Nonlinear Expectations and Risk Measures

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Resume

How an investor measures his daily risky choices in a financial market? This a fundamentally important problem. According to the well–known expected utility theory under von Neumann and Morgenstern's axioms an agent prefers a random choice X to another one Y can be described by

$$E[u(X)] \geq E[u(Y)].$$

where u is a fixed increasing function, called utility function and $E[\cdot]$ is the expectation in a probability space $(\Omega, \mathcal{F}, P)$. But statistical tests and theoretical analysis show that this expectation has to be nonlinear in many important situations. This phenomenon becomes even more obvious in the situation where the statistic data is incomplete: it is often the case for investors in a financial market.

In this lectures, we will study this theory nonlinear expectation within the model of continuous time. Let $(\Omega, \mathcal{F})$ be a measurable space and let $L_b(\mathcal{F})$ be the space of $\mathcal{F}$–measurable and bounded real functions. A nonlinear expectation is a continuous functional

$$\mathcal{E}[\cdot] : L_b(\mathcal{F}) \longrightarrow R$$

that is order preserving (i.e., $\mathcal{E}[X_1] \geq \mathcal{E}[X_2], \quad \text{if} \quad X_1 \geq X_2$) and constant preserving (i.e., $\mathcal{E}[c] = c$).

If furthermore $\mathcal{E}[\cdot]$ is a linear functional, then it is a classical expectation under the (additive) probability measure P on $(\Omega, \mathcal{F})$ induced by

$$P(A) := \mathcal{E}[1_A], \; A \in \mathcal{F}. \tag{0.1}$$

In this case we have

$$\mathcal{E}[X] = \int_\Omega X(\omega) dP(\omega).$$

It is well–known that there is a 1–1 correspondence between linear expectation and additive probability measures. But this 1–1 correspondence fails in nonlinear situation. In general, given a nonlinear expectation $\mathcal{E}[\cdot]$, one can still derive a non additive probability measure P by (0.1). But there exist an infinite number of nonlinear expectations satisfying the same relation. There is a simple example: let $\mathcal{E}[\cdot]$ be a (linear or nonlinear) expectation. We define the following nonlinear expectation

$$\mathcal{E}_f[X] := f^{-1}(\mathcal{E}[f(X)]),$$

where f is an arbitrary strictly increasing and continuous function defined on R with $f(x) \equiv x$, for $x \in [0, 1]$. It is easy to check that $\mathcal{E}_f[1_A] = \mathcal{E}[1_A]$, for all $A \in \mathcal{F}$. Clearly, in nonlinear situations, the notion of expectation is more characteristic than that of non additive measures. Thus in nonlinear situations the notion of expectation is more characteristic than that of non additive measures. We refer to [4, Chen-Epstein2002] for a deeper investigation.

Let $\mathcal{E}$ and $\mathcal{E}'$ be two nonlinear expectations. $\mathcal{E}'$ is said to be dominated by $\mathcal{E}'$ if

$$\mathcal{E}'[X] - \mathcal{E}'[Y] \leq \mathcal{E}[X - Y], \; \forall X, Y. \tag{0.2}$$

$\mathcal{E}$ is said to be self–dominated, or subadditive, if $\mathcal{E}$ is dominated by itself. This notion of dominations will play an important role in this lecture.

In dynamic situation, a basic notion is the conditional expectation under a given filtration $\mathcal{F}_t$. This notion permits us to use the up–date information $\mathcal{F}_t$ to obtain the best estimate of a given random variable. The well-known martingale theory is fundamentally based on this notion (see [17, Dellachirie-Meyer1982]). As in linear situations, the conditional nonlinear expectation of a random variable X under $\mathcal{F}_t$ is an $\mathcal{F}_t$–measurable random variable $\mathcal{E}[X/\mathcal{F}_t]$ satisfying

$$\mathcal{E}[1_A \mathcal{E}[X/\mathcal{F}_t]] = \mathcal{E}[1_A X], \; \forall A \in \mathcal{F}_t.$$

An nonlinear expectation $\mathcal{E}[\cdot]$ is called $\mathcal{F}_t$–consistent if such $\mathcal{E}[X/\mathcal{F}_t]$ exists for all $t \geq 0$ and $X \in L_b(\mathcal{F})$. In nonlinear situations, there exist non–consistence expectations. If $\mathcal{E}[\cdot]$ is $\mathcal{F}_t$–consistent, we then can develope the related nonlinear martingale theory, parallel to the classical one.

In this topic the following problems are theoretically meaningful and practically important:

P1. Can we find a simple mechanism, which enable us to generate a large kind of filtration–consistent nonlinear expectation?

P2. For a given filtration consistent nonlinear expectation, is there a simple mechanism that determines the value of this expectation?

Problem **P1** was investigated in [37] where a notion of g–expectation was introduced under the framework of the natural filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ generated by a d–dimensional Brownian motion $(B_t)_{0\leq t\leq T}$ in a probability space $(\Omega, \mathcal{F}, P)$. It is defined as follows. For each $\mathcal{F}_T$ measurable and L^2–integrable random variable X, we solve the following BSDE:

$$\begin{aligned} -dY_t^X &= g(t, Z_t^X)dt - Z_t^X B_t, \quad t \in [0,T], \\ Y_T^X &= X. \end{aligned} \tag{0.3}$$

Here the mechanism is the function $g : (\omega, t, z) \in \Omega \times [0,T] \times R^d \longmapsto R$. It satisfies the usual conditions for BSDE, i.e., Lipschitz and Linear growth in z and $\mathcal{F}_t$–adapted. In addition we assume that $g(t,0) \equiv 0$. The g–expectation of X is defined by

$$\mathcal{E}_g[X] := Y_0^X.$$

We can check that this is an $\mathcal{F}_t$–consistent nonlinear expectation. In fact the corresponding conditional g–expectation of X given $\mathcal{F}_t$ is nothing else but $\mathcal{E}_g[X|\mathcal{F}_t] = Y_t^X$. It is worth to point out that the expectation $E_Q[\cdot]$ under the probability Q derived via the well–known Girsanov transforation

$$\frac{dQ}{dP} = \exp\left\{\int_0^T b_s B_s - \frac{1}{2}\int_0^T |b_s|^2 ds\right\}$$

is in fact a linear case of the g–expectation in which $g(t,z) = \langle b_t, z\rangle$. In the case where g is nonlinear in z, the notion of g–expectations can be considered as a nonlinear Girsanov transformation. Thus a large kind of $\mathcal{F}_t$–consistent nonlinear expectation can be generated by a simple mechanism g. Once this function g is obtained, then the corresponding nonlinear expectation is uniquely determined by solving BSDE (0.3). We recall that in recent 10 years many numerical methods, algorithms and the related numerical analysis i.e., convergence and convergence rate ...

For a $\mathcal{F}_t$–consistent nonlinear expectation, one can introduce the notion of nonlinear martingales, submartingales and supermartingales. It is then natural to ask whether the abundant results in the classical martingale theory have their counterparts under the framework of g–expectations. Many results have been obtained in this direction, among them the decomposition theorem of g–super or submartingales of Doob–Meyer's type has been proved for square–integrable situation by [38] (Peng1999) and [8], [10], [9].

A natural question closely related to Problem **P2** is: is the notion of g–expectations large enough to cover all regular $\mathcal{F}_t$–consistent nonlinear expectation? In recent lecture [5] we have the following result: if an $\mathcal{F}_t$–consistent nonlinear expectation $\mathcal{E}$ is $\mathcal{E}_{g^\mu}$ dominated, with $g^\mu(z) := \mu|z|$, for some large enough $\mu > 0$, then there exists a unique function g such that $\mathcal{E}[X] = \mathcal{E}_g[X]$, for all X. Nonlinear Doob–Meyer decomposition mentioned above plays a crucial role in the proof of this result.

But on the other hand, we will show that $\mathcal{E}_g[\cdot]$ is a quasi nonlinear expectation, i.e., the fully nonlinear situation can not be covered. Thus to solve Problem **P2**, we must find a new mechanism to generate a wider kind of nonlinear expectations.

In this lecture we will use a nonlinear Markov semigroup (or Markov chain) $(\mathcal{T}_t)_{t\geq 0}$ to generate an filtration–consistent nonlinear expectation $\mathcal{E}[\cdot]$. In other words, the infinitesimal generator $\mathcal{A}$ of $(\mathcal{T}_t)_{t\geq 0}$ is the generator of the corresponding nonlinear expectations. In this situation, if $\mathcal{A}$ is quasilinear (resp. fully nonlinear) then $\mathcal{E}[\cdot]$ is also quasilinear (resp. fully nonlinear). Briefly, our procedure is as follows:

1. We use a self–dominated nonlinear Markov semigroup $\mathcal{T}_t^*$ to a self–dominated and $\mathcal{F}_t$–consistent nonlinear expectation $\mathcal{E}^*$. In this step, we will introduce an extension of Kolmogorov consistent theorem for a family finite dimensional nonlinear distributions are induced by the Markov semigroup $\mathcal{T}_t^*$. The condition of self domination of $\mathcal{T}^*$ permits us to induce a norm under which $\mathcal{E}^*[\cdot]$ and $\mathcal{E}^*[\cdot|\mathcal{F}_t]$ are continuous.

2. For an arbitrary $\mathcal{T}_t^*$–dominated Markov semigroup $\mathcal{T}_t$ we can apply the same topology induced by $\mathcal{T}^*$ to generate the corresponding $\mathcal{F}_t$–consistent nonlinear expectation $\mathcal{E}[\cdot]$ which is $\mathcal{E}^*$–dominated. This $\mathcal{E}[\cdot]$ is therefor continuous under the given norm.

Let $g(z)$, $z \in R^d$ be a real Lipschitz function with Lipschitz constant $\mu > 0$. Then $\mathcal{E}_g$ is $\mathcal{E}_{g^\mu}$ dominated. So is the related nonlinear Markov chains. This implies that a large part of g–expectations can be also generated by the above approach. In this lecture we will also give some typical class of fully nonlinear Markov semigroups. They are either self dominated or dominated by some other self dominated fully nonlinear Markov semigroups. Thus the way to generate filtration consistent nonlinear expectations is largely extended.

On the other hand, since the classical linear Markov semigroup are self dominated. Thus they are also within our new framework. In fact in this special situation this method corresponds the classical L^1 theory. We recall that the notion of g–expectations is essentially an L^2–theory.

Another advantage of this domination approach is that, unlike in BSDE theory, no prior probability space is required. In fact, the continuity and completeness of the generated nonlinear expectation is under the norm induced by the given self dominated Markov semigroup. This constitute a new 'probability space'.

We will also study the existence and uniqueness of BSDE under this new "probability space". This extends BSDE theory to fully nonlinear situations.

Contents

- Existence and Uniqueness of BSDE
- Comparison theorem
- g–expectations
- Girsanov transformation
- Nonlinear Feynman–Kac formula
- Nonlinear Doob–Meyer decompositions
- Inverse problem: how to find the function g?

3. Fully nonlinear situations

- Distributions of Random variables and stochastic processes
- Family of finite–dimensional nonlinear distributions of a process
- Generalized Kolmogorov consistent theorem
- Nonlinear Markov Chains
- Nonlinear expectation generated by Nonlinear Markov Chains
- Conditional nonlinear expectations under $\mathcal{F}_t$

4. Applications to risk measures in finance

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