Random points, convex bodies, and approximation

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Abstract: Let K be a convex body in d-dimensional space, and choose n points, $x_1, ..., x_n$ randomly, independently, and uniformly from K. The convex hull of these points is a random polytope, to be denoted by K_n. It is clear that K_n gets closer and closer to K as n goes to infinity with high probability. The question is how good this approximation is. There has been a lot of research in this direction which is going to be the main topic of this lecture series. We will also investigate the (expected) number of k-dimensional faces of K_n, and compare best approximation with random approximation. It is also interesting to see whether the behaviour of random points and lattice points in convex bodies is similar or not.

1st Lecture. Following Rényi and Sulanke (1963) we determine the expectation $E(K, n)$ of the area of $K \setminus K_n$ when $K \subset \mathbb{R}^2$ is the unit square and when $K \subset \mathbb{R}^2$ is the unit disk. It turns out that the expectations are of order $(\log n)/n$ and of order $n^{-2/3}$, respectively. What's the reason for such a different behaviour? We are going to explain this in the first two lectures. A key notion is the function $v : K \to \mathbb{R}$ defined by

$$v(x) = \min\{\text{Vol}(K \cap H) : x \in H, H \text{ is a halfspace}\}.$$

Thus $v(x)$ is the volume of the smallest cap containing x. Define, further

$$K(v \leq t) = \{x \in K : v(x) \leq t\}.$$

This is a kind of inner parallel body to K which is often called "the wet part" of K since, assuming K is a 3-dimensional convex body containing t units of water, $K(v \leq t)$ is the part of the inside that gets wet when K is rotated in various positions.

2nd Lecture. Properties of the function v, and of the "wet part" $K(v \leq t)$ will be stated and proved. Observe that both v and $K(v \leq t)$ are invariant (or equivariant) under non-degenerate affine transformations, and so is $E(K, n)$. So we may assume that $\text{Vol}(K) = 1$. Under this condition we will show that, as n goes to infinity, the expectation $E(K, n)$ is of the same order as $K(v \leq 1/n)$.

This result can be used to determine the expectation $E(K, n)$ for smooth convex bodies and polytopes, even asymptotically as n gets large. I'll try to indicate how this can be done. Also, the expected number of vertices and facets of K_n can be determined.

3rd Lecture. The random polytope K_n approximates K better and better as n increases. The expectation of the missed volume $E(K, n)$, and the expected number of vertices of K_n are known. So one can compare best approximation with random approximation. Here best appoximation is meant the following way: Given the convex body $K \subset \mathbb{R}^d$ find a polytope $P \subset K$ with n vertices such that the missed volume $\mathrm{Vol}(K \setminus P)$ is minimal. This is a classical question in convexity theory. We will show that random approximation is almost as good as best approximation.

Time allowing we will consider the following question as well. Let $B(r) \subset \mathbb{R}^2$ denote the disk of radius r centred at the origin, r is large, and denote by P_r the convex hull of the integer points in $B(r)$. Clearly, P_r is a convex polygon. How many vertices does it have? How large is the missed area, that is, $\mathrm{Area}(B(r) \setminus P_r)$?

4th Lecture. For a convex body $K \subset \mathbb{R}^2$ let $p(n, K)$ denote the probability that n random, independent, and uniform points from K are in convex position, i.e., none of them lies in the convex hull of the others. Sylvester's question from 1864 is the special case $n = 4$. I am going to explain how one can determine the asymptotic behavior of $p(n, K)$. It turns out that, as n goes to infinity, $n^2 \sqrt[n]{p(n, K)}$ tends to a finite and positive limit. Moreover, the following result is true: if the n random points are in convex position, then their convex hull is, with high probability, very close to a fixed convex subset of K.

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