Groups and Graphs
Section IV: the zig-zag product

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THE ZIG-ZAG PRODUCT

The zig-zag product is a technique for composing graphs, developed by Reingold, Vadhan and Wigderson, that allow recursive constructions of regular graphs of fixed degree, which for appropriate parameters yield families of expanders.

The nature of this construction is entirely graph theoretical. But some relations with Cayle graph are in a sense natural.

The zig-zag product is the most important general way for the construction of families of expanders that may not be Cayley graphs; as such it avoids use of algebraic groups, representations, number theory...

Let $G = (V_G, E_G)$ be a d-regular (multi)-graph,
$H = (V_H, E_H)$ a graph on d vertices.

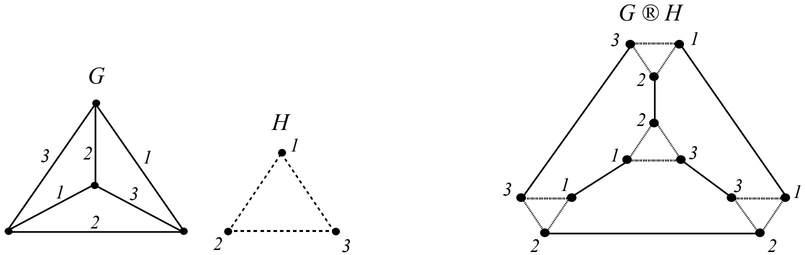
Write $V_H = \{1, 2, \ldots, d\}$. Then, to every $v \in V_G$ assigne a labeling $v(1), v(2), \ldots, v(d)$ to the set of edges incident to v in G.

The *R-product* $G \circledR H$ is the graph with vertex set $V_G \times V_H$ and two type of edges:

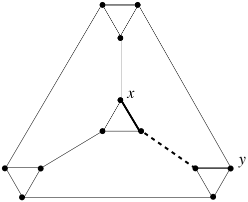
vertices $(u, i), (w, j) \in V_G \times V_H$ are adjacent if

- $v = w$ and $\{i, j\} \in E_H$ (H-type edge)
- $v \neq w$, $\{v, w\} \in V_G$, and $v(i) = w(j) = \{v, w\}$ (G-type edges)

This product depends on the labelings $v(i)$. However, it is always the case that $G \circledR H$ is a graph (possibly having loops, if G has). If H is k-regular, then $G \circledR H$ is $(k + 1)$-regular graph on $|V_G|d$ vertices example: a R-product of $G = K_4$ by $H = K_3$



- $G = (V_G, E_G)$ a d-regular (multi-)graph, $H = (V_H, E_H)$ a graph on d vertices.
- Form a R-product $G \circledR H$. A 3-path in $G \circledR H$ is a path of length 3 whose first and last edges are of H-type, while the central edge is of G-type.
- The zig-zag product $G ⓩ H$ has vertex set $V_G \times V_H$;
- $\{(v,i),(w,j)\}$ is an edge if there is a 3-path in $G \circledR H$ from (v,i) to (w,j)



$\{x, y\}$ is an edge in the zig-zag product

- The definition of $G \circledR H$ ensures that $G ⓩ H$ is in fact a graph. Moreover, if H is k-regular, then $G ⓩ H$ is k^2-regular

To describe the expanding properties of a zig-zag product, we need to fix some notation.

- For Γ a k-regular graph let $\delta = \delta(\Gamma) = \mu/d$, where μ is the largest absolute value of the eigenvalues $\neq k$ of $A(\Gamma)$.
- For $n, k \in \mathbb{N}$, $\epsilon > 0$, we say that Γ is a (n, k, ϵ)-graph if Γ is k-regular, on n vertices, and $\delta(\Gamma) \leq \epsilon$.

If, for $i \geq 1$, Γ_i is a (n_i, k, δ_i)-graph, with $\lim_{i\to\infty} n_i = \infty$ and there exists $\delta < 1$ such that $\delta_i \leq \delta$ for every $i \geq 1$, then $(\Gamma_i)_{i\geq 1}$ is a family of expanders.

Theorem (Reingold, Vadhan and Wigderson)

Let $G = (V_G, E_G)$ be a (n, d, δ_1)-graph, and $H = (V_H, E_H)$ be a (d, k, δ_2)-graph. Then a zig-zag product $G \textcircled{z} H$ is a $(nd, k^2, f(\delta_1, \delta_2))$-graph, with

(1) $f(\delta_1, \delta_2) < 1$ *if* $\delta_1, \delta_2 < 1$;

(2) $f(\delta_1, \delta_2) \leq \delta_1 + \delta_2$.

- From (1) it follows that if $(G_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ are families of graphs, with $(H_n)_{n\in\mathbb{N}}$ a family of expanders of degree k, and there exists $\delta < 1$ such that $\delta(G_n) \leq \delta$ for all $n \in \mathbb{N}$, then $(G_n \textcircled{z} H_n)_{n\in\mathbb{N}}$ is also a family of expanders, of degree k^2.
- Point (2) allows recursive construction of the members of families of expanders.

Fix $d \geq 2$ and H a $(d^4, d, 1/4)$-graph (may be found by direct search).
Set $G_1 = H^2$ (the multi-graph whose adjacency matrix is $A(H)^2$), and inductively

$$G_i = G_{i=1}^2 \text{ⓩ} H$$

Theorem (R.V.W.)

For every $n \geq 1$, G_n is a $(d^{4n}, d^2, 1/2)$-graph. Hence, $(G_n)_{n\geq 1}$ is a family of expanders.

Proof.

If Γ is a graph and $A = A(\Gamma)$, then the eigenvalues of A^2 are just the squares of those of A, hence if Γ is a (n, d, δ)-graph, Γ^2 is a (n, d^2, δ^2)-multigraph.
Hence, G_1 is a $(d^4, d^2, 1/2)$-multigraph.
Assume that, for $n \geq 1$, G_n is a $(d^{4n}, d^2, 1/2)$-graph. Then G_n^2 is a $(d^{4n}, d^4, 1/4)$-graph. By point (2) of the zig-zag Theorem, G_{n+1} is a (d^{4n+1}, d^4, δ)-graph, where $\delta \leq 1/4 + 1/4 = 1/2$. □

The connection with Cayely graphs is given by the following

Theorem (Alon, Lubotzky, Wigderson)

Let $G = B \rtimes H$ be a semidirect-product of finite groups B and H. Suppose that there exists $x \in B$ such that its H-orbit $x^H = \{x^h \mid h \in H\}$ is a symmetrical set of generators of B, and let T be a set of generators of H. Then

- $R = \{(x, 1)\} \cup \{(1, h) \mid h \in T\}$ *is a set of generators of G, and*

$$\Gamma[G, R] = \Gamma[B, x^H] \circledR \Gamma[H, T]$$

- $S = \{(x^{t^{-1}}), ts) \mid t, s \in T\}$ *is a set of generators of G, and*

$$\Gamma[G, S] = \Gamma[B, x^H] \textcircled{z} \Gamma[H, T].$$

Example 1. For every prime $p \geq 3$ let $H_p = SL(2,p)$, T_p the set of generators of H_p, with $|T_p| = 3$, such that $\Gamma[H_p, T_p]$ is a family of expanders.

Let $B_p = GF(p)^2$ be the natural module for H_p (as additive group). Then, H_p acts transitively on $B_p^* = B_p - \{0\}$, thus, for any $0 \neq x \in B_p$, $x^{H_p} = B_p^*$.

Thus, $\Gamma[B_p, x^H]$ is the complete graph K_{p^2}, whose eigenvalues are $p^2 - 1$ and -1 (with multiplicity $p^2 = 1$), hence

$$\delta([B_p, x^H]) = 1/p^2 < 1/2.$$

For each p ler $G_p = B_p \rtimes H_p$ and $S_p = \{(x^t, t^{-1}s, \mid t, s \in T_p\}$. Then, by the previous Theorem,

$$\Gamma[G_p, S_p] = K_{p^2} ⓩ \Gamma[H_p, T_p]$$

and, by the zig-zag Theorem, $\Gamma[G_p, S_p]$ is a family of expanders (of degree 9).

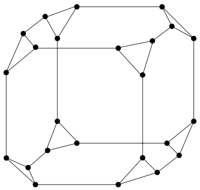
Example 2. Another case in which the the property that B is generated by a single H-conjugacy class x^H is ensured, isthat of a it standard wreath product $G = C_2 wr H$, where C_2 is cyclic group of order 2.

Then $G = B \rtimes H$, where B, the base of the wreath product is the direct product $B = Dir_{h \in H} X_h$, where $X_h \simeq C_2$, and H acts on B by regularly permuting the components X_h.

If x is a generator of $X_1 \simeq C_2$, then x^H is a set of generators of B (with $|x^H| = |H|$) and it is clearly symmetric. Then, if $|H| = d$, the graph $\Gamma[B, x^H]$ is the d-hypercube Q_d. Let T be a symmetric set of generators of H.

If $R = \{(x, 1)\} \cup \{(1, h) \mid h \in T\}$; then, by the previous Theorem

$$\Gamma[G, R] = Q_d \circledR \Gamma[H, T].$$



case H cyclic of order $d = 3$

If $S = \{(x^{t^{-1}}), ts) \mid t, s \in T\}$, then by the previous Theorem,

$$\Gamma[G, S] = Q_d \textcircled{z} \Gamma[H, T].$$

This, however, does not lead to families of expanders, because the normalized principal spectral gap (Q_d is bipartite, and this notion of parameter δ is needed) of Q_d is $2/d$ which tend to zero as $d \to \infty$

But zig-zag product may be adapted so that in the semidirect product $B \rtimes H$ what one needs is to have, as groups grow, a fixed number of H-orbits in B whose union generates B.

Theorem (A.L.W.)

let $G_n = B_n \rtimes H_n$ an infinite family of semi-direct products, with $(H_n)_{n \in \mathbb{N}}$ a family of Cayley expanders of degree d. Suppose that there exists $1 \leq c \in \mathbb{N}$, $0 < \beta < 1$ such that, for every n, there are c elements $b_{n,1}, \ldots, b_{n,c}$ in B_n, the union of whose H-orbits generate G and

$$\delta(\Gamma[B, b_{n,1}^H \cup \cdots \cup b_{n,c}^H]) \leq \beta.$$

Then $(G_n)_{n \in \mathbb{N}}$ is a family of Cayley expanders of degree cd^2.

Theorem (A.L.W.)

Let p be a prime. Then there exists $0 < \beta_p < 1$ such that, for every finite group H and every irreducible $GF(p)H$-module B, there exist two elements $a, b \in B$ such that

$$\delta(\Gamma[B, a^H \cup b^H]) \leq \beta_p.$$

Explicit algorithms to find such elements a, b are known only in some special cases.

If H, K are permutation groups, let $H \wr K$ denote the (permutational) wreath product. Then, for H a permutation group, define recursively

$$\wr^1 H = H \qquad \wr^{n+1} H = (\wr^n H) \wr H.$$

Theorem (Rozenman, Shalev, Wigderson)

There exists d sufficiently large, such that the family of wreath powers

$$G_n = \wr^n Alt(d)$$

is a family of Cayley expanders.

Proof uses the previous Theorem, the zig-zag Theorem, and the following results

Lemma (Kassabov)

For a sufficiently large d there exists a small system of generators T of $Alt(d)$ such that

$$\delta(\Gamma[Alt(d), T]) \leq 1/1000.$$

Lemma (Nikolov)

Every element of G_n is a commutator.

Apart of these, the rest of the proof is by (clever but) rather elementary group theory.

SOME READING

- **Expander graphs and their applications**, by S. Hoory, N. Linial and A. Wigderson, 2006.
- **Elementary Number Theory, Group Theory, and Ramanujan Graphs**, by G. Davidoff, P. Sarnak and A. Valette.
- **Growth in Groups: ideas and perspectives**, by H. A. Helfgott, 2012.
- **Expander graphs in pure and applied mathematics**, by A. Lubotzky, 2011.
- **Approximate Groups**. by L. van den Dries, 2015.
- T. Tao **blog**.