid of this problem. The basic idea on which this modification relies is given by the following result [1].

Theorem 25. *With reference to* (55)*, if* $\det(B) > 0$ *then there exists a nonsingular matrix* $T \in \mathbb{R}^{r\times r}$ *such that*

$$TBT^{-1} = LU,$$

with L a lower triangular matrix with constant diagonal entries all equal to $\sqrt[r]{\det(B)}$*, and U an upper triangular matrix with unit diagonal.*

Consequently, after a variable transformation, one can repeat all the arguments in Section 4.1 with the new matrices L and U, and B replaced by TBT^{-1}. In particular, in [1] a simple Matlab function is provided, which computes a matrix T which is upper bidiagonal with unit diagonal entries. This is the matrix that we shall consider for the comparisons.

4.3. The blended iteration

In the case of the blended iteration (35), (40)–(42), by using the choice (47) for the parameter γ, one obtains that the parameters (44)–(46) become, if

$$\lambda_1 = |\lambda_1| \mathrm{e}^{\mathrm{i}\phi_1} \quad \left(|\phi_1| < \frac{\pi}{2}\right), \tag{60}$$

is the eigenvalue of minimum modulus of the matrix C [3]:

$$\rho^* = 1 - \cos\phi_1, \quad \tilde{\rho} = 2\gamma\rho^*, \quad \tilde{\rho}_\infty = 2\gamma^{-1}\rho^*. \tag{61}$$

Therefore, from (39), (43)–(46), (60)–(61), and (37), it follows that the blended iteration is always 0-convergent and L-convergent with index $\nu_\infty = 1$.

4.4. Comparisons and concluding remarks

Tables 1, 2, and 3 summarize the results obtained for the three iterative procedures described in Sections 4.1, 4.2, and 4.3, respectively (in Table 3 we also list the optimal parameter γ, according to (47)). For all iterations, $\rho_\infty = 0$, whereas $\nu_\infty = 1$ for the blended iteration. By comparing the convergence parameters listed in the tables, we observe that:

Table 1
Convergence parameters for the triangular splitting

r	Radau IIA methods				r	Gauss–Legendre methods			
	ρ^*	$\tilde{\rho}$	ν_∞	$\tilde{\rho}_\infty$		ρ^*	$\tilde{\rho}$	ν_∞	$\tilde{\rho}_\infty$
2	0.1837	0.1500	2	0.9000	2	0.1429	0.0833	2	1.0000
3	0.3726	0.1853	3	0.6229	3	0.3032	0.1098	3	0.6189
4	0.5064	0.1728	4	0.5696	4	0.4351	0.1126	4	0.5517
5	0.6103	0.1496	5	0.5448	5	0.5457	0.1058	5	0.5239
6	0.7007	0.1300	6	0.5291	6	0.6432	0.0973	6	0.5080
7	0.7844	0.1145	7	0.5178	7	0.7325	0.0894	7	0.4972
8	0.8637	0.1022	8	0.5089	8	0.8158	0.0822	8	0.4893
9	0.9396	0.0921	9	0.5018	9	0.8946	0.0760	9	0.4831
10	1.0125	0.0839	10	0.4958	10	0.9696	0.0705	10	0.4780

Table 2
Convergence parameters for the modified triangular splitting

r	Radau IIA methods				r	Gauss–Legendre methods			
	ρ^*	$\tilde{\rho}$	ν_∞	$\tilde{\rho}_\infty$		ρ^*	$\tilde{\rho}$	ν_∞	$\tilde{\rho}_\infty$
2	0.1835	0.1498	2	0.8990	2	0.1340	0.0774	2	0.9282
3	0.3138	0.1375	3	0.4873	3	0.2537	0.0856	3	0.4817
4	0.4137	0.1236	4	0.3713	4	0.3492	0.0803	4	0.3884
5	0.4949	0.1090	5	0.2870	5	0.4223	0.0730	5	0.3375
6	0.5744	0.1027	6	0.2736	6	0.4861	0.0702	6	0.2791
7	0.6473	0.1032	7	0.3408	7	0.5461	0.0704	7	0.2445
8	0.7182	0.1034	8	0.3568	8	0.6060	0.0701	8	0.3048
9	0.7856	0.1056	9	0.3568	9	0.6690	0.0723	9	0.3162
10	0.8480	0.1104	10	0.3466	10	0.7324	0.0763	10	0.3127

Table 3
Convergence parameters for the blended iteration ($\nu_\infty = 1$)

r	Radau IIA methods				r	Gauss–Legendre methods			
	γ	ρ^*	$\tilde{\rho}$	$\tilde{\rho}_\infty$		γ	ρ^*	$\tilde{\rho}$	$\tilde{\rho}_\infty$
2	0.4082	0.1835	0.1498	0.8990	2	0.2887	0.1340	0.0774	0.9282
3	0.2462	0.3398	0.1674	2.7602	3	0.1967	0.2765	0.1088	2.8105
4	0.1738	0.4416	0.1535	5.0817	4	0.1475	0.3793	0.1119	5.1423
5	0.1334	0.5123	0.1367	7.6799	5	0.1173	0.4544	0.1066	7.7454
6	0.1079	0.5644	0.1217	10.4654	6	0.0971	0.5114	0.0993	10.5330
7	0.0903	0.6045	0.1092	13.3872	7	0.0827	0.5561	0.0919	13.4554
8	0.0776	0.6366	0.0988	16.4133	8	0.0718	0.5921	0.0851	16.4813
9	0.0679	0.6628	0.0900	19.5222	9	0.0635	0.6218	0.0789	19.5895
10	0.0603	0.6847	0.0826	22.6987	10	0.0568	0.6467	0.0735	22.7649

- both the modified triangular splitting and the blended iteration are more efficient than the original triangular splitting: as matter of fact, all convergence parameters of the former iterations are smaller than those of the latter, for both classes of methods;
- the blended iteration is surely more efficient than the modified triangular splitting for r large enough: indeed, for both classes of methods, all the convergence parameters of the former iteration become smaller than those of the latter one, as r increases.

However, by also considering that, for the blended iteration:

- for the smallest values of r the parameters ρ^* and $\tilde{\rho}$ are only slightly larger than those of the modified triangular splitting;
- we have L-convergence with index 1, whereas the modified triangular splitting is L-convergent with index r;

we can then conclude that the blended iteration is the most efficient general procedure, among those here examined.

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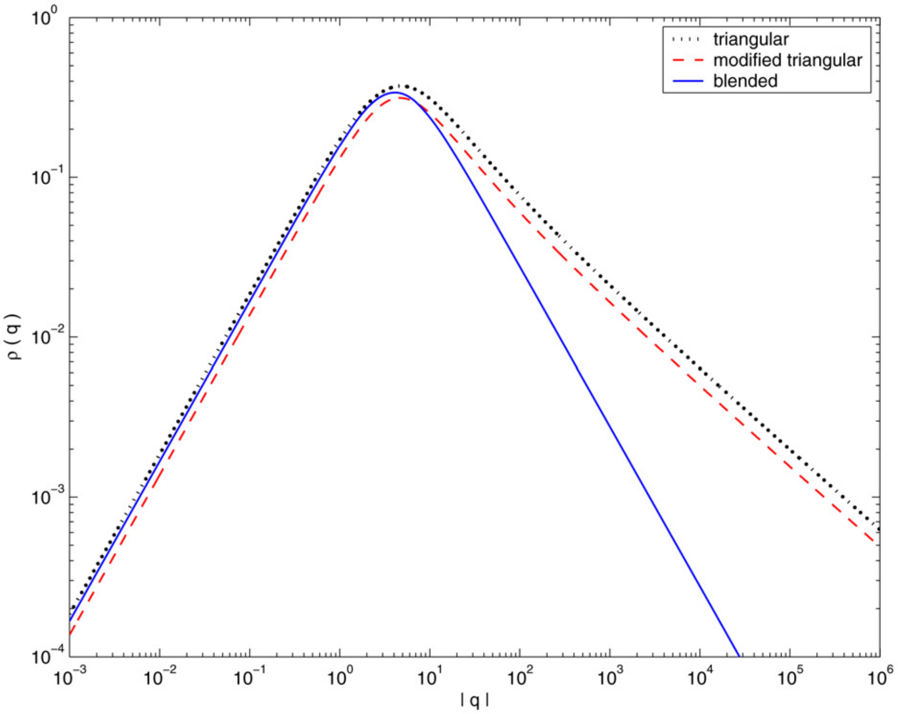

Fig. 1. Spectral radius $\rho(q)$ for the three splittings solving the discrete problem generated by the Radau IIA method with blocksize $r = 3$ (order $p = 5$).

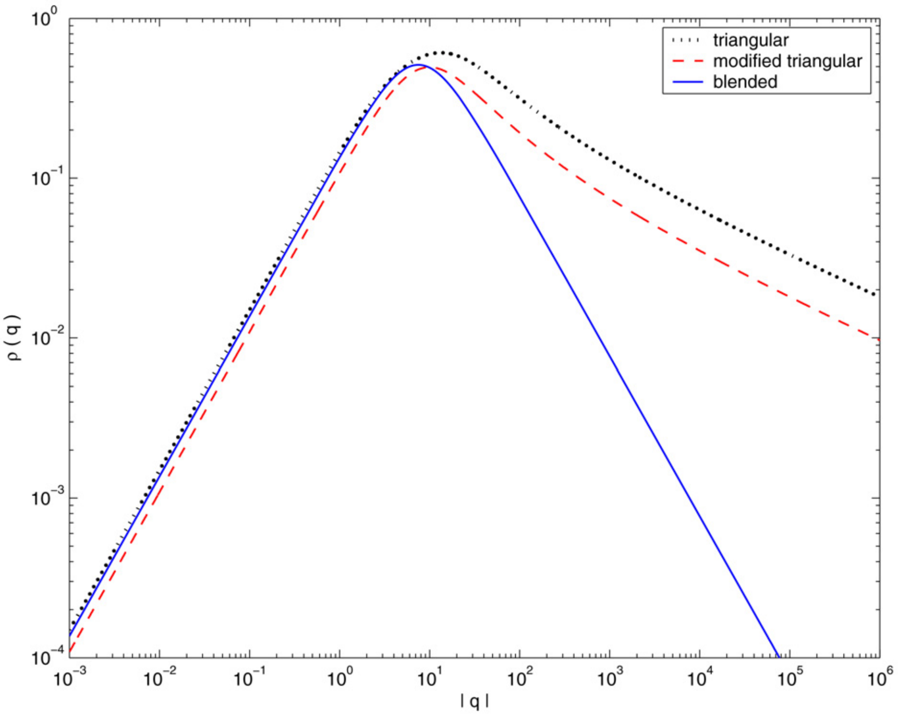

Fig. 2. Spectral radius $\rho(q)$ for the three splittings solving the discrete problem generated by the Radau IIA method with blocksize $r = 5$ (order $p = 9$).

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For completeness, in Figs. 1 and 2 we plot the spectral radius $\rho(q)$, for q belonging to the positive imaginary axis, for the three above splittings, for the Radau IIA methods with blocksize $r = 3$ and $r = 5$, in order to verify the way $\rho(q)$ converges towards 0, as $q \to \infty$, predicted by (30) and (39), respectively.

In this paper, we have described a complete linear convergence analysis which provides a fairly complete framework for studying iterative procedures aimed to solve the discrete problems generated by block implicit methods. The theory completes the original formulation made by van der Houwen et al. in the nineties. Moreover, the theory is extended to discuss more general iterative procedures, such as the blended iteration induced by the blended implementation of block implicit methods, and to the case of block methods used for solving special second order problems. Examples of application of the theory prove its usefulness in ranking the convergence properties of the corresponding iterative procedures.

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Please cite this article in press as: L. Brugnano, C. Magherini, Recent advances in linear analysis of convergence for splittings for solving ODE problems, Applied Numerical Mathematics (2008), doi:10.1016/j.apnum.2008.03.008