Mathematical foundation of turbulent viscous flows

CIME SUMMER SCHOOL

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Organizers

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Confirmed Speakers

P. Constantin (University of Chicago)
G. Gallavotti (Università di Roma 1, La Sapienza e Accademia dei Lincei)
A.V. Kazhikov (Lavrentyev Institute of Hydrodynamics, Novosibirsk)
Y. Meyer (Ecole Normale Supérieure de Cachan et Institut de France)
S. Ukai (Yokohama University)

Scientifical report

ABOUT THE THEME

The aim of this international summer is to contribute in the comprehension of some mathematical problems arising in the study of various model evolution equations of Fluid Mechanics and Gas Dynamic.

Since the derivation by L. Euler (and later on by H. Navier and G. Stokes), more than two centuries ago, of the Fluid mechanics and Gas Dynamics equations and, by L. Boltzmann, of the Boltzmann equation, much progress has been achieved on the understanding of these models, but many fundamental questions are still unresolved, like the existence and uniqueness of solutions to the corresponding equations and their asymptotic behaviour.

Not only do these physical models raise intriguing mathematical problems which are interesting for their own sake, but, considering the large number of applications to different fields (such as meteorology, astrophysics, aeronautics, thermodynamics or plasma physics) the study of these models from a pure mathematical point of view, plays a crucial role in Applied Mathematics. More generally, Nonlinear Partial Differential Equations have become such a vast subject that it gave raise to many fruitful connections with different sciences like Physics, Mechanics, Chemistry, Engineering Sciences with a considerable number of applications to industrial problems.

Kinetic models arise in various branches of Physics and the most famous one is certainly the Boltzmann equation that describes the evolution of rarefied gases. Since its introduction, this model equation is being increasingly used in lasers and plasma physics and its main industrial application concerns flights in a rarefied atmosphere (reentry problems).

As far as the Navier-Stokes equations they describe the motion of a viscous incompressible fluid and were introducted by H. Navier in France and G. Stokes in England, independently, in the first half of the nineteenth century. The nature of a turbulent motion of a fluid, an ocean for instance, or the creation of vortex inside it, are two typical problems related to these Navier-Stokes equations and are still far for being understood. One of the most intriguing unresolved questions concerning the Navier-Stokes equations (and closely related to turbulence phenomena) is the regularity of solutions to the initial value problem. More precisely, given a smooth data at time zero, will the solution of the Navier-Stokes equations continue to be smooth for all time? This question was posed in 1934 by J. Leray and is still without answer (neither in the positive nor in the negative).

The famous mathematician Steve Smale includes the uniqueness and regularity question for the Navier-Stokes equations as one of the 18 major open problems for this century (S. Smale: Mathematical problems for the next century, The Mathematical Intelligencer, 20 (2) (1998), 7–15.).

There is no uniqueness proof except for small time intervals and it has been questioned whether the Navier-Stokes equations really describe general flows. But there is no proof for nonuniqueness either.

Uniqueness of the solutions of the equations of motion is the cornerstone of classical determinism. If more than one solution were associated to the same initial data, the committed determinist will say that the space of the solutions is too large, beyond the real physical possibility, and that uniqueness can be restored if the unphysical solutions are excluded. On the other hand, anarchists will be happy to conclude that the laws of motion are not verified and that chaos reigns. More precisely, a non uniqueness result would represent such an insulting paradoxe to classical determinism, that the introduction of a more sophisticated model for the study of the motion of a viscous fluid would be certainly justified.

Thirty years ago, Marvin Shinbrot wrote :"*Without the d'Alembert (and other paradoxes), who would have thought it necessary to study more intricate models than the ideal fluid ? However, it is usually through paradoxes that mathematical work has the greatest influence on physics. In terms of existence and uniqueness theory, this means that the most important thing to discover is what is not true. When one proves the Navier-Stokes equations have solutions, the physicist yawns. If one can prove these solutions are not unique (say), he opens his eyes instead of his mouth. Thus, when we prove existence theorems, we are only telling the world where paradoxes are not and perhaps sweeping away some of the mist that surrounds the area where they are.*" (M. Shinbrot: Lectures on fluid mechanics, Gordon and Breach, Science Publishers, New York, London, Paris (1973)).

If the problem of the uniqueness relates to the predictive power aspect of the theory, the existence issue touches the question of the self-consistency of the physical model involved in the Navier-Stokes equations; if no solution exists, then the theory is empty.

In the nineteenth century the existence problems arising from mathematical physics were studied with the aim of finding exact solutions to the corresponding equations. This was only possible in particular cases. For instance, very few exact solutions of the Navier-Stokes equations were found, and almost all of these do not involve the specifically nonlinear aspects of the problem, since the corresponding nonlinear terms in the Navier-Stokes equations vanish.

In the twenty century the strategy changed. Instead of explicit formulas in particular cases, the problems were studied in all their generality. This led to the concept of weak solutions. The price to pay is that only the existence of the solutions can be ensured. In fact, the construction of weak solutions as the limit of a subsequence of approximations leaves open the possibility that there is more than one distinct limit, even for the same sequence of approximations.

The uniqueness question is among the most important unsolved problem in fluid mechanics. In fact, not only for the solutions of the Navier-Stokes equations, but also for the solutions of the Boltzmann equation of rarefied gases or Enskog equation of dense gases, such a uniqueness result is not available either.

A question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the Navier-Stokes equations blow up in finite time ? One may imagine that blow up of initially regular solutions never happens, or it becomes more likely as the initial norm increases, or that there is blow-up, but only on a very thin set, of probability zero.

Nobody knows the answer of all these problems and the Clay Fondation is offering a prize of one million dollars for it, as can be checked at http://www.claymath.org.

ABOUT THE SPEAKERS

The five speakers are internationally well-known in the field of Applied Mathematics. As far as the two european ones, Profs. Y. Meyer and G. Gallavotti, they both belong to two prestigious institutions, respectively the French Academy of Science (Institut de France) and the Italian Academy (Accademia Nazionale dei Lincei).

Prof. P. Constantin will give lectures on Euler equations of ideal incompressible fluids and models associated to them and discuss the blow up problem as well as on the Navier-Stokes equations of viscous fluids and will describe some of the major mathematical questions of turbulence theory.

Prof. G. Gallavotti will lecture on incompressible fluids and strange attractors

Prof. A. Kazikhov will introduce the audience to the theory of strong approximation of weak limits via the method of averaging with applications to Navier-Stokes equations

Prof. Y. Meyer's series of lectures will aim to illustrate the following paradigm: proving size estimates on solutions of nonlinear evolution equations often relies on understanding some unexpected cancellation properties. These cancellations will be either imposed on the initial condition, or be satisfied by the solution itself whenever it is localized in space or time variable. The four problems which will be studied are (1) the existence of global solutions (no blowup in finite time), (2) the rate of energy decay as time tends to infinity, (3) the localization of solutions with respect to the space variable, (4) the localization of the vorticity and the celebrated coherent structures.

Finally, Prof. S. Ukai will present the asymptotic analysis theory of fluid equations. More precisely he will present recent results, including the topics on the Cauchy Kovalevskaya technique for the Boltzmann-Grad limit of the Newtonian equation, the multi-scale analysis giving the compressible and incompressible limits of the Boltzmann equation and the analysis of their initial layers.

ABOUT THE AUDIENCE

We expect a huge number of European (in particular Italian and French) Ph. D. students as well as confirmed researchers from all over the world (in particular from Japan).