Lectures on Control Using Logic and Switching

Lecture 3

Coordination of Groups of Mobile Autonomous Agents Using Nearest Neighbor Rules

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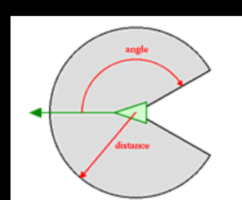
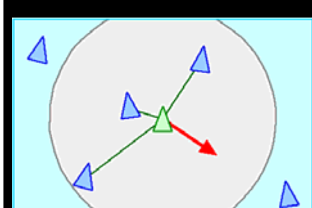
J. Lin {Archer} and Ali Jadbabaie

11:30 – 12:30
June 22, 2004

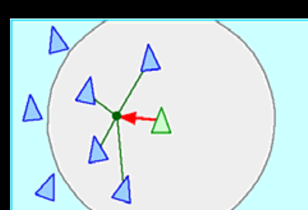
CRAIG REYNOLDS - 1987



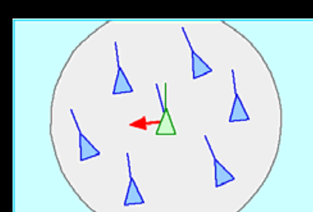
neighborhood



separation



alignment



cohesion

The problem which follows is motivated by simulation results reported in the paper:

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Novel Type of Phase Transition in a System of Self-Driven Particles

Tamás Vicsek,[1,2] András Czirók,[1] Eshel Ben-Jacob,[3] Inon Cohen,[3] and Ofer Shochet[3]
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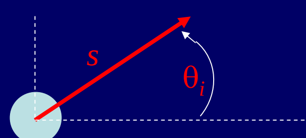
(Received 25 April 1994)

A simple model with a novel type of dynamics is introduced in order to investigate the emergence of self-ordered motion in systems of particles with biologically motivated interaction. In our model particles are driven with a constant absolute velocity and at each time step assume the average direction of motion of the particles in their neighborhood with some random perturbation (η) added. We present numerical evidence that this model results in a kinetic phase transition from no transport (zero average velocity, $|\mathbf{v}_a| = 0$) to finite net transport through spontaneous symmetry breaking of the rotational symmetry. The transition is continuous, since $|\mathbf{v}_a|$ is found to scale as $(\eta_c - \eta)^\beta$ with $\beta \simeq 0.45$.

PACS numbers: 87.10.+e, 64.60.–i

Vicsek et al. simulated a flock of *n* agents {particles} all moving in the plane at the same speed *s*, but with different headings $\theta_1, \theta_2, \ldots \theta_n$



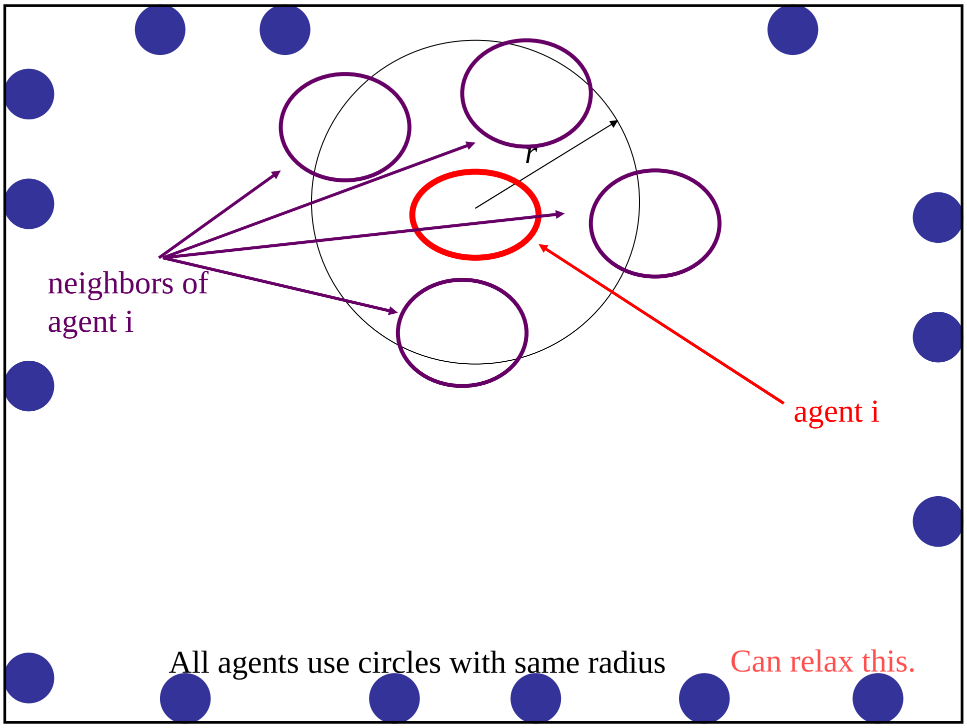
s = speed

θ_i = heading

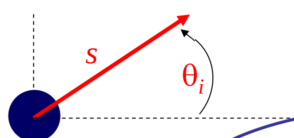
Each agent's heading is updated using a local rule based on the average of its own current heading plus the headings of its "neighbors."

Vicsek's simulations demonstrate that these nearest neighbor rules can cause all agents to eventually move in the same direction despite the absence of centralized coordination and despite the fact that each agent's set of neighbors changes with time.

The aim of this talk is provide a theoretical explanation for this observed behavior.



HEADING UPDATE EQUATIONS



s = speed

θ_i = heading

$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t)\right)$$

$\mathcal{N}_i(t)$ = set of indices of agents i's "neighbors" at time t

$n_i(t)$ = number of indices in $\mathcal{N}_i(t)$

Average at time t of headings of neighbors of agent i with its own heading.

NEIGHBOR CONFIGURATIONS

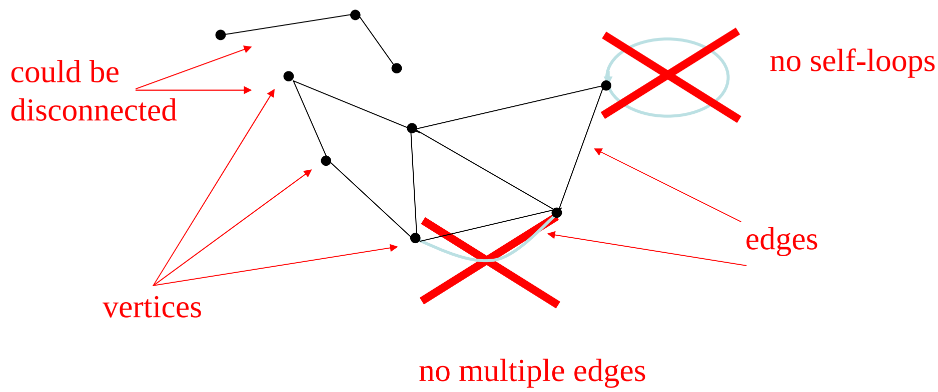
$\mathcal{P}$ = index set of all possible neighbor configurations.

$\mathcal{V}$ = agent index set $=\{1, 2, \ldots, n\}$

For each $p \in \mathcal{P}$

$\mathbb{G}_p = \{\mathcal{V}, \mathcal{E}_p\}$ - a simple graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}_p$

$(i, j) \in \mathcal{E}_p \Leftrightarrow$ agents i and j are neighbors in configuration p

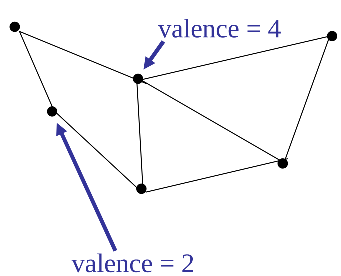


Matrix Representation of $\mathbb{G}_p = \{\mathcal{V}, \mathcal{E}_p\}$

adjacency matrix $A_p = [a_{ij}(p)]_{n\times n}$

$a_{ij}(p) = 1$ if i and j are neighbors
$a_{ij}(p) = 0$ otherwise

D_p = diagonal $\{d_1(p), d_2(p), \ldots., d_n(p)\}_{n\times n}$ $d_i(p)$ = valence of vertex i



State Space Equation

adjacency matrix $A_p = [a_{ij}(p)]_{n\times n}$

$a_{ij}(p) = 1$ if i and j are neighbors
$a_{ij}(p) = 0$ otherwise

D_p = diagonal $\{d_1(p), d_2(p), \ldots, d_n(p)\}_{n\times n}$ $d_i(p)$ = valence of vertex i

$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t)\right) \qquad \theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \vdots \\ \theta_n(t) \end{bmatrix}$$

$$\theta(t+1) = (I + D_{\sigma(t)})^{-1}(I + A_{\sigma(t)})\theta(t)$$

$\sigma(t)$ = index in $\mathcal{P}$ of neighbor configuration at time t

Yes we have yet another switched linear system here!

For convergence of the θ_i to a common steady state heading to occur along a particular trajectory, it turns out to be sufficient that the graphs $\mathbb{G}_p$ encountered along the way be connected. More precisely……………….

Let $\mathcal{P}_C$ denote the set of indices in $\mathcal{P}$ which are the indices of the connected graphs in $\{\mathbb{G}_p : p \in \mathcal{P}\}$.

THEOREM

If along a particular trajectory, σ takes values only in $\mathcal{P}_C$, then the θ_i all converge to a common steady-state heading θ_{ss} whose value depends on σ and the initial values of the θ_i.

Connectivity is not necessary for convergence to a common heading. In fact one can construct convergent trajectories along which connected graphs are never encountered.

How does one go about proving this theorem?

Quadratic Lyapunov Functions will not work here !

A switched linear system → $\theta(t+1) = (I + D_{\sigma(t)})^{-1}(I + A_{\sigma(t)})\theta(t)$

adjacency matrix $A_p = [a_{ij}(p)]_{n\times n}$

$a_{ij}(p) = 1$ if i and j are neighbors
$a_{ij}(p) = 0$ otherwise

D_p = diagonal $\{d_1(p), d_2(p), \ldots, d_n(p)\}_{n\times n}$ $d_i(p)$ = valence of vertex i

$$\boxed{F_p = (I + D_p)^{-1}(I + A_p), \quad p \in \mathcal{P}}$$

1. non-negative entries

2. row sums all = 1

stochastic $F_p\mathbf{1} = \mathbf{1}$ $\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

3. positive diagonal entries

Suppose $p \in \mathcal{P}_C$; $\mathbb{G}_p$ is connected.

adjacency matrix $A_p = [a_{ij}(p)]_{n\times n}$

$a_{ij}(p) = 1$ if i and j are neighbors
$a_{ij}(p) = 0$ otherwise

D_p = diagonal $\{d_1(p), d_2(p), \ldots, d_n(p)\}_{n\times n}$ $d_i(p)$ = valence of vertex i

$$\boxed{F_p = (I + D_p)^{-1}(I + A_p), \quad p \in \mathcal{P}}$$

If e_i is the ith unit column vector in $\mathbb{R}^n$, then the row indices of the nonzero entries of $(I+A_p)^{n-1}e_i$ are the vertices of $\mathbb{G}_p$ reachable from vertex i.

In a connected graph, every vertex is reachable from every other vertex.

So $(I+A_p)^{n-1}e_i > 0$; that is, $(I+A_p)^{n-1}e_i$ has all positive entries

Since this must be true of every unit vector e_i, $(I+A_p)^{n-1} > 0$

Any non-negative matrix $M_{n\times n}$ such that $M^{n-1} > 0$ is called primitive

$\mathbb{G}_p$ connected implies $I + A_p$ primitive $\mathbb{G}_p$ connected implies F_p primitive

Suppose $p \in \mathcal{P}_C$; $\mathbb{G}_p$ is connected.

adjacency matrix $A_p = [a_{ij}(p)]_{n\times n}$

$a_{ij}(p) = 1$ if i and j are neighbors
$a_{ij}(p) = 0$ otherwise

D_p = diagonal $\{d_1(p), d_2(p), \ldots, d_n(p)\}_{n\times n}$ $d_i(p)$ = valence of vertex i

$$\boxed{F_p = (I + D_p)^{-1}(I + A_p), \quad p \in \mathcal{P}}$$

The eigenvalue of a primitive matrix $M_{n\times n}$ which is largest in magnitude, has multiplicity one and is the only eigenvalue possessing a positive eigenvector.

Since $F_p\mathbf{1} = \mathbf{1}$, this means that F_p must have one eigenvalue at 1 and all the rest must be smaller in magnitude.

Therefore $\lim_{i\to\infty} F_p^i = \mathbf{1}c$ for some row vector c; i.e., F_p is ergodic

$\mathbb{G}_p$ connected implies F_p ergodic $\mathbb{G}_p$ connected implies F_p primitive

Suppose $p \in \mathcal{P}_C$; $\mathbb{G}_p$ is connected.

adjacency matrix $A_p = [a_{ij}(p)]_{n\times n}$

$a_{ij}(p) = 1$ if i and j are neighbors
$a_{ij}(p) = 0$ otherwise

D_p = diagonal $\{d_1(p), d_2(p), \ldots, d_n(p)\}_{n\times n}$ $d_i(p)$ = valence of vertex i

$$\boxed{F_p = (I + D_p)^{-1}(I + A_p), \quad p \in \mathcal{P}}$$

Wolfowitz [1963] Let $M_1, M_2, \cdots, M_m$ be a finite set of ergodic matrices with the property that every finite product $M_{i_1}M_{i_2}\cdots M_{i_k}$ is also ergodic. Then every infinite product $M_{i_1}M_{i_2}\cdots$ converges to a rank one matrix of the form $\mathbf{1}c$ for some row vector c.

$\mathbb{G}_p$ connected implies F_p ergodic

If along a particular trajectory, σ takes values only in $\mathcal{P}_C$, then the θ_i all converge to a common steady-state heading θ_{ss} whose value depends on σ and the initial values of the θ_i.

$$\theta(t+1) = F_{\sigma(t)}\theta(t) \qquad \theta(t) = (F_{\sigma(t-1)}F_{\sigma(t-1)}\cdots F_{\sigma(1)}F_{\sigma(0)})\theta(0)$$

Using the fact that all F_p, $p \in \mathcal{P}$, have positive diagonals, it can be shown that any finite length product $F_{p_1}F_{p_2}\cdots F_{p_k}$ will be ergodic provided at least one F_p in the product is ergodic.

So by Wolfowitz's Theorem, the product of the F_σ converges to $\mathbf{1}c$ for some row vector c.

$$\lim_{t\to\infty}\theta(t) = \theta_{ss}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \qquad \theta_{ss} = c\theta(0)$$

Wolfowitz [1963] Let $M_1, M_2, \cdots, M_m$ be a finite set of ergodic matrices with the property that every finite product $M_{i_1}M_{i_2}\cdots M_{i_k}$ is also ergodic. Then every infinite product $M_{i_1}M_{i_2}\cdots$ converges to a rank one matrix of the form $\mathbf{1}c$ for some row vector c.

$\mathbb{G}_p$ connected implies F_p ergodic

Why Not Use Quadratic Lyapunov Functions?

$$F_p\mathbf{1} = \mathbf{1}\mathbf{1}, \qquad p \in \mathcal{P} \qquad \mathbf{1} = \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

$$F_p = S\begin{bmatrix}M_p & 0\\ h_p & 1\end{bmatrix}S^{-1}$$

$$F_{p_i}\cdots F_{p_2}F_{p_1} = S\begin{bmatrix}M_{p_i}\cdots M_{p_2}M_{p_1} & 0\\ k_{pi} & 1\end{bmatrix}S^{-1}$$

$$\lim_{i\to\infty}F_{p_i}\cdots F_{p_2}F_{p_1} = \mathbf{1}c \qquad \Leftrightarrow \qquad \lim_{i\to\infty}M_{p_i}\cdots M_{p_2}M_{p_1} = 0$$

If the $p \in \mathcal{P}_C$, then the $\mathbb{G}_p$ are connected and the M_p are stable.

Suppose for some pos. def. P, that $M_p'PM_p - P$ pos. def. for all $p \in \mathcal{P}_C$

No such P exists for connected, 10 vertex $\mathbb{G}_p$ with 9 or 10 edges!

Weaker Assumption on σ

If along a particular trajectory, σ is a switching signal for which there exists a positive integer T large enough so that σ takes at least one value in $\mathcal{P}_C$ in each interval of length T, then the θ_i all converge to a common steady-state heading θ_{ss} whose value depends on σ and the initial values of the θ_i.

$$\theta(t+1) = F_{\sigma(t)}\theta(t) \qquad \theta(t) = \Phi(t,0)\theta(0)$$

$$\Phi(t,\tau) = F_{\sigma(t-1)}F_{\sigma(t-2)}\cdots F_{\sigma(\tau)}$$

Any finite length product $F_{p_1}F_{p_2}\cdots F_{p_k}$ is ergodic provided at least one F_p in the product is ergodic….True because the F_p have positive diagonals.

$$\underbrace{F_{\sigma(kT-1)}\cdots F_{\sigma((k-1)T)}}_{\Phi(kT,\,(k-1)T)}\cdots\cdots\cdots\underbrace{F_{\sigma(3T-1)}\cdots F_{\sigma(2T)}}_{\Phi(3T,\,2T)}\underbrace{F_{\sigma(2T-1)}\cdots F_{\sigma(T)}}_{\Phi(2T,\,T)}\underbrace{F_{\sigma(T-1)}\cdots F_{\sigma(0)}}_{\Phi(T,\,0)}$$

Therefore each block $\Phi(kT,(k-1)T)$, $k \geq 1$, is an ergodic matrix.

So by Wolfowitz's Theorem, the product of the $\Phi(kT,(k-1)T)$ converges to $\mathbf{1}c$ for some row vector c.

Weaker Assumption on σ

If along a particular trajectory, σ is a switching signal for which there exists a positive integer T large enough so that σ takes at least one value in $\mathcal{P}_C$ in each interval of length T, then the θ_i all converge to a common steady-state heading θ_{ss} whose value depends on σ and the initial values of the θ_i.

$$\theta(t) = \Phi(t,0)\theta(0)$$

$$\Phi(t,\tau) = F_{\sigma(t-1)}F_{\sigma(t-2)}\cdots F_{\sigma(\tau)}$$

$$\theta \to \mathbf{1}\theta_{ss}$$

$$\theta_{ss} = c\theta(0)$$

$$\lim_{k\to\infty}\left(\Phi(kT,0) - \mathbf{1}c\right) = 0$$

$$F_p\mathbf{1} = \mathbf{1},\ p \in \mathcal{P} \quad\longrightarrow\quad \Phi(t,\tau)\mathbf{1} = \mathbf{1}, \quad t \geq \tau \geq 0$$

$$\Phi(t,\tau)\mathbf{1}c = \mathbf{1}c, \quad t \geq \tau \geq 0$$

$$\boxed{\begin{aligned}\mathbf{1}c &= \Phi(t,k_tT)\mathbf{1}c, \quad t \geq 0\\ \Phi(t,0) &= \Phi(t,k_tT)\Phi(k_tT,0), \quad t \geq 0\end{aligned}}$$

k_t = largest value of k such that $kT \leq t$

$$\Phi(t,0) - \mathbf{1}c = \Phi(t,k_tT)\left(\Phi(k_tT,0) - \mathbf{1}c\right), \quad t \geq 0$$

bounded function of t

$$\boxed{\lim_{t\to\infty}\Phi(t,0) = \mathbf{1}c}$$

goes to 0 as $t \to \infty$

Generalization

By the union of a collection of simple graphs

$$\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$$

with the same vertex set $\mathcal{V}$, is meant the graph whose vertex set is $\mathcal{V}$ and whose edge set is the union of the edge sets of all the graphs in the collection.

Call such a collection jointly connected if the collection's union is a connected graph.

Lemma: If $\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$ *is jointly connected, then*

$$F_{p_1} F_{p_2} \cdots F_{p_m}$$

is ergodic

Still Weaker Assumption on σ

If along a particular trajectory, σ is a switching signal for which there exists a positive integer T large enough so that the collection of graphs encountered on each successive interval of length T is jointly connected, then the θ_i all converge to a common steady-state heading θ_{ss} whose value depends on σ and the initial values of the θ_i.

$$\theta(t+1) = F_{\sigma(t)}\theta(t) \qquad \theta(t) = \Phi(t,0)\theta(0)$$

$$\Phi(t,\tau) = F_{\sigma(t-1)}F_{\sigma(t-2)}\cdots F_{\sigma(\tau)}$$

$\{\mathbb{G}_{p_1}, \mathbb{G}_{p_2}, \ldots, \mathbb{G}_{p_m}\}$ jointly connected implies $F_{p_1} F_{p_2} \cdots F_{p_m}$ ergodic

$$\underbrace{F_{\sigma(kT-1)}\cdots F_{\sigma((k-1)T)}}_{\Phi(kT,\,(k-1)T)}\cdots\cdots\cdots\underbrace{F_{\sigma(3T-1)}\cdots F_{\sigma(2T)}}_{\Phi(3T,\,2T)}\underbrace{F_{\sigma(2T-1)}\cdots F_{\sigma(T)}}_{\Phi(2T,\,T)}\underbrace{F_{\sigma(T-1)}\cdots F_{\sigma(0)}}_{\Phi(T,\,0)}$$

Therefore each block $\Phi(kT,(k-1)T)$, $k \geq 1$, is an ergodic matrix.

So by Wolfowitz's Theorem, the product of the $\Phi(kT,(k-1)T)$ converges to $\mathbf{1}c$ for some row vector c.

Result

Suppose that along a particular trajectory, σ is a switching signal for which there exists an infinite sequence of contiguous, non-empty, bounded time intervals $[t_j, t_{j+1})$, $j \geq 0$, starting at $t_0 = 0$, with the property that on each such interval,

$$\{\mathbb{G}_{\sigma(t_j)}, \mathbb{G}_{\sigma(t_j+1)}, \ldots, \mathbb{G}_{\sigma(t_{j+1}-1)}\}$$

is a jointly connected collection. Then the θ_i all converge to a common steady-state heading θ_{ss} whose value depends on σ and the initial values of the θ_i.

SIMPLIFICATION

Can re-write Vicsek's heading update equations

$$\theta_i(t+1) = \frac{1}{1+n_i(t)}\left(\theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t)\right), \quad i \in \{1, 2, \ldots, n\}$$

as

$$\theta_i(t+1) = \theta_i(t) - \frac{1}{1+n_i(t)}\left(n_i(t)\theta_i(t) - \sum_{j \in \mathcal{N}_i(t)} \theta_j(t)\right), \quad i \in \{1, 2, \ldots, n\}$$

In state-space form

$$\theta(t+1) \quad = \quad \theta(t) - (I + D_{\sigma(t)})^{-1} L_{\sigma(t)} \theta(t)$$

Replace

$L_p = D_p - A_p, \quad p \in \mathcal{P}$ Laplacian of $\mathbb{G}_p$

D_p = diagonal valence matrix $\qquad$ A_p = adjacency matrix

$L_p \theta = 0$ means $\dot{\theta} = -(I + D(t))^{-1} e \qquad e = L(t)\theta$ of its neighbors

$$\theta(t+1) \quad = \quad \theta(t) - (1 + d)^{-1} L_{\sigma(t)} \theta(t)$$

1. Using definitions of D_p, A_p and of suitably defined incidence matrix B_p of $\mathbb{G}_p$ can show that $L_p = B_p B_p'$
2. From definition of L_p, it follows that $L_p\mathbf{1} = 0$.
3. Multiplicity of eigenvalue 0 of L_p equals number of connected components of $\mathbb{G}_p$.
4. For $p \in \mathcal{P}_C$, all eigenvalues of L_p are less than or equal to n

For each $p \in \mathcal{P}_C$, the singular values of $(1 + d)^{-1}L_p$ are all strict smaller than the largest {which is 1} if $d \geq n$.

Thus uniform heading is achieved asymptotically along any trajectory on which σ takes values only in $\mathcal{P}_C$.

$L_p = D_p - A_p, \quad p \in \mathcal{P}$ Laplacian of $\mathbb{G}_p$

$$\theta(t+1) = \theta(t) - (1+d)^{-1} L_{\sigma(t)} \theta(t)$$

LEADER FOLLOWING

Now suppose a group consisting of n of Vicsek's agents plus one additional agent labeled 0 which acts as the group's leader.

$$\theta_i(t+1) = \frac{1}{1 + n_i(t) + b_i(t)} \left(\theta_i(t) + \sum_{j \in \mathcal{N}_i(t)} \theta_j(t) + b_i(t)\theta_0 \right)$$

$\mathcal{N}_i(t)$ = set of indices of agents i's neighbors at time t

$n_i(t)$ = number of indices in $\mathcal{N}_i(t)$

θ_0 = heading of agent 0 = constant

$b_i(t)$ = 1 if the leader is in agent i's neighborhood, and = 0 otherwise

LEADER FOLLOWING

$$\theta_i(t+1) = \frac{1}{1+n_i(t)+b_i(t)}\left(\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t) + b_i(t)\theta_0\right)$$

$\mathcal{P}^*$ = index set of all possible neighbor configurations on n +1 agents.

$$\theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \vdots \\ \theta_n(t) \end{bmatrix}$$

$\mathbb{G}^*_p$ = graph on n + 1 vertices for neighbor relationship $p \in \mathcal{P}^*$

$\mathbb{G}_p$ = subgraph of $\mathbb{G}^*_p$ obtained by deleting vertex 0 and all edges incident on vertex 0.

$$\theta(t+1) \quad = \quad (I + D_{\sigma(t)} + B_{\sigma(t)})^{-1}((I + A_{\sigma(t)})\theta(t) + B_{\sigma(t)}\mathbf{1}\theta_0)$$

σ(t) = index in $\mathcal{P}^*$ of neighbor configuration at time t

B_p = diag. matrix whose ith diag. element = 1 if agent 0 is a neighbor of agent i and 0 otherwise, $p \in \mathcal{P}^*$.

LEADER FOLLOWING

THEOREM

If along a particular trajectory, σ is a switching signal for which there exists a positive integer T large enough so that σ takes at least one value in $\mathcal{P}^*_C$ in each interval of length T, then the θ_i all converge to θ_0 .

$\mathcal{P}^*$ = index set of all possible neighbor configurations on n +1 agents.

$\mathbb{G}^*_p$ = graph on n + 1 vertices for neighbor relationship $p \in \mathcal{P}^*$

$\mathbb{G}_p$ = subgraph of $\mathbb{G}^*_p$ obtained by deleting vertex 0 and all edges incident on vertex 0.

$$\theta(t+1) \quad = \quad (I + D_{\sigma(t)} + B_{\sigma(t)})^{-1}((I + A_{\sigma(t)})\theta(t) + B_{\sigma(t)}\mathbf{1}\theta_0)$$

Let $\mathcal{P}^*_C$ denote the set of indices in $\mathcal{P}^*$ which are the indices of the connected graphs in $\{\mathbb{G}^*_p \colon p \in \mathcal{P}^*\}$.

Leader Following in Continuous Time

$$\theta_i(t+1) = \frac{1}{1+n_i(t)+b_i(t)}\left(\theta_i(t) + \sum_{j\in\mathcal{N}_i(t)} \theta_j(t) + b_i(t)\theta_0\right)$$

$$u_i(t) = -\frac{1}{1+n_i(t)+b_i(t)}\left((n_i(t)-b_i(t))\theta_i(t) - \sum_{j\in\mathcal{N}_i(t)} \theta_j(t) - b_i(t)\theta_0\right)$$

$$\theta_i(t+1) = \theta_i(t) + u_i(t)$$

$$\dot{\theta}_i = u_i$$

$\tau_D =$ dwell time

$t \in [\, t_{ik},\ t_{ik} + \tau_D)$

$$u_i(t) = -\frac{1}{1+n_i(t_{ik})+b_i(t_{ik})}\left((n_i(t_{ik})-b_i(t_{ik}))\theta_i(t) - \sum_{j\in\mathcal{N}_i(t_{ik})} \theta_j(t) - b_i(t_{ik})\theta_0\right)$$

Can also allow agents to use different dwell times provided they are rationally related.

Leader Following in Continuous Time

$$\dot{\theta} = -(I + D_\sigma + B_\sigma)^{-1}((L_\sigma + B_\sigma)\theta - B_\sigma\ \theta_0)$$

For $p \in \mathcal{P}^*$:
L_p = Laplacian of $\mathbb{G}_p$
D_p = diag. matrix of valences of $\mathbb{G}_p$
B_p = diag. matrix of 1's and 0's with $b_{ii} = 1$ iff $(i,0)$ is edge of $\mathbb{G}^*_p$

$\sigma : [0, \infty) \to \mathcal{P}^*$ = piecewise constant with dwell time τ_D

$$\dot{\theta}_i = u_i$$

$\tau_D =$ dwell time

$t \in [\, t_{ik},\ t_{ik} + \tau_D)$

$$u_i(t) = -\frac{1}{1+n_i(t_{ik})+b_i(t_{ik})}\left((n_i(t_{ik})-b_i(t_{ik}))\theta_i(t) - \sum_{j\in\mathcal{N}_i(t_{ik})} \theta_j(t) - b_i(t_{ik})\theta_0\right)$$

Leader Following in Continuous Time

$$\dot\theta = -(I + D_\sigma + B_\sigma)^{-1}((L_\sigma + B_\sigma)\theta - B_\sigma\ \theta_0)$$

For $p \in \mathcal{P}^*$:
L_p = Laplacian of $\mathbb{G}_p$
D_p = diag. matrix of valences of $\mathbb{G}_p$
B_p = diag. matrix of 1's and 0's with $b_{ii} = 1$ iff $(i,1)$ is edge of $\mathbb{G}^*_p$

$\sigma:[0, \infty) \to \mathcal{P}^*$ = piecewise constant with dwell time τ_D

THEOREM

Suppose that along along a particular trajectory, σ is a switching signal whose switching times $t_1, t_2, \ldots$ satisfy $t_{i+1} - t_i \geq \tau_D$, $i \geq 1$. If in addition there is an infinite subsequence of switching times at which σ's value is in $\mathcal{P}^*_C$, then the θ_i all converge to θ_0.

· The constraints on σ are significantly weaker than those imposed for the discrete-time case.

· The theorem holds for any positive dwell time.

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Lemma: For all $t > 0$,

$$\left\| e^{-(I+D_p+B_p)^{-1})(L_p+B_p)t} \right\| \leq 1, \qquad p \in \mathcal{P}^*$$

$$\left\| e^{-(I+D_p+B_p)^{-1})(L_p+B_p)t} \right\| < 1, \qquad p \in \mathcal{P}^*_C$$

where ||·|| denotes the induced matrix infinity norm.

Uniform bound: $\left\| e^{-(I+D_p+B_p)^{-1})(L_p+B_p)\tau_D} \right\| \leq \lambda < 1, \qquad p \in \mathcal{P}^*_C$

Observations

New data structures, models, etc.are needed to represent large groups of mobile autonomous agents at various degrees of granularity, for purposes of simulations, management, analysis and control.

Such representations will exploit tools from both graph theory and from the theory of dynamical systems

At least initially, individual agent descriptions using simple kinematic and dynamic models will suffice.

System complexity will stem more from the number of agent models being studied than from the detailed properties of the individual agent models.

New concepts of robustness, stability, etc are needed to understand such systems – to address issues such as cascade failure, security reliability, coordination, etc.