Optimal Investment in Incomplete Financial Markets

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Abstract

We give a review of classical and recent results on maximization of expected utility for an investor who has the possibility of trading in a financial market. Emphasis will be given to the duality theory related to this convex optimization problem.

For expository reasons we first consider the classical case where the underlying probability space Ω is finite. This setting has the advantage that the technical difficulties of the proofs are reduced to a minimum, which allows for a clearer insight into the basic ideas, in particular the crucial role played by the Legendre-transform. In this setting we state and prove an existence and uniqueness theorem for the optimal investment strategy, and its relation to the dual problem; the latter consists in finding an equivalent martingale measure optimal with respect to the conjugate of the utility function. We also discuss economic interpretations of these theorems.

We then pass to the general case of an arbitrage-free financial market modeled by an $\mathbb{R}^d$-valued semi-martingale. In this case some regularity conditions have to be imposed in order to obtain an existence result for the primal problem of finding the optimal investment, as well as for a proper duality theory. It turns out that one may give a *necessary and sufficient* condition, namely a mild condition on the asymptotic behavior of the utility function, its so-called *reasonable asymptotic elasticity*. This property allows for an economic interpretation motivating the term "reasonable". The remarkable fact is that this regularity condition only pertains to the behavior of the utility function, while we do not have to impose any regularity conditions on

the stochastic process modeling the financial market (to be precise: of course, we have to require the arbitrage-freeness of this process in a proper sense; also we have to assume in one of the cases considered below that this process is locally bounded; but otherwise it may be an arbitrary $\mathbb{R}^d$-valued semi-martingale).

We state two general existence and duality results pertaining to the setting of optimizing expected utility of terminal consumption. We also survey some of the ramifications of these results allowing for intermediate consumption, state-dependent utility, random endowment, non-smooth utility functions and transaction costs.

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