Unitary representations and complex analysis

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One of the great classical theorems of representation theory is the Borel-Weil Theorem, which realizes any irreducible unitary representation of a compact connected Lie group as the space of holomorphic sections of a complex line bundle on a homogeneous space. These lectures concern two closely related questions:

1. Why should something like the Borel-Weil Theorem be true?
2. Can it usefully be extended to other Lie groups?

The first of these questions leads almost immediately to the more general one of how one should "expect" to construct irreducible representations in general, and so to things like Mackey's theory of "induced" representations for Lie groups. A useful language for all of these matters is the "orbit method" of Kirillov and Kostant. I will briefly describe the orbit method, and how to cast the Borel-Weil theorem in that language.

The orbit-method formulation of the Borel-Weil Theorem extends extends easily (as a statement without a proof) to many other groups, suggesting at first that one should look for irreducible representations on higher Dolbeault cohomology groups with coefficients in complex line bundles on homogeneous spaces. A fundamental difficulty is that the homogeneous spaces are almost always noncompact. As a consequence, the Dolbeault cohomology spaces are generally infinite-dimensional, and are not easily related to Hilbert spaces.

I will discuss some of the (many) cases in which these difficulties have been overcome: typically by building a unitary representation in some unrelated manner, and managing by calculation to relate it to Dolbeault cohomology. This approach is at least aesthetically unsatisfactory, and there has been a great deal of work on finding more direct geometric constructions of unitary representations. One of the greatest successes is the work of Atiyah and Schmid constructing discrete series representations of reductive Lie groups on L^2 versions of Dolbeault cohomology.

I will conclude by suggesting an alternative framework for these questions, in which Dolbeault cohomology is replaced by Dolbeault cohomology with compact support. It seems reasonable to hope that in this way one can find straightforward and direct constructions of unitary representations, and that at the same time there will be simple connections with classical Dolbeault cohomology.