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The Conditioning of Toeplitz Band Matrices*

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Abstract—In this paper, practical conditions to check the well-conditioning of a family of non-singular Toeplitz band matrices are obtained. All the results are based on the location of the zeros of a polynomial associated with the given family of Toeplitz matrices.

The same analysis is also used to derive uniform componentwise bounds for the entries of the inverse matrices in such family.

Keywords—Toeplitz band matrices, Family of matrices, Well conditioning, Polynomial zeros.

1. INTRODUCTION

In the study of a new class of linear multistep methods for the solution of ordinary differential equations, namely Boundary Value Methods (BVMs) [1–3], we were concerned with Toeplitz band matrices. In particular, the convergence results for these methods, along with the linear stability theory, can be stated provided that one is able to discuss the behavior of the inverses of Toeplitz band matrices belonging to the family $\{T_n\}$,

$$T_n = \begin{pmatrix} a_0 & \dots & a_k & & \\ \vdots & \ddots & & \ddots & \\ a_{-m} & & \ddots & & a_k \\ & \ddots & & \ddots & \vdots \\ & & a_{-m} & \dots & a_0 \end{pmatrix}_{n\times n}, \tag{1}$$

where a_j is the generic entry on the j^{th} diagonal, the index 0 denotes the main diagonal, a positive index denotes an upper diagonal, and a negative index a lower diagonal. Without loss of generality, here we assume that $a_{-m}a_k \neq 0$.

Several papers have been devoted to Toeplitz band matrices, in particular to algorithms for their inversion [4–7]. The problem of their conditioning was previously treated, for example, in [8,9], while important results concerning the asymptotic eigenvalue distribution of Toeplitz band matrices can be found in [10,11].

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We shall study under what conditions the family of matrices $\{T_n\}$ is "well-conditioned" according to the following definition.

DEFINITION 1. *A family of matrices $\{T_n\}$ is said to be well conditioned if the condition numbers $\{\kappa(T_n)\}$ are uniformly bounded with respect to n. It is said to be weakly well conditioned if $\kappa(T_n)$ grows as a small power of n.*

Moreover, a matrix T_n will be said (weakly) well conditioned if it belongs to a (weakly) well-conditioned family of matrices.

REMARK 1. Observe that since

$$\kappa(T_n) = \|T_n\| \; \|T_n^{-1}\|$$

and, for Toeplitz band matrices, $\|T_n\|$ is always uniformly bounded with respect to n, then the family of matrices $\{T_n\}$ will be well conditioned (weakly well conditioned), if and only if the elements in the sequence $\{\|T_n^{-1}\|\}$ are uniformly bounded with respect to n (grow as a small power of n).

The problem of the conditioning of a family of Toeplitz band matrices is discussed in Section 2 for the simpler case of triangular matrices. Then, the obtained results are extended to the more general case in Section 3. Some of the results presented in this section are known, but are here rederived in a novel way.

Finally, in Section 4 we consider componentwise bounds for the entries of $|T_n^{-1}|$, the matrix having as entries the absolute values of the corresponding entries of T_n^{-1}. Some particular cases, which are relevant for the study of the convergence and of the linear stability theory for BVMs [2], will be analyzed in more details.

1.1. Notations

All the results presented in this paper will be obtained by analyzing the following polynomial of degree $m + k$ associated with the matrices $\{T_n\}$ defined as in (1):

$$p(z) = \sum_{i=-m}^{k} a_i z^{m+i}. \tag{2}$$

Let $z_1, \dots, z_{m+k}$ be its zeros, where

$$0 < |z_1| \le \cdots \le |z_m| \le \cdots \le |z_{m+k}| < \infty. \tag{3}$$

We say that the polynomial (2) is of type (s, u, l) if it has:

- s zeros with modulus smaller than 1,
- u zeros with unit modulus,
- l zeros with modulus larger than 1.

This notation has already been used by Miller in [12], where a general criterion to check the type of a given polynomial is provided.

Associated with the zeros in (3), we also define the matrix

$$D = \begin{pmatrix} D_1 & \\ & D_2 \end{pmatrix},$$

where

$$D_1 = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_m \end{pmatrix}, \qquad D_2 = \begin{pmatrix} z_{m+1} & & \\ & \ddots & \\ & & z_{m+k} \end{pmatrix},$$

and the Casorati matrix [13] which, in the case of simple zeros, may be written as

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix},$$

where

$$W_{11} = \begin{pmatrix} 1 & \dots & 1 \\ z_1 & \dots & z_m \\ \vdots & & \vdots \\ z_1^{k-1} & \dots & z_m^{k-1} \end{pmatrix}, \qquad W_{12} = \begin{pmatrix} 1 & \dots & 1 \\ z_{m+1} & \dots & z_{m+k} \\ \vdots & & \vdots \\ z_{m+1}^{k-1} & \dots & z_{m+k}^{k-1} \end{pmatrix},$$

$$W_{21} = \begin{pmatrix} z_1^k & \dots & z_m^k \\ \vdots & & \vdots \\ z_1^{m+k-1} & \dots & z_m^{m+k-1} \end{pmatrix}, \qquad W_{22} = \begin{pmatrix} z_{m+1}^k & \dots & z_{m+k}^k \\ \vdots & & \vdots \\ z_{m+1}^{m+k-1} & \dots & z_{m+k}^{m+k-1} \end{pmatrix}.$$

2. CONDITIONING OF TRIANGULAR TOEPLITZ BAND MATRICES

In this section, we shall present the results concerning the conditioning of triangular Toeplitz band matrices, obtained by using standard arguments on linear difference equations. Let us suppose that the matrix (1) is lower triangular, that is $k = 0$:

$$T_n = L_n := \begin{pmatrix} a_0 & & & \\ \vdots & \ddots & & \\ a_{-m} & & \ddots & \\ & \ddots & & \ddots & \\ & & a_{-m} & \dots & a_0 \end{pmatrix}_{n\times n}. \tag{4}$$

In this case, the associated polynomial (2) is given by

$$p_l(z) = \sum_{i=-m}^{0} a_i z^{m+i}. \tag{5}$$

The following result holds true.

THEOREM 1. *The family of matrices $\{L_n\}$ defined as in (4) is*

(i) *well conditioned iff the polynomial (5) is of type $(m, 0, 0)$,*

(ii) *weakly well conditioned iff the polynomial (5) is of type $(m_1, m_2, 0)$, with $m_1 + m_2 = m$.*

PROOF. It is well known that the set of Toeplitz triangular matrices is closed with respect to the operation of inversion. Then, it is sufficient to examine the entries on the first column of L_n^{-1}. In fact, the matrix L_n will be well conditioned iff the entries of the first column of L_n^{-1} go to zero with the row-index, and weakly well conditioned iff these entries depend polynomially on the row-index.

The r^{th} entry on the first column of L_n^{-1}, say y_r, satisfies the difference equation

$$\begin{aligned} &\sum_{i=-m}^{0} a_i y_{r+i} = 0, && r = 2, \dots, n, \\ &y_{2-m} = \dots = y_0 = 0, && y_1 = \frac{1}{a_0}. \end{aligned} \tag{6}$$

The general solution of (6) is given by

$$y_{r+1} = \sum_{i=1}^{q} z_i^r \sum_{s=0}^{m_i-1} c_{is} r^s, \qquad r = 0, 1, 2, \dots, \tag{7}$$

where z_i is a zero of multiplicity m_i of the polynomial (5) and the zeros are ordered for increasing moduli, according to (3). The thesis follows by observing that, since from (6) and (7) it is possible to prove that $c_{q,m_q-1} \neq 0$, for large values of r it is $|y_r| \propto |z_q^r r^{m_q-1}|$ which goes to zero iff $|z_q| < 1$, and is $O(r^{m_q-1})$ iff $|z_q| = 1$ (here, we have supposed, without loss of generality, that $m_q = \max\{m_i : m_i \text{ multiplicity of } z_i, |z_i| = |z_q|\}$). ∎

Similarly, we can handle the case where the matrices $\{T_n\}$ are upper triangular; that is when $m = 0$,

$$T_n = U_n := \begin{pmatrix} a_0 & \dots & a_k & & \\ & \ddots & & \ddots & \\ & & \ddots & & a_k \\ & & & \ddots & \vdots \\ & & & & a_0 \end{pmatrix}_{n\times n}. \tag{8}$$

In this case, the associated polynomial (2) is given by

$$p_u(z) = \sum_{i=0}^{k} a_i z^i \tag{9}$$

and the following result holds true.

THEOREM 2. *The family of matrices* $\{U_n\}$ *defined as in (8) is*

(i) *well conditioned iff the polynomial (9) is of type* $(0, 0, k)$,

(ii) *weakly well conditioned iff the polynomial (9) is of type* $(0, k_1, k_2)$, *with* $k_1 + k_2 = k$.

PROOF. The proof is obtained by applying the previous Theorem 1 to the family $\{U_n^{\mathsf{T}}\}$. ∎

By considering the norms $\|\cdot\|_1$ or $\|\cdot\|_\infty$ from Theorems 1 and 2, the next result follows.

COROLLARY 1. *Let* $\{T_n\}$ *be a weakly well-conditioned family of triangular Toeplitz band matrices. Then,* $\kappa(T_n) = O(n^\mu)$, *where* μ *is the highest among the multiplicities of the zeros of unit modulus of the associated polynomial* $p(z)$.

EXAMPLE. Consider the following triangular matrix:

$$L_n = \begin{pmatrix} 2 & & & & & \\ 3 & \ddots & & & & \\ -1 & \ddots & \ddots & & & \\ -3 & \ddots & \ddots & \ddots & & \\ -1 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & -1 & -3 & -1 & 3 \quad 2 \end{pmatrix}_{n\times n}.$$

The associated polynomial is $p_l(z) = (z-1)(2z+1)(z+1)^2$. Moreover, for large n, one can verify that $\kappa_\infty(L_n) \approx 2.5n^2$, that is $\kappa_\infty(L_n) = O(n^2)$, as predicted by Corollary 1.

3. CONDITIONING OF GENERAL TOEPLITZ BAND MATRICES

In this section, we shall discuss the conditioning of a family of general Toeplitz band matrices $\{T_n\}$ defined as in (1). As before, $p(z)$ is the associated polynomial (2) whose zeros z_i, $i = 1, \ldots, m+k$, are ordered according to (3). Then, we factor $p(z)$ as follows:

$$p(z) = p_l(z)p_u(z), \tag{10}$$

where

$$p_l(z) = \prod_{i=1}^{m}(z - z_i) =: \sum_{i=-m}^{0} b_i z^{m+i}, \qquad b_0 = 1, \tag{11}$$

and

$$p_u(z) = a_k \prod_{i=1}^{k}(z - z_{m+i}) =: \sum_{i=0}^{k} c_i z^i, \qquad c_k = a_k. \tag{12}$$

It can be verified that

$$T_n = L_n U_n + E_n, \tag{13}$$

where

$$L_n = \begin{pmatrix} b_0 & & & \\ \vdots & \ddots & & \\ b_{-m} & & \ddots & \\ & \ddots & & \ddots & \\ & & b_{-m} & \cdots & b_0 \end{pmatrix}_{n\times n} \qquad U_n = \begin{pmatrix} c_0 & \cdots & c_k & & \\ & \ddots & & \ddots & \\ & & \ddots & & c_k \\ & & & \ddots & \vdots \\ & & & & c_0 \end{pmatrix}_{n\times n},$$

and E_n is an $n \times n$ matrix which has only the principal $m \times k$ submatrix with nonzero entries. Moreover, all these entries are independent of n.

Because of these arguments, we can always consider the matrix E_n in (13) as a perturbation of the product $L_n U_n$.

In a way similar to (13), we want now to see when it is possible to write

$$T_n^{-1} = U_n^{-1} L_n^{-1} + H_n, \tag{14}$$

with the matrix H_n which can be regarded as a perturbation of the product $U_n^{-1} L_n^{-1}$. Now, in fact, this is not always the case, as the following example clearly shows.

EXAMPLE. Consider the Toeplitz tridiagonal matrix

$$T_n = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & 0 \end{pmatrix}_{n\times n}.$$

Then

$$\begin{aligned} T_n &= \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}_{n\times n} \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}_{n\times n} - e_1 e_1^{\mathrm{T}} \\ &=: L_n U_n - e_1 e_1^{\mathrm{T}}, \end{aligned}$$

where e_1 is the first vector of the canonical base on $\mathbb{R}^n$.

For even n, equation (14) holds since both the matrices L_nU_n and T_n are weakly well conditioned. On the contrary, for odd n, T_n is always singular, while L_nU_n is nonsingular.

In the following, we shall always suppose the matrices $\{T_n\}$ to be nonsingular. In this hypothesis, we will obtain sufficient conditions to guarantee the (weakly) well-conditioning of T_n provided that the product L_nU_n is (weakly) well conditioned.

This will be done in Section 3.2, after the statement of the preliminary results of Section 3.1.

3.1. Preliminary Results

In order to give the main result, we first need to prove the following two lemmas.

LEMMA 1. *Let*

$$A = \begin{pmatrix} A_{11} & O_k \\ A_{21} & A_{22} \end{pmatrix}$$

be a nonsingular $(n+k)\times(n+k)$ *matrix, where* O_k *is a* $k\times k$ *null block and* A_{21} *is a square block of size* n. *Moreover, let*

$$A^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

be its inverse, where now X_{12} *is* $n\times n$ *and* X_{21} *is* $k\times k$. *If* A_{21} *is nonsingular, then* X_{21} *is also nonsingular, and*

$$A_{21}^{-1} = X_{12} - X_{11}X_{21}^{-1}X_{22}.$$

PROOF. Let us now consider the following block permutation matrix:

$$P = \begin{pmatrix} O_{n,k} & I_n \\ I_k & O_{n,k}^{\top} \end{pmatrix},$$

where $O_{n,k}$ denotes the $n\times k$ null matrix. Suppose A_{21} to be nonsingular. One then has

$$\begin{aligned} PA &= \begin{pmatrix} A_{21} & A_{22} \\ A_{11} & O_k \end{pmatrix} \\ &= \begin{pmatrix} A_{21} & O_{n,k} \\ A_{11} & S \end{pmatrix}\begin{pmatrix} I_n & A_{21}^{-1}A_{22} \\ O_{n,k}^{\top} & I_k \end{pmatrix}, \end{aligned}$$

where $S = -A_{11}A_{21}^{-1}A_{22}$. We observe that since both the matrices A and A_{21} are nonsingular, then S is nonsingular. It follows that

$$\begin{aligned} A^{-1} &= \begin{pmatrix} I_n & -A_{21}^{-1}A_{22} \\ O_{n,k}^{\top} & I_k \end{pmatrix}\begin{pmatrix} A_{21}^{-1} & O_{n,k} \\ -S^{-1}A_{11}A_{21}^{-1} & S^{-1} \end{pmatrix} P \\ &= \begin{pmatrix} -A_{21}^{-1}A_{22}S^{-1} & A_{21}^{-1} + A_{21}^{-1}A_{22}S^{-1}A_{11}A_{21}^{-1} \\ S^{-1} & -S^{-1}A_{11}A_{21}^{-1} \end{pmatrix}. \end{aligned}$$

The thesis then follows by direct identification: $X_{21} = S^{-1}$ and

$$A_{21}^{-1} = X_{21} - A_{21}^{-1}A_{22}S^{-1}A_{11}A_{21}^{-1} = X_{21} - X_{11}X_{21}^{-1}X_{22}.$$ ∎

LEMMA 2. *Let* T_n *in (1) be nonsingular and* $p(z)$ *in (2) be the polynomial associated with the family* $\{T_n\}$. *If* $p(z)$ *is of type* (m_1, m_2, k) *or* (m, k_1, k_2), *where* $m_1 + m_2 = m$ *and* $k_1 + k_2 = k$, *then for* $n \gg 0$ *the* $(i,j)^{\text{th}}$ *entry of* $H_n = T_n^{-1} - U_n^{-1}L_n^{-1}$ *behaves as* $O(|(i^{\mu-1}z_m^i)(j^{\nu-1}z_{m+1}^{-j})|)$, *where*

$$\mu = \max\{\mu_i : \mu_i \text{ multiplicity of } z_i, |z_i| = |z_m|\}$$

and

$$\nu = \max\{\nu_i : \nu_i \text{ multiplicity of } z_i, |z_i| = |z_{m+1}|\}.$$

PROOF. For the sake of brevity, we shall prove the result only in the simpler case where the zeros of $p(z)$ are all simple (and then $\mu = \nu = 1$). In the case of multiple zeros the proof can still be obtained by using similar arguments, but it is more entangled. Let us suppose that $p(z)$ is (m_1, m_2, k): in the case where $p(z)$ is of type (m, k_1, k_2), it will be sufficient to consider the matrix T_n^{T}, whose associated polynomial is of type (k_2, k_1, m). Then, we define the following lower block triangular matrix:

$$A_T = \begin{pmatrix} & C_1 & & & & \\ a_0 & \dots & a_k & & & \\ \vdots & \ddots & & \ddots & & \\ a_{-m} & & \ddots & & \ddots & \\ & \ddots & & \ddots & & a_k \\ & & \ddots & & \ddots & \vdots & \ddots \\ & & & a_{-m} & \dots & a_0 & \dots & a_k \end{pmatrix}_{(n+k)\times(n+k)} = \begin{pmatrix} A_{11} & O_k \\ T_n & A_{22} \end{pmatrix},$$

where C_1 is a nonsingular $k \times k$ block, which will be specified later, and

$$A_{11} = \begin{pmatrix} C_1 & O_{k,n-k} \end{pmatrix}, \qquad A_{22} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & & \vdots \\ a_k & \ddots & \vdots \\ \vdots & \ddots & 0 \\ a_1 & \dots & a_k \end{pmatrix}_{n\times k}$$

Since C_1 is nonsingular, the matrix A_T is invertible and

$$A_T^{-1} = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}, \tag{15}$$

where X_{12} is $n \times n$, and X_{21} is $k \times k$. Moreover, since T_n is also nonsingular, from Lemma 1 one also has that X_{21} is nonsingular and

$$T_n^{-1} = X_{12} - X_{11}X_{21}^{-1}X_{22}. \tag{16}$$

Similarly, we define

$$A_{LU} = \begin{pmatrix} A_{11} & O_k \\ L_nU_n & A_{22} \end{pmatrix}$$

whose entries are the same as those of A_T, except for an $m \times k$ block on rows $k+1, \dots, k+m$, columns $1, \dots, k$,

$$A_{LU}^{-1} = \begin{pmatrix} \hat{X}_{11} & X_{12} \\ \hat{X}_{21} & X_{22} \end{pmatrix},$$

and

$$U_n^{-1}L_n^{-1} = X_{12} - \hat{X}_{11}\hat{X}_{21}^{-1}X_{22}.$$

It follows that

$$H_n = T_n^{-1} - U_n^{-1}L_n^{-1} = \left(\hat{X}_{11}\hat{X}_{21}^{-1} - X_{11}X_{21}^{-1}\right) X_{22}. \tag{17}$$

We shall now show that the $(i,j)^{\text{th}}$ entry of H_n is $O(|z_m|^i/|z_{m+1}|^j)$.

Let for simplicity $x_i^{(j)}$ be the $(i,j)^{\text{th}}$ entry of A_T^{-1}. The first k columns of this matrix are obtained by solving the block linear system

$$A_T \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix} = \begin{pmatrix} I_k \\ O_{n,k} \end{pmatrix}.$$

Observe that the unknown entries on column j satisfy the difference equation

$$a_{-m}x_i^{(j)} + \cdots + a_0 x_{i+m}^{(j)} + \cdots + a_k x_{i+m+k}^{(j)} = 0, \tag{18}$$

with the initial conditions $x_1^{(j)}, \dots, x_{m+k}^{(j)}$ fixed.

Since we have supposed that the zeros of the polynomial $p(z)$ are simple, then the solution of (18) can be written as

$$x_i^{(j)} = E_{m+k}^{\top} D^{i-1} \begin{pmatrix} b_1^{(j)} \\ \vdots \\ b_{m+k}^{(j)} \end{pmatrix}, \qquad i = 1, 2, \dots, n+k, \tag{19}$$

where $E_i = (1, \dots, 1)_i^{\top}$, and

$$\begin{pmatrix} b_1^{(j)} \\ \vdots \\ b_{m+k}^{(j)} \end{pmatrix} =: \begin{pmatrix} B_1^{(j)} \\ B_2^{(j)} \end{pmatrix} = W^{-1} \begin{pmatrix} x_1^{(j)} \\ \vdots \\ x_{m+k}^{(j)} \end{pmatrix}.$$

In the last relation, according to the definitions of W and D, the vectors $B_1^{(j)}$ and $B_2^{(j)}$ have length m and k, respectively.

The entries $\hat{x}_i^{(j)}$ of A_{LU}^{-1} satisfy the same difference equation (18), but with initial conditions $x_1^{(j)}, \dots, x_k^{(j)}, \hat{x}_{k+1}^{(j)}, \dots, \hat{x}_{k+m}^{(j)}$ (because of the definition of A_{LU}). Hence

$$\hat{x}_i^{(j)} = E_{m+k}^{\top} D^{i-1} \begin{pmatrix} \hat{B}_1^{(j)} \\ \hat{B}_2^{(j)} \end{pmatrix}, \qquad i = 1, 2, \dots, n+k, \tag{20}$$

where

$$\begin{pmatrix} \hat{B}_1^{(j)} \\ \hat{B}_2^{(j)} \end{pmatrix} = W^{-1} \begin{pmatrix} x_1^{(j)} \\ \vdots \\ x_k^{(j)} \\ \hat{x}_{k+1}^{(j)} \\ \vdots \\ \hat{x}_{k+m}^{(j)} \end{pmatrix}.$$

Then, if $e_r^{(s)}$ is the r^{th} vector of the canonical base on $\mathbb{R}^s$ and for $l = 1, 2$ we denote by $B_l^{(1:k)} = (B_l^{(1)}, B_l^{(2)}, \dots, B_l^{(k)})$, from (19) and (20) it follows that for $i = 1, \dots, n$,

$$\left(x_i^{(1)}, \dots, x_i^{(k)}\right) = \left(e_i^{(n)}\right)^{\top} X_{11} = E_{m+k}^{\top} D^{i-1} \begin{pmatrix} B_1^{(1:k)} \\ B_2^{(1:k)} \end{pmatrix} \tag{21}$$

and

$$\left(\hat{x}_i^{(1)}, \dots, \hat{x}_i^{(k)}\right) = \left(e_i^{(n)}\right)^{\top} \hat{X}_{11} = E_{m+k}^{\top} D^{i-1} \begin{pmatrix} \hat{B}_1^{(1:k)} \\ \hat{B}_2^{(1:k)} \end{pmatrix}. \tag{22}$$

With similar arguments, if we assume $n \gg 0$, one has

$$X_{21} = \begin{pmatrix} W_{11} & W_{12} \end{pmatrix} D^n \begin{pmatrix} B_1^{(1:k)} \\ B_2^{(1:k)} \end{pmatrix} \approx W_{12} D_2^n B_2^{(1:k)}, \tag{23}$$

$$\hat{X}_{21} = \begin{pmatrix} W_{11} & W_{12} \end{pmatrix} D^n \begin{pmatrix} \hat{B}_1^{(1:k)} \\ \hat{B}_2^{(1:k)} \end{pmatrix} \approx W_{12} D_2^n \hat{B}_2^{(1:k)}. \tag{24}$$

The neglected terms are, in both cases, $O(|z_m/z_{m+1}|^n)$. The $k \times k$ matrices $B_2^{(1:k)}$ and $\hat{B}_2^{(1:k)}$ depend on the initial conditions and hence on C_1, which is an arbitrary $k \times k$ nonsingular matrix. This matrix can always be chosen in order to have both $B_2^{(1:k)}$ and $\hat{B}_2^{(1:k)}$ nonsingular. It follows that, for n sufficiently large, also X_{21} and $\hat{X}_{21}$ are always nonsingular, since both W_{12} and D_2 are nonsingular.

For what concerns the matrix X_{22}, let us denote by $x_{n+i}^{(k+j)}$ its $(i,j)^{\text{th}}$ entry, for $j = 1, \ldots, n$ and $i = 1, \ldots, k$. The entries on column j satisfy the difference equation (18) with initial conditions

$$x_{1-m+j}^{(k+j)} = \cdots = x_{k+j-1}^{(k+j)} = 0, \qquad x_{k+j}^{(k+j)} = \frac{1}{a_k}$$

whose solution is

$$x_i^{(k+j)} = E_{m+k}^{\mathsf{T}} D^{i+m-j-1} \begin{pmatrix} B_1^{(\cdot)} \\ B_2^{(\cdot)} \end{pmatrix},$$

where

$$\begin{pmatrix} B_1^{(\cdot)} \\ B_2^{(\cdot)} \end{pmatrix} = W^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_k^{-1} \end{pmatrix}.$$

Observe that in the previous formula the vectors $B_1^{(\cdot)}$ and $B_2^{(\cdot)}$ do not depend on j. Then, for $j = 1, \ldots, n$, one has

$$\begin{pmatrix} x_{n+1}^{(k+j)} \\ \vdots \\ x_{n+k}^{(k+j)} \end{pmatrix} = X_{22} e_j^{(n)} = \begin{pmatrix} W_{11} & W_{12} \end{pmatrix} D^{n+m-j} \begin{pmatrix} B_1^{(\cdot)} \\ B_2^{(\cdot)} \end{pmatrix}. \tag{25}$$

From the relations (17) and (21)–(25), it follows that the $(i,j)^{\text{th}}$ entry of H_n is given by

$$\begin{aligned}
h_{ij} &= \left(e_i^{(n)}\right)^{\mathsf{T}} \left(\hat{X}_{11}\hat{X}_{21}^{-1} - X_{11}X_{21}^{-1}\right) X_{22} e_j^{(n)} \\
&\approx E_{m+k}^{\mathsf{T}} D^{i-1} \left(\begin{pmatrix} \hat{B}_1^{(1:k)} \\ \hat{B}_2^{(1:k)} \end{pmatrix} \left(\hat{B}_2^{(1:k)}\right)^{-1} - \begin{pmatrix} B_1^{(1:k)} \\ B_2^{(1:k)} \end{pmatrix} \left(B_2^{(1:k)}\right)^{-1} \right) \\
&\quad \times D_2^{-n} W_{12}^{-1} \left(W_{11} D_1^{n+m-j} B_1^{(\cdot)} + W_{12} D_2^{n+m-j} B_2^{(\cdot)} \right) \\
&= E_{m+k}^{\mathsf{T}} D_1^{i-1} \left(\hat{B}_1^{(1:k)} \left(\hat{B}_2^{(1:k)}\right)^{-1} - B_1^{(1:k)} \left(B_2^{(1:k)}\right)^{-1} \right) \\
&\quad \times \left(D_2^{-n} W_{12}^{-1} W_{11} D_1^{n+m-j} B_1^{(\cdot)} + D_2^{m-j} B_2^{(\cdot)} \right) \\
&\approx E_{m+k}^{\mathsf{T}} D_1^{i-1} \left(\hat{B}_1^{(1:k)} \left(\hat{B}_2^{(1:k)}\right)^{-1} - B_1^{(1:k)} \left(B_2^{(1:k)}\right)^{-1} \right) D_2^{m-j} B_2^{(\cdot)} \\
&= O\left(\frac{|z_m|^i}{|z_{m+1}|^j} \right).
\end{aligned}$$

∎

REMARK 2. From the relation (16) we observe that T_n is nonsingular iff the block X_{21} of A_T is nonsingular. But for n sufficiently large, from the relation (23) it follows that this is always true, if $p(z)$ is of type (m_1, m_2, k) or (m, k_1, k_2).

3.2. Main Result

In the following, $\|\cdot\|$ denotes either $\|\cdot\|_1$ or $\|\cdot\|_\infty$.

THEOREM 3. *Let $\{T_n\}$ be a family of nonsingular Toeplitz band matrices defined as in (1) and $p(z)$ in (2) be the associated polynomial. Then, the family of matrices $\{T_n\}$ is*

(i) *well conditioned if $p(z)$ is of type $(m, 0, k)$;*

(ii) *weakly well conditioned if $p(z)$ is of type (m_1, m_2, k) or (m, k_1, k_2), where $m_1 + m_2 = m$ and $k_1 + k_2 = k$. In this case, $\kappa(T_n)$ grows at most as $O(n^\mu)$, where μ is the highest multiplicity among the zeros of unit modulus.*

PROOF. From (10)–(14), we have that

$$T_n^{-1} = U_n^{-1} L_n^{-1} + H_n,$$

where L_n and U_n are Toeplitz triangular matrices. If $p(z)$ is of type $(m, 0, k)$, then $p_l(z)$ is of type $(m, 0, 0)$, $p_u(z)$ is of type $(0, 0, k)$, and therefore the family of matrices $\{L_n U_n\}$ is well conditioned. Moreover, from Lemma 2 it follows that also the elements of the sequence $\{\|H_n\|\}$ are uniformly bounded with respect to n.

Let now $p(z)$ be of type (m_1, m_2, k) or (m, k_1, k_2). Then, from Corollary 1 and Lemma 2 it follows that the family $\{L_n U_n\}$ is weakly well conditioned and both $\|U_n^{-1} L_n^{-1}\|$ and $\|H_n\|$ grow at most as $O(n^\mu)$, where μ is the highest multiplicity among the zeros of unit modulus. The thesis then follows from relation (14). ∎

REMARK 3. We observe that statement (i) in Theorem 3 has already been derived, by using results on infinite Toeplitz matrices, by Gohberg and Fel'dman [8].

EXAMPLES. Let us consider the matrix

$$T_n^{(1)} = \begin{pmatrix} 1 & 1 & & & \\ -\frac{7}{4} & \ddots & \ddots & & \\ \frac{1}{2} & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & 1 \\ & & \frac{1}{2} & -\frac{7}{4} & 1 \end{pmatrix}_{n\times n} \tag{26}$$

The associated polynomial is

$$p_1(z) = \left(z - \frac{1}{2}\right)^2 (z + 2),$$

which is of type $(2, 0, 1)$. Since $T_n^{(1)}$ has two lower off-diagonals, it follows that $\kappa_\infty(T_n^{(1)})$ is uniformly bounded with respect to n, as it can be seen in Figure 1.

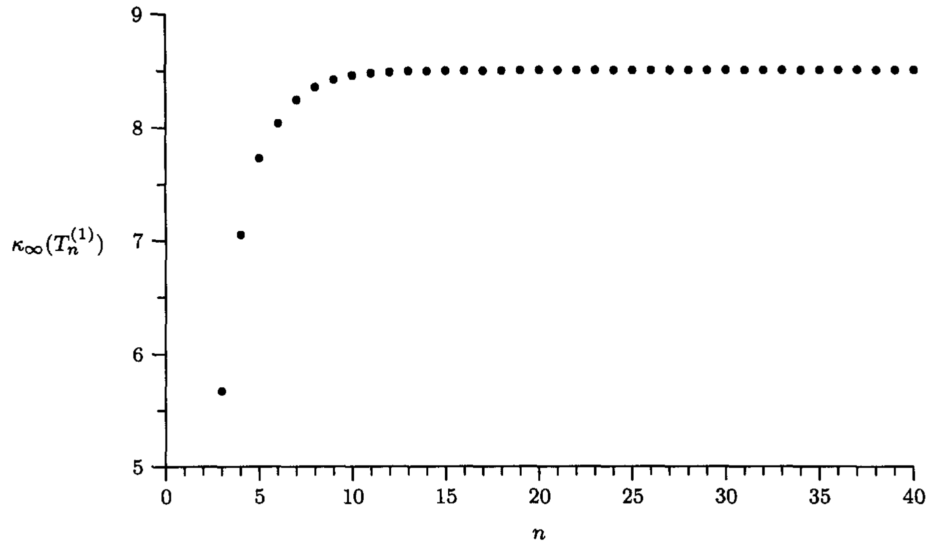

Figure 1. Condition number of the matrix (26).

Consider now the matrix

$$T_n^{(2)} = \begin{pmatrix} 2 & 1 & & & & & \\ -2 & \ddots & \ddots & & & & \\ -4 & \ddots & \ddots & \ddots & & & \\ 1 & \ddots & \ddots & \ddots & \ddots & & \\ 2 & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ & & 2 & 1 & -4 & -2 & 2 \end{pmatrix}_{n\times n} \tag{27}$$

The associated polynomial is

$$p_2(z) = (z+1)^2(z-1)^2(z+2),$$

which is of type $(0,4,1)$. Since $T_n^{(2)}$ has four lower off-diagonals and $p_2(z)$ has two zeros of multiplicity two on the unit circle, it follows that $\kappa_\infty(T_n^{(2)})$ grows at most as $O(n^2)$, as it can be seen in Figure 2.

4. COMPONENTWISE BOUNDS FOR THE INVERSE

In this last section, we shall derive uniform componentwise bounds for the entries of T_n^{-1} in the following two cases:

1. $p(z)$ is of type $(m, 0, k)$;
2. $p(z)$ is of type (m_1, m_2, k), $m_1 + m_2 = m$, being simple the zeros of unit modulus.

These bounds are important to discuss the convergence and the linear stability theory of BVMs. In [2], the polynomials of the first kind have been called *$S_{m,k}$-polynomials*, while those of the second kind have been called *$N_{m,k}$-polynomials*. Observe that when $k = 0$, an $S_{m,k}$-polynomial reduces to a Schur polynomial and, similarly, an $N_{m,k}$-polynomial becomes a Von Neumann polynomial.

Moreover, we observe that the obtained results can be extended in a straightforward way to discuss the case where $p(z)$ is of type (m, k_1, k_2), again being simply the roots of unit modulus.

We first state the following lemma.

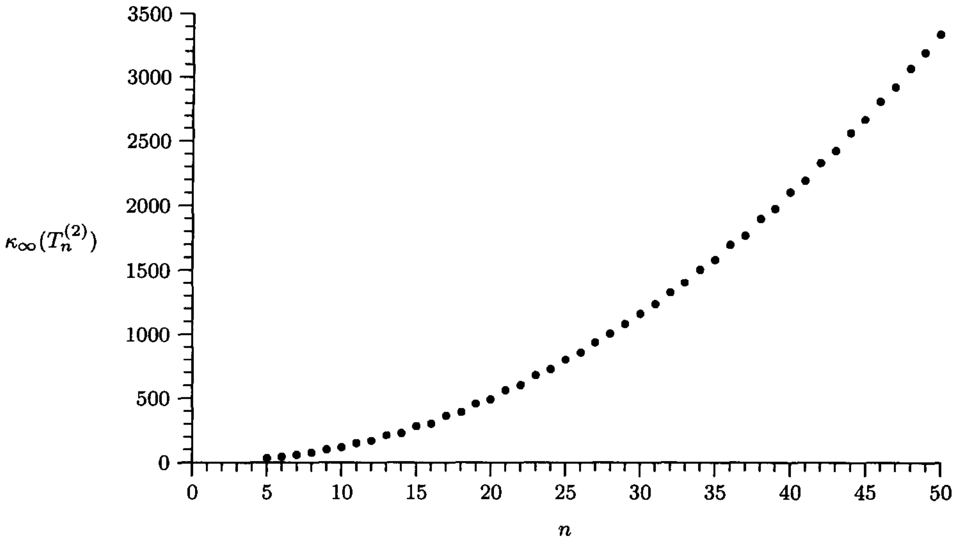

Figure 2. Condition number of the matrix (27).

LEMMA 3. *Let $\xi_1 \in \mathbb{R}$, $0 < \xi_1 < 1$, and $r \in \mathbb{N}$. Then there exist $\alpha > 0$ and $\xi_1 \le \xi_2 < 1$ such that*

$$n^r \xi_1^n \le \alpha^r \xi_2^n, \qquad \text{for all } n \ge 1.$$

PROOF. It is sufficient to use a value $\alpha > -r/(\mathrm{e} \log \xi_1)$ so that we can choose $\xi_2 = \xi_1 \mathrm{e}^{\mathrm{r}(\mathrm{e}\alpha)^{-1}} < 1$. Moreover, for all $x > 0$ one has $x\mathrm{e}^{-1} \ge \log \mathrm{x}$ (the equality holds only for $x = \mathrm{e}$). It follows that $n(\alpha \mathrm{e})^{-1} \ge \log(\mathrm{n}\alpha^{-1})$, that is $\mathrm{e}^{\mathrm{n}(\alpha\mathrm{e})^{-1}} \ge \mathrm{n}\alpha^{-1}$, and hence

$$\alpha^r \xi_2^n = \left(\alpha \mathrm{e}^{\mathrm{n}(\mathrm{e}\alpha)^{-1}}\right)^r \xi_1^n \ge n^r \xi_1^n. \qquad \blacksquare$$

Let us now define the following two strictly lower triangular matrices:

$$C_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \dots & 1 & 0 \end{pmatrix}_{n\times n}, \qquad \Delta_n = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \gamma & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \gamma^{n-1} & \dots & \gamma & 0 \end{pmatrix}_{n\times n}$$

Then, the next result holds true.

THEOREM 4. *Let $\{T_n\}$ be a family of nonsingular Toeplitz band matrices defined as in (1), and suppose that the associated polynomial $p(z)$ is of type $(m-s, s, k)$ with simple zeros of unit modulus. Then the matrix $|T_n^{-1}|$ can be bounded componentwise as follows:*

1. $|T_n^{-1}| \le \alpha(I_n + \Delta_n + \Delta_n^\top)$, *when* $s = 0$;
2. $|T_n^{-1}| \le \alpha(I_n + C_n + \Delta_n^\top)$, *when* $s > 0$,

where $\alpha > 0$ and $0 < \gamma < 1$ can be chosen independently of n.

PROOF. From (14), it follows that

$$|T_n^{-1}| \le |U_n^{-1}|\,|L_n^{-1}| + |H_n|,$$

so that we obtain the result by considering upper bounds for the elements on the right-hand side. From the arguments used in the proofs of Theorems 1 and 2, and Lemmas 2 and 3, it follows that there exist $0 < \gamma < 1$ and positive scalars α_1, α_2, α_3, all independent of n, such that

$$|U_n^{-1}| \leq \alpha_1 \left(I_n + \Delta_n^\top\right),$$
$$|L_n^{-1}| \leq \alpha_2 \begin{cases} (I_n + \Delta_n), & \text{if } s = 0, \\ (I_n + C_n), & \text{if } s \neq 0, \end{cases}$$
$$|H_n| \leq \alpha_3 \begin{cases} S_n, & \text{if } s = 0, \\ N_n, & \text{if } s \neq 0, \end{cases}$$

where $S_n = (\gamma^{i+j-2})_{i,j=1,\dots,n}$ and $N_n = (\gamma^{j-1})_{i,j=1,\dots,n}$. From the last relation, it follows that

$$|H_n| \leq \alpha_3 \begin{cases} \left(I_n + \Delta_n + \Delta_n^\top\right) & \text{if } s = 0, \\ \left(I_n + C_n + \Delta_n^\top\right), & \text{if } s \neq 0, \end{cases}$$

which is in the desired form. Moreover, one has

$$\Delta_n^\top C_n = \begin{pmatrix} \sum_{i=1}^{n-1} \gamma^i & \sum_{i=2}^{n-1} \gamma^i & \sum_{i=3}^{n-1} \gamma^i & \dots & \gamma^{n-1} & 0 \\ \sum_{i=1}^{n-2} \gamma^i & \sum_{i=1}^{n-2} \gamma^i & \sum_{i=2}^{n-2} \gamma^i & \dots & \gamma^{n-2} & 0 \\ \sum_{i=1}^{n-3} \gamma^i & \sum_{i=1}^{n-3} \gamma^i & \sum_{i=1}^{n-3} \gamma^i & \dots & \gamma^{n-3} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \gamma & \gamma & \gamma & \dots & \gamma & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$
$$\leq \frac{\gamma}{1-\gamma} \left(I_n + C_n + \Delta_n^\top\right),$$

where the following bound:

$$\sum_{i=r}^{s} \gamma^i = \gamma^r \frac{1 - \gamma^{s-r+1}}{1-\gamma} \leq \frac{\gamma^r}{1-\gamma}, \qquad r, s = 1, \dots, n-1,$$

is used in order to obtain the last inequality. Similarly, it is possible to obtain

$$\Delta_n^\top \Delta_n \leq \frac{\gamma^2}{1-\gamma^2} \left(I_n + \Delta_n + \Delta_n^\top\right).$$

Then one has

$$|T_n^{-1}| \leq \alpha_1\alpha_2 \left(I_n + \Delta_n^\top\right)\left(I_n + \Delta_n\right) + \alpha_3 \left(I_n + \Delta_n + \Delta_n^\top\right)$$
$$\leq \left(\frac{\alpha_1\alpha_2}{1-\gamma^2} + \alpha_3\right)\left(I_n + \Delta_n + \Delta_n^\top\right), \qquad \text{if } s = 0,$$

and

$$|T_n^{-1}| \leq \alpha_1\alpha_2 \left(I_n + \Delta_n^\top\right)\left(I_n + C_n\right) + \alpha_3 \left(I_n + C_n + \Delta_n^\top\right)$$
$$\leq \left(\frac{\alpha_1\alpha_2}{1-\gamma} + \alpha_3\right)\left(I_n + C_n + \Delta_n^\top\right), \qquad \text{if } s \neq 0.$$

The thesis then follows by setting

$$\alpha = \begin{cases} \alpha_1\alpha_2(1-\gamma^2)^{-1} + \alpha_3, & \text{when } s = 0, \\ \alpha_1\alpha_2(1-\gamma)^{-1} + \alpha_3, & \text{when } s \neq 0. \end{cases}$$

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