Random sets (in particular Boolean models)

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Introduction

Random sets are mathematical models to describe complex spatial data as they arise in modern applications in numerous form, as pictures, maps, digital images, etc. Whenever the geometric structure of an image is essential, a description by a set-valued random variable seems to be appropriate. Mathematically, a random set can simply be defined to be a measurable mapping Z from some abstract probability space $(\Omega, \mathbf{A}, \mathbb{P})$ into a class $\mathcal{F}$ of sets, where the latter is supplied with a σ-algebra $\mathbb{F}$. Here, on the one hand, $\mathcal{F}$ and $\mathbb{F}$ have to be reasonably large to represent structures occurring in practice at least approximately and to allow the basic geometric operations like intersection and union to be measurable operations within the class. On the other hand, $\mathbb{F}$ has to be reasonably small in order to have enough nontrivial examples of distributions on $(\mathcal{F}, \mathbb{F})$. A pair $(\mathcal{F}, \mathbb{F})$ which fulfills both requirements is given by the class $\mathcal{F}$ of all closed subsets of a topological space T and by the Borel σ-algebra $\mathbb{F} = \mathcal{B}(\mathcal{F})$ with respect to the topology of closed convergence on $\mathcal{F}$. Throughout the following, we will concentrate on this case. Random open sets can be defined and discussed in a similar manner, but random closed sets cover more cases which are of interest in applications, since they include random compact sets and, in particular, random finite or locally finite sets (simple point processes). Furthermore, we only work with random subsets $Z \subset \mathbb{R}^d$, other topological spaces T will only be mentioned occasionally. Some aspects of the theory remain unchanged in more general spaces, others make use of the vector space structure of $\mathbb{R}^d$ (or even the finite dimension).

Having clarified now the basic setting for our considerations, the challenge still remains to find a reasonably large class of distributions on $\mathcal{F}$ (we frequently suppress the σ-algebra, in the following), for example to allow some statistical analysis for random sets. In the classical situation of real random variables, various distributions can be constructed using the distribution function as a tool, but there is also the family of normal distributions which plays a central role and from which further distributions can be obtained by variation. In stochastic geometry, there is an analog of the distribution function (the capacity functional of a random closed set), but it cannot be used for the explicit construction of distributions, in general. However, there is a basic random set model, the Boolean model, which allows to calculate various geometric parameters and therefore can be used to fit a specific random set to given spatial data. Actually, Boolean models build a whole family of random sets which is still very rich in structure. Their definition is based on Poisson processes, which play a role in stochastic geometry comparable to the role of the normal distribution in classical statistics.

Boolean models can be classified according to their invariance properties. Those which are stationary and isotropic (thus invariant in distribution with respect to rigid motions) are the best studied ones and there is a large variety of formulas for them. Boolean models which are only stationary (invariant in distribution with respect to translations) have been the object of research for the last years. Boolean models without any invariance properties (or with only partial invariances) are the most complex ones and were studied only recently. In the following, we concentrate on Boolean models, as the basic random sets in stochastic geometry and as ingredients for more general classes of random sets. We present the results for Boolean models with increasing generality, first for stationary and isotropic models, then without the isotropy condition and finally without any invariance assumptions. The section headings are:

1. Random sets, particle processes and Boolean models
2. Mean values of additive functionals
3. Directional data, local densities, nonstationary Boolean models
4. Contact distributions

Since Boolean models arise as union sets of (Poisson) particle processes, geometric functionals compatible with unions are of particular interest (additive functionals) and expectation formulas for such functionals immediately lead to formulas of integral geometric type. For stationary and isotropic models, we will thus make use of kinematic formulas, the general case requires formulas from translative integral geometry. This part will therefore frequently refer to results explained in the contribution [S] by Rolf Schneider but will also be based on results for point processes (see the chapter [B] by Adrian Baddeley).

The use of integral geometric results has also some influence on the choice of the set class to start with. We will therefore often work with Boolean models having convex grains, but more general results will be mentioned, too.

1 Random sets, particle processes and Boolean models

In this section, we define Boolean models and explain their role in stochastic geometry. We begin however with introducing the two basic notions in stochastic geometry, random closed sets (RACS) and point processes of compact sets (particle processes).

1.1 Random closed sets

Since we already described the ideas behind the notion of a random closed set, we will be quite brief in this subsection and mainly collect the technical setup which we use in the following. From now on, $\mathcal{F}$ denotes the class of closed subsets of $\mathbb{R}^n$ (including the empty set $\emptyset$). Subclasses which we frequently use are the compact sets $\mathcal{C}$ and the convex bodies $\mathcal{K}$ (again both including $\emptyset$). We supply $\mathcal{F}$ with the σ-algebra $\mathbb{F}$ generated by the sets

$$\mathcal{F}_C, \qquad C \in \mathcal{C}.$$

Here and in the following we use the notation

$$\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \emptyset\}$$

for any subset $A \subset \mathbb{R}^n$; and we similarly define

$$\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \emptyset\}.$$

There are various other classes which generate the same σ-algebra $\mathbb{F}$, for example

$$\{\mathcal{F}^C : C \in \mathcal{C}\}, \quad \{\mathcal{F}_G : G \subset \mathbb{R}^d \text{ open}\}, \quad \{\mathcal{F}^G : G \subset \mathbb{R}^d \text{ open}\}.$$

Consequently, also $\{\mathcal{F}^C : C \in \mathcal{C}\} \cup \{\mathcal{F}_G : G \subset \mathbb{R}^d \text{ open}\}$ generates $\mathbb{F}$. This set class is of interest since it can serve as a sub-basis of a topology on $\mathcal{F}$, the **topology of closed convergence**. For a sequence $F_j \to F$, convergence in this topology means that each $x \in F$ is limit of a sequence $x_j \to x$, $x_j \in F_j$ (for almost all j) and that each limit point $x = \lim x_{j_k}$ of a (converging) subsequence $x_{j_k} \in F_{j_k}$ lies in F. As it turns out, $\mathbb{F}$ is the Borel σ-algebra with respect to this topology.

The subclasses $\mathcal{C}$ and $\mathcal{K}$, and others which arise later, are supplied with the induced σ-algebras. On $\mathcal{C}$ (and similarly on $\mathcal{K}$), the induced σ-algebra coincides with the Borel σ-algebra generated by the Hausdorff metric. Note however that the topology of closed convergence on $\mathcal{C}$ is weaker than the Hausdorff metric topology. For example, the sequence $B^n + kx, k \in \mathbb{N}$, of balls does not converge in the Hausdorff metric, but it converges in $\mathcal{F}$ (namely to $\emptyset$). Thus, $\mathcal{C}$ and $\mathcal{K}$ are neither closed nor open in $\mathcal{F}$, but they are Borel subsets. The latter is also true for all the subsets of $\mathcal{F}$ which come up later, without that we will mention it in all cases.

The choice of the σ-algebra (respectively the topology) on $\mathcal{F}$ was motivated by the desire to make the standard set transformations, which map closed sets into closed sets, continuous and therefore measurable. As it turns out, all geometric transformations which will occur are measurable, but only a few are continuous, the others have a semi-continuity property. We mention only $(F, F') \to F \cup F'$ (which is continuous), $(F, F') \to F \cap F'$ (which is upper semi-continuous) and $F \to \partial F$ (which is lower semi-continuous). The real- and measure-valued functionals on convex bodies which arise in integral geometry are mostly continuous and therefore measurable. This refers in particular to the continuous valuations like the intrinsic volumes, the curvature measures, the mixed measures and the mixed functionals. Their additive extensions to the convex ring $\mathcal{R}$ are no longer continuous but measurable (this is a deeper result using the existence of measurable selections). We will also use the set class

$$\mathcal{S} := \{F \in \mathcal{F} : F \cap K \in \mathcal{R}, \quad \text{for all } K \in \mathcal{K}\},$$

in the following. $\mathcal{S}$ consists of all locally finite unions of convex bodies (extended polyconvex sets). As it was already mentioned in [S], the curvature measures as well as the mixed measures are locally defined and therefore have a (unique additive) extension to $\mathcal{S}$. The extended measures are measurable (as functions on $\mathcal{S}$).

To finish this list of technical pre-requisites, we consider rigid motions (rotations, translations) of closed sets. The action of the group G_n on $\mathcal{F}$ is continuous and thus $(g, F) \mapsto gF$ is measurable, as is $(\vartheta, F) \mapsto \vartheta F, \vartheta \in SO_n$, and $(x, F) \mapsto xF := F + x, x \in \mathbb{R}^n$.

We now come to our basic definitions. A **random closed set (RACS)** Z (in $\mathbb{R}^n$) is a measurable mapping

$$Z : (\Omega, \mathbf{A}, \mathbb{P}) \to (\mathcal{F}, \mathcal{B}(\mathcal{F})).$$

As usual, the image measure

$$\mathbb{P}_Z := Z(\mathbb{P})$$

is the **distribution** of Z. We write $Z \sim Z'$, if $\mathbb{P}_Z = \mathbb{P}_{Z'}$ (equality in distribution).

Further probabilistic notions will be used without detailed explanation, as long as they are standard. For example, RACS $Z_1, ..., Z_k$ or $Z_1, Z_2, ...$ are (stochastically) **independent** if

$$\mathbb{P}_{(Z_1,...,Z_k)} = \mathbb{P}_{Z_1} \otimes \cdots \otimes \mathbb{P}_{Z_k},$$

respectively

$$\mathbb{P}_{(Z_1,Z_k,...)} = \bigotimes_{i=1}^{\infty} \mathbb{P}_{Z_i}.$$

The measurable transforms mentioned above produce random sets. Thus if Z, Z' are RACS, the following are also RACS: $Z \cup Z', Z \cap Z', gZ$ (for $g \in G_n$).

A RACS Z is **stationary** if $Z \sim xZ$, for all $x \in \mathbb{R}^n$, and **isotropic** if $Z \sim \vartheta Z$, for all $\vartheta \in SO_n$. At the beginning, we will concentrate on RACS which are stationary or even stationary and isotropic. Here is a first result on stationary RACS.

1.1.1 **Theorem 1.** *A stationary RACS Z is almost surely either empty or unbounded.*

If we replace Z by its closed convex hull Z' (which is a stationary convex RACS), the theorem follows from the fact, that Z' (almost surely) only takes values in $\{\emptyset, \mathbb{R}^n\}$. The reader is invited to think about a proof of this simple result.

Although we will not use it in full detail, we want to mention a fundamental result of Choquet. It concerns the **capacity functional** T_Z of a RACS Z,

$$T_Z : \mathcal{C} \to [0,1], \qquad T_Z(C) := \mathbb{P}(Z \cap C \neq \emptyset).$$

Generally, a real functional T on $\mathcal{C}$ is called a Choquet capacity, if it fulfills $0 \leq T \leq 1, T(\emptyset) = 0$ and $T(C_i) \to T(C)$, for every decreasing sequence $C_i \searrow C$. The mapping T is alternating of infinite order, if

$$S_k(C_0; C_1, ..., C_k) \geq 0, \qquad \text{for all } C_0, C_1, ..., C_k \in \mathcal{C}, k \in \mathbb{N}_0.$$

Here, $S_0(C_0) := 1 - T(C_0)$ and

$$S_k(C_0; C_1, ..., C_k) := S_{k-1}(C_0; C_1, ..., C_{k-1}) - S_{k-1}(C_0 \cup C_k; C_1, ..., C_{k-1}),$$

for $k \in \mathbb{N}$.

choquet **Theorem 2 (Choquet). (a)** *The capacity functional T_Z of a RACS Z is an alternating Choquet capacity of infinite order.*

(b) *If T is an alternating Choquet capacity of infinite order, then there is a RACS Z with $T = T_Z$.*

(c) *If $T_Z = T_{Z'}$, then $Z \sim Z'$.*

(a) follows directly from the definition of T_Z. The uniqueness result (c) will be useful for us. It is a consequence of the fact that the class of complements $\mathcal{F}^C, C \in \mathcal{C}$, is $\cap$-stable and generates $\mathcal{B}(\mathcal{F})$. (b) is the genuine result of Choquet and has a lengthy and complicated proof, but following the usual lines of extension theorems in measure theory.

The capacity functional T_Z of a RACS Z can be viewed as the analog of the distribution function of a real random variable. Theorem 2 thus parallels the continuity and monotonicity properties of distribution functions as well as the corresponding characterization and uniqueness results.

For a stationary RACS Z, the value $p := T_Z(\{x\})$ is independent of $x \in \mathbb{R}^n$, since

$$T_Z(\{x\}) = \mathbb{P}(x \in Z) = \mathbb{P}(0 \in Z - x) = \mathbb{P}(0 \in Z) = T_Z(\{0\}).$$

Moreover,

$$\begin{aligned}
\mathbb{E}\, \lambda_n(Z \cap A) &= \mathbb{E} \int_{\mathbb{R}^n} \mathbf{1}_A(x) \mathbf{1}_Z(x)\, \lambda_n(\mathrm{d}x) \\
&= \int_{\mathbb{R}^n} \mathbf{1}_A(x) \mathbb{E}\, \mathbf{1}_{Z-x}(0)\, \lambda_n(\mathrm{d}x) \\
&= \mathbb{E}\, \mathbf{1}_Z(0) \int_{\mathbb{R}^n} \mathbf{1}_A(x)\, \lambda_n(\mathrm{d}x) \\
&= p\, \lambda_n(A),
\end{aligned}$$

for $A \in \mathcal{B}(\mathbb{R}^n)$, due to the stationarity and Fubini's theorem. p is therefore called the **volume fraction** of Z.

Hints to the literature. The theory of random closed sets was developed independently by Kendall [Ke74] and Matheron [Ma72]. A first detailed exposition appeared in [Ma75]; for a more recent presentation, see [SW00].

1.2 Particle processes

The space $\mathcal{F}' := \mathcal{F} \setminus \{\emptyset\}$ of nonempty closed sets is a locally compact space with countable base. Therefore, one can define and consider point processes X on $\mathcal{F}'$ (see [B]). Formally, a (simple) point process X is a measurable mapping $X : (\Omega, \mathbf{A}, \mathbb{P}) \to (\mathsf{N}, \mathcal{N})$, where N denotes the collection of all locally finite subsets of $\mathcal{F}'$ and $\mathcal{N}$ is the σ-algebra generated by the counting functions

$$N \mapsto \text{card } (N \cap \mathcal{A}),$$

for $N \in \mathsf{N}$ and $\mathcal{A} \in \mathcal{B}(\mathcal{F}')$. Alternatively, N can be described as the collection of simple counting measures on $\mathcal{F}'$, the counting functions then have the form $N \mapsto N(\mathcal{A})$. Here, a Borel measure N on $\mathcal{F}'$ is a **counting measure** if it is integer-valued and locally finite, that is finite on all compact subsets of $\mathcal{F}'$. For the latter it is sufficient that

$$N(\mathcal{F}_C) < \infty, \qquad \text{for all } C \in \mathcal{C}. \tag{1}$$

LF

The counting measure N is **simple**, if $N(\{F\}) \leq 1$, for all $F \in \mathcal{F}'$, that means there are no multiple points occurring in N. For the following, it is convenient to use both interpretations of (simple) point processes X on $\mathcal{F}'$ simultaneously. Thus, we will interpret X as a random countable collection of closed sets, but will also view it as a random measure on $\mathcal{F}'$, such that expressions like $X(\mathcal{A})$, $\mathcal{A} \in \mathcal{B}(\mathcal{F}')$, make sense (and describe the number of 'points' in X which lie in $\mathcal{A}$).

For a point process X on $\mathcal{F}'$, let Θ be the **intensity measure**. For a Borel set $\mathcal{A} \in \mathcal{B}(\mathcal{F}')$, $\Theta(\mathcal{A})$ gives the mean number of points in X which lie in $\mathcal{A}$. In the language of counting measures,

$$\Theta(\mathcal{A}) = \mathbb{E}X(\mathcal{A}).$$

Whereas X is, by definition, locally finite (at least almost surely), the intensity measure Θ need not be. However, we will make the corresponding assumption throughout the following, that is, we generally assume

$$\Theta(\mathcal{F}_C) < \infty, \qquad \text{for all } C \in \mathcal{C}. \tag{2}$$

LF2

We also assume that Θ is not the zero measure since then the point process X would be empty (with probability 1), a case which is not very interesting, but also has to be excluded in some of the later results.

Although there are a number of interesting results for point processes on $\mathcal{F}'$, we now restrict our attention to point processes on $\mathcal{C}' := \mathcal{C} \setminus \{\emptyset\}$, that is, to point processes X on $\mathcal{F}'$ which are concentrated on $\mathcal{C}'$. The latter is the case, if and only if Θ is concentrated on $\mathcal{C}'$. We call such point processes **particle processes**. Why did we make this detour via point processes on $\mathcal{F}'$? Since $\mathcal{C}'$ with the Hausdorff metric is also a locally compact space with countable base, we could have defined a particle process directly as a point process on the metric space $\mathcal{C}'$. One reason is that the sets $\mathcal{F}_C, C \in \mathcal{C}$, are compact in $\mathcal{F}'$, but the corresponding sets $\mathcal{C}_C := \mathcal{F}_C \cap \mathcal{C}', C \in \mathcal{C}$, are not compact in the Hausdorff metric. Thus the condition of local finiteness for measures on $\mathcal{C}'$ would be weaker than (1) and not sufficient for our later purposes. A second aspect is that there is another important family of point processes in stochastic geometry which should be at least mentioned here, the **q-flat processes**. These are point processes of closed sets which are concentrated on the space $\mathcal{E}_q^n$ of q-flats (q-dimensional affine subspaces). $\mathcal{E}_q^n$ is also a measurable subset of $\mathcal{F}'$. Processes of flats are very interesting objects and show also close connections to convex geometry. Some results of this kind are discussed in [S]. Due to lack of time, we will not consider them further. Another very interesting class, which will not be treated here in detail, are the **random mosaics**. A random mosaic can be defined as a particle process X where the particles are convex polytopes which tile the space. Alternatively, but mathematically less informative, one can consider the union of the boundaries of the tiles and call the RACS Z made up by these boundaries a random mosaic. Because there is a strong dependence between the cells of a random mosaic, random mosaics and Boolean models are far from each other, but there are some connections which we will mention later on.

The definition of invariance properties of a particle process X is now straight-forward. Rigid motions g act in a natural way on collections of sets and on (random) measures η (on $\mathcal{C}'$). Namely,

$$g\mathcal{A} := \{gK : K \in \mathcal{A}\} \quad \text{and} \quad g\eta(\mathcal{A}) := \eta(g^{-1}\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\mathcal{C}').$$

Therefore, X is called **stationary** (respectively **isotropic**) if $X \sim xX$, for all translations x, (respectively $X \sim \vartheta X$, for all rotations ϑ). Here, we use distributions of particle processes and equality in distribution in the obvious way, without copying the definitions which we have described in more detail for RACS.

Particle processes can also be interpreted as marked point processes in $\mathbb{R}^n$ with mark space $\mathcal{C}'$, if we associate with each particle K a pair (x, K') such that $K = x + K'$. The idea is that x represents the 'location' of K, whereas K' represents the 'form'. Such a representation is especially helpful in the stationary case and we will use it directly or indirectly throughout the following. Apparently, there is no natural decomposition of this kind, any suitable center map $c : \mathcal{C}' \to \mathbb{R}^n$ will produce a corresponding pair $(c(K), K - c(K))$. In the following we work with one specific center map, which is compatible with rigid motions, namely we choose $c(K)$ to be the **midpoint of the circumsphere** of K. The marks are then concentrated on $\mathcal{C}_0 := \{K \in \mathcal{C}' : c(K) = 0\}$. For some applications, different center maps have been used (for example lower tangent points of the particles). In the case of convex particles, the Steiner point is also a natural choice.

For stationary particle processes, the representation as marked point process leads to a decomposition of the intensity measure (we remind the reader that we always assume $\Theta \not\equiv 0$).

decomposition **Theorem 3.** *For a stationary particle process X, the intensity measure Θ is translation invariant and has a decomposition*

$$\Theta(\mathcal{A}) = \gamma \int_{\mathcal{C}_0} \int_{\mathbb{R}^n} \mathbf{1}_{\mathcal{A}}(x + K)\, \lambda_n(\mathrm{d}x)\mathbb{Q}(\mathrm{d}K), \qquad \mathcal{A} \in \mathcal{B}(\mathcal{C}'), \tag{3}$$

decomp

with a constant $\gamma > 0$ and a probability measure $\mathbb{Q}$ on $\mathcal{C}_0$.

If X is isotropic, then $\mathbb{Q}$ is rotation invariant.

We call γ the **intensity** of X and $\mathbb{Q}$ the **grain distribution**. The marked point process $\Psi := \{(c(K), K - c(K)) : K \in X\}$ is sometimes called a **germ-grain process** since we can think of the particles K as grains grown around a germ.

We shortly indicate the proof of Theorem 3. The translation invariance of Θ is obvious. The image measure Θ' of Θ under $K \mapsto (c(K), K - c(K))$ is a measure on $\mathbb{R}^n \times \mathcal{C}_0$. The translation invariance of Θ implies that $\Theta' = \lambda_n \otimes \rho$ with some measure ρ. The local finiteness of Θ yields that ρ is finite. (3) thus follows with $\gamma := \rho(\mathcal{C}_0)$ and $\mathbb{Q} := \frac{1}{\gamma}\rho$. If X is isotropic, Θ is rotation invariant, and therefore Θ' is rotation invariant in the second component. Thus, $\mathbb{Q}$ is rotation invariant.

The representation (3) is unique with respect to the center map c which we used. A different center map c' can produce a different representation of X as a marked point process, and therefore a different decomposition of Θ. As we shall show below, this does not affect the intensity γ which will be the same for each representation, but it will affect the grain distribution $\mathbb{Q}'$ which will then live on a different space $\mathcal{C}_1 := \{C \in \mathcal{C}' : c'(C) = 0\}$. However, on can express $\mathbb{Q}$ and $\mathbb{Q}'$ as image measures of each other under a certain transformation which is connected with the center maps c, c'. The fact that γ depends only on X and not on c follows from the representation

$$\gamma = \lim_{r \to \infty} \frac{1}{\lambda_n(rB^n)} \Theta(\mathcal{F}_{rB^n}). \tag{4}$$

asymp

(4) is a consequence of (3), since

$$\begin{aligned}\frac{1}{\lambda_n(rB^n)}\,\Theta(\mathcal{F}_{rB^n}) &= \frac{\gamma}{\kappa_n r^n}\int_{\mathcal{C}_0}\int_{\mathbb{R}^n}\mathbf{1}_{\mathcal{F}_{rB^n}}(x+K)\,\lambda_n(\mathrm{d}x)\,\mathbb{Q}(\mathrm{d}K)\\ &= \frac{\gamma}{\kappa_n r^n}\int_{\mathcal{C}_0}\lambda_n(K+rB^n)\,\mathbb{Q}(\mathrm{d}K)\\ &= \frac{\gamma}{\kappa_n}\int_{\mathcal{C}_0}\lambda_n(\frac{1}{r}K+B^n)\,\mathbb{Q}(\mathrm{d}K).\end{aligned}$$

For $r\to\infty$, we have $\lambda_n(\frac{1}{r}K+B^n)\to\lambda_n(B^n)=\kappa_n$ und thus the result follows from Lebesgue's dominated convergence theorem.

In view of (4) we may interpret γ as the mean number of particles in X per unit volume of $\mathbb{R}^n$; we also speak of the **particle density**.

So far, we have worked with particle processes in general, now we want to mention a special class of them, the Poisson processes. A particle process X (with intensity measure Θ) is a **Poisson (particle) process**, if

$$\mathbb{P}(X(\mathcal{A})=k)=\mathrm{e}^{-\Theta(\mathcal{A})}\frac{\Theta(\mathcal{A})^k}{k!},\qquad \text{for } k\in\mathbb{N}_0,\ \mathcal{A}\in\mathcal{B}(\mathcal{C}'),\tag{5}$$

Poisson1

and, for mutually disjoint $\mathcal{A}_1,\mathcal{A}_2,...\in\mathcal{B}(\mathcal{C}')$,

$$X(\mathcal{A}_1),X(\mathcal{A}_2),...\text{ are independent.}\tag{6}$$

Poisson2

Conditions (5) and (6) are not independent. In fact, if we only consider intensity measures Θ without atoms, (5) and (6) are equivalent (see [SW00], for more details).

Poisson processes actually can be defined on quite arbitrary (measurable) spaces and each measure Θ (which fulfills a suitable finiteness condition) gives rise to a Poisson process which is uniquely determined in distribution and which has Θ as intensity measure. Hence, knowing the intensity measure Θ already determines the whole (Poisson) process. This uniqueness property makes the class of Poisson processes so important for results in stochastic geometry, but also for the statistical analysis of random point fields. In particular, the uniqueness implies that a Poisson process is stationary (isotropic), if and only if Θ is translation invariant (rotation invariant). We refer to [B], for further properties of Poisson processes (in $\mathbb{R}^n$).

If we use the representation of a Poisson particle process X as a marked point process Ψ (based on the center map c), the underlying point process $\Phi:=\{c(K):K\in X\}$ will be a Poisson process. Vice versa, we can start with a Poisson process Φ in $\mathbb{R}^n$ (with intensity measure Ξ) and can add to each point $\xi_i\in\Phi$ a random set Z_i independently (from each other and from X) with a given distribution $\mathbb{Q}$ on $\mathcal{C}_0$, say. Then, $X:=\{\xi_1+Z_1,\xi_2+Z_2,...\}$ is a Poisson particle process and the intensity measure Θ of X is the image of $\Xi\otimes\mathbb{Q}$ under $(x,K)\mapsto x+K$. In general, however, not every Poisson particle process X is obtained from a Poisson process Φ on $\mathbb{R}^n$ by independent marking, since for the intensity measure Θ the image under $K\mapsto(c(K),K-c(K))$ need not be a product measure (we discuss this phenomenon further in Section 4).

For a stationary Poisson particle process X the situation is simpler, since then we can apply Theorem 3.

decomposition2 **Theorem 4.** *For a stationary Poisson process X on $\mathcal{C}$, let γ be the intensity and $\mathbb{Q}$ the grain distribution. If Φ denotes the stationary Poisson process on $\mathbb{R}^n$ with intensity measure $\gamma\,\lambda_n$, then X is obtained from Φ by independent marking and $\mathbb{Q}$ is the corresponding mark distribution.*

X is isotropic, if and only if $\mathbb{Q}$ is rotation invariant.

Hints to the literature. There are numerous books on point processes and many of them work in general spaces (e.g. [DV88]). Point processes of geometric objects are treated in [Ma75], [KS92] and [SW00].

1.3 Boolean models

Having now defined the two basic notions in stochastic geometry, the RACS and the particle processes, we can start looking for examples. What are interesting random sets which can serve as models for random structures as they appear in practical applications? At this stage we notice that Theorem 2 is not as helpful as its real-valued counterpart. Whereas distribution functions on the real line are easy to construct and lead to a large variety of explicit (families of) distributions, the conditions for an alternating Choquet capacity of infinite order are more complex and the procedure to define a corresponding distribution on $\mathcal{F}$ is far from being constructive. However, particle processes are easier to construct and then we can use the following simple fact.

PP-RS **Theorem 5.** *If X is a particle process, then*

$$Z := \bigcup_{K \in X} K$$

is a RACS. Moreover, if X is stationary (isotropic), then Z is stationary (isotropic).

The proof is simple and left to the reader. A bit more challenging (but still simple) is a reverse statement: Each RACS Z is the union set of a particle process X, and if Z is stationary (isotropic), X can be chosen to be stationary (isotropic).

If the particle process X is concentrated on $\mathcal{K}$ (we speak of convex particles then), the union set Z takes its values in $\mathcal{S}$. We call Z a **random $\mathcal{S}$-set**. For random $\mathcal{S}$-sets Z there is also a reverse statement, which is not as obvious anymore.

RS-PP **Theorem 6.** *Each random $\mathcal{S}$-set Z is the union set of a process X of convex particles, and if Z is stationary (isotropic), X can be chosen to be stationary (isotropic).*

It is now easy to construct some examples of random $\mathcal{S}$-sets. For instance, let ξ be a nonnegative real random variable and K a fixed convex body, then $Z_0 := \xi K$ is a random convex body. The collection $X := \{z + \xi K : z \in \mathbb{Z}^n\}$ is then a particle process and its union set Z a random $\mathcal{S}$-set. In order to make X and Z stationary, we can add a uniform random translation $\tau \in [0,1]^n$ (independently of ξ), the distribution of τ thus being the Lebesgue measure on the unit cube $[0,1]^n$. In addition, we can make X and Z isotropic by applying a subsequent random rotation $\vartheta \in SO_n$ (again independently), the distribution of which is given by the Haar probability measure on SO_n. Although the resulting random set Z is now stationary and isotropic, it looks pretty regular. There are some obvious modifications which would add some more randomness to this construction. Using an enumeration $z_1, z_2, ...$ of $\mathbb{Z}^n$, we could replace $z_i + \xi K$ by $z_i + \xi_i K$, where $\xi_1, \xi_2, ...$ are independent copies of ξ, or even by $z_i + \xi_i K_i$, where we use a sequence $K_1, K_2, ...$ of convex bodies. Of course, we could even start with $X := \{z_i + Z_i : i = 1, 2, ...\}$, where $Z_1, Z_2, ...$ is a sequence of (independent or dependent) random sets with values in $\mathcal{K}, \mathcal{R}$ or $\mathcal{C}$ (in the latter case, the union set Z will be a RACS, but in general not a random $\mathcal{S}$-set).

Even with these modifications, the outcomes will be too regular to be useful for practical applications. But we can use the principle just described also to produce more interesting examples. Namely, we can start with a point process Φ in $\mathbb{R}^n$, choose a (measurable) enumeration $\Phi = \{\xi_1, \xi_2, ...\}$, and then 'attach' random (compact or convex) sets $Z_1, Z_2, ...$ to the points and consider

$$Z := \bigcup_{i=1}^{\infty} (\xi_i + Z_i). \tag{7}$$ germgrain

We will only consider the case where the Z_i are i.i.d. random compact sets, Z is then called a **germ-grain model**, the Z_i are called the **grains** of Z and their common distribution $\mathbb{Q}$ is called the **distribution of the typical grain** (or grain distribution). If $\mathbb{Q}(\mathcal{K}) = 1$, the germ-grain model Z has **convex grains**. If Φ is stationary, then Z is stationary, and if Φ is in addition isotropic and $\mathbb{P}_{Z_1}$ is rotation invariant, then Z is isotropic. Since there are many well-studied classes of point processes in $\mathbb{R}^n$, we can thus produce a large variety of random sets Z. However, since the particles of the process $X := \{\xi_1 + Z_1, \xi_2 + Z_2, ...\}$ may overlap, it is in general difficult to calculate geometric functionals of Z, even for well-established point processes Φ. The exception is the class of Poisson processes, for which a rich variety of formulas for the union sets Z are known. This is the reason why we will concentrate on Poisson processes in the following.

A RACS Z is a **Boolean model** if it is the union set of a Poisson particle process X. In particular, if we start with a Poisson process Φ in $\mathbb{R}^n$, the corresponding germ-grain model with grain distribution $\mathbb{Q}$ is a Boolean model. Not every Boolean model arises in this way since not every Poisson particle process X is obtained from a Poisson process in $\mathbb{R}^n$ by independent marking (as we mentioned already). Vice versa, for a stationary Boolean model, the center map c produces a representation as germ-grain model, but of a special kind, namely with grain distribution $\mathbb{Q}$ concentrated on $\mathcal{C}_0$. The correspondence between Boolean models Z and Poisson particle processes X is one-to-one, as we will show now. Our argument is based on the fact that the capacity functional T_Z of a Boolean model Z can be expressed in terms of the intensity measure Θ of X. Namely

$$\begin{aligned} T_Z(C) &= \mathbb{P}(Z \cap C \neq \emptyset) = \mathbb{P}(X(\mathcal{F}_C) > 0) \\ &= 1 - \mathbb{P}(X(\mathcal{F}_C) = 0) \\ &= 1 - \mathrm{e}^{-\Theta(\mathcal{F}_C)}, \end{aligned} \tag{8}$$ cf

for $C \in \mathcal{C}$. Hence, if Z is the union set of another Poisson particle process X' as well (with intensity measure Θ'), we obtain

$$\Theta(\mathcal{F}_C) = \Theta'(\mathcal{F}_C), \qquad C \in \mathcal{C},$$

and therefore $\Theta = \Theta'$ (this implication is not immediate, but needs a bit of work; see [SW00], for details). From $\Theta = \Theta'$, we get $X \sim X'$ and hence the following result.

uniqueness **Theorem 7.** *Let X, X' be Poisson particle processes with the same union set,*

$$\bigcup_{K \in X} K \sim \bigcup_{K' \in X'} K'.$$

Then,

$$X \sim X'.$$

The fact that $\Theta = \Theta'$ implies $X \sim X'$ can be deduced from general results in point process theory. It follows however also from Theorem 2 (in its general version, for RACS in a topological space T), since $\Theta = \Theta'$ implies $\mathbb{P}(X(A) = 0) = \mathbb{P}(X'(A) = 0)$ (from (5)). Therefore the (locally finite) RACS X and X' in $T := \mathcal{C}'$ have the same capacity functional.

For the remainder of this section, we concentrate on stationary Boolean models Z and their representation (7), yielding the intensity γ and the grain distribution $\mathbb{Q}$. Such stationary Boolean models can easily be simulated and produce interesting RACS even for simple distributions $\mathbb{Q}$ (for example, in the plane, for circles with random radii). In order to fit such a model to given (spatial) data, it is important to express geometric quantities of Z in terms of γ and $\mathbb{Q}$. For the capacity functional T_Z, such a result follows now from Theorem 3 in conjunction with Theorem 7. For a set $A \subset \mathbb{R}^n$, we use A^* to denote the reflection of A in the origin.

BMdecomp **Theorem 8.** *Let Z be a stationary Boolean model with intensity γ and grain distribution $\mathbb{Q}$. Then, the capacity functional T_Z of Z has the form*

$$T_Z(C) = 1 - \exp\left(-\gamma \int_{\mathcal{C}_0} \lambda_n(K + C^*)\mathbb{Q}(\mathrm{d}K)\right), \qquad C \in \mathcal{C}, \tag{9}$$ decomp2

and the volume fraction p of Z fulfills

$$p = 1 - \exp\left(-\gamma \int_{\mathcal{C}_0} \lambda_n(K)\mathbb{Q}(\mathrm{d}K)\right). \tag{10}$$ volumefraction

Equation (9) follows from (3) since $(K+x)\cap C \neq \emptyset$ is equivalent to $x \in K^* + C$, and thus

$$\begin{aligned}\int_{\mathbb{R}^n} \mathbf{1}_{\mathcal{F}_C}(K + x)\,\lambda_n(\mathrm{d}x) &= \int_{\mathbb{R}^n} \mathbf{1}_{K^*+C}(x)\,\lambda_n(\mathrm{d}x)\\ &= \lambda_n(K + C^*).\end{aligned}$$

Putting $C = \{0\}$ in (9) yields (10).

The capacity functional is of interest for statistical purposes, since it is easily estimated, using modern image analysing equipment. To be more precise, consider

$$T_Z(C) = \mathbb{P}(Z \cap C \neq \emptyset)$$

for a random set Z and a given 'test set' C. Since $\mathbb{P}(Z \cap C \neq \emptyset)$ equals the volume fraction of $Z + C^*$, a simple estimator of $T_Z(C)$ arises from counting pixels of a digitized image of $(Z + C^*) \cap [0,1]^n$.

Which information on γ and $\mathbb{Q}$ is contained in $T_Z(C)$? We can get a more precise answer if we choose C to be convex and assume that the Boolean model has convex grains. Then $\mathbb{Q}$ is concentrated on $\mathcal{K}_0 := \{K \in \mathcal{K} : c(K) = 0\}$ and we can use the mixed volume expansion for convex bodies. For example, if C is a ball $rB^n, r > 0$, the Steiner formula (see [S]) gives us

$$\begin{aligned}T_Z(rB^n) &= 1 - \exp\left(-\gamma \int_{\mathcal{K}_0} \lambda_n(K + rB^n)\mathbb{Q}(\mathrm{d}K)\right)\\ &= 1 - \exp\left(-\gamma \sum_{i=0}^{n} r^{n-i}\kappa_{n-i} \int_{\mathcal{K}_0} V_i(K)\mathbb{Q}(\mathrm{d}K)\right)\\ &= 1 - \exp\left(-\sum_{i=0}^{n} r^{n-i}\kappa_{n-i}\overline{V}_i(X)\right),\end{aligned}$$

where the **quermass densities** $\overline{V}_i(X)$ of X are defined by

$$\overline{V}_i(X) := \gamma \int_{\mathcal{K}_0} V_i(K)\mathbb{Q}(\mathrm{d}K), \qquad i = 0, ..., n.$$

Here $\overline{V}_0(X) = \gamma$. Hence, if we estimate $T_Z(rB^n)$ from realizations of Z, for different values of r, we obtain an empirical function $\hat{f} : r \mapsto \hat{T}_Z(rB^n)$. Fitting a polynomial (of order n) to

$$-\ln(1-\hat{f})$$

gives us estimators for $\overline{V}_0(X), ..., \overline{V}_n(X)$ and thus for γ and the mean values

$$\int_{\mathcal{K}_0} V_i(K)\mathbb{Q}(\mathrm{d}K), \qquad i = 1, ..., n.$$

Hints to the literature. In addition to the books [Ma75] and [SW00], which we already mentioned, we refer to [Mo97] and [SKM95] for results on Boolean models.

2 Mean values of additive functionals

The formula for the capacity functional, which we just gave in the case of convex grains, connects certain geometrical mean values of the Boolean model, the volume density of $Z + rB^n$, with mean values of the underlying Poisson process X, the quermass densities. Formulas of this kind can be used for the statistical estimation of particle quantities and in particular, for the estimation of the intensity γ. In this section, we discuss related formulas for other geometric quantities, like the surface area. Since Z is defined as the union set of the particles in X, functionals which are adapted to unions and intersections are of particular interest. Therefore, we concentrate on additive functionals and their mean values. We start with a general result for Boolean models. Then we consider general RACS and particle processes and discuss and compare different possible approaches for mean values of additive functionals. Results from integral geometry will be especially helpful here. In the last subsection, we come back to Boolean models and give explicit results for the quermass densities in the stationary and isotropic case.

2.1 A general formula for Boolean models

We consider a Boolean model Z in $\mathbb{R}^n$ with convex grains and an additive (and measurable) functional φ on $\mathcal{R}$. Additivity in the sequel always includes the convention $\varphi(\emptyset) = 0$. How can we define a mean value of φ for Z? Since Z may be unbounded (for example in the stationary case), it does not make much sense to work with $\varphi(Z)$, namely because Z is then not in the convex ring anymore and extensions of φ to $\mathcal{S}$, if they exist, usually yield $\varphi(Z) = \infty$. It seems more promising to consider $\varphi(Z \cap K_0)$ instead, where $K_0 \in \mathcal{K}$ is a suitably chosen bounded set. We can think of K_0 as a **sampling window** in which we observe the realizations of Z. This corresponds to many practical situations, where natural sampling windows arise in the form of boxes (rectangles) or balls (circles), for example as boundaries in photographic or microscopic images. Hence the question arises how to express $\mathbb{E}\,\varphi(Z \cap K_0)$ in terms of the intensity measure Θ.

We first describe a corresponding result in rather loose form before we give a rigorous formulation and proof. By definition of Z and the additivity of φ (used in form of the inclusion-exclusion formula),

$$\begin{aligned}\varphi(Z \cap K_0) &= \varphi\left(\bigcup_{K \in X} (K \cap K_0)\right) \\ &= \sum_{k=1}^{N} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} \varphi\big(K_0 \cap K_{i_1} \cap \cdots \cap K_{i_k}\big).\end{aligned}$$

Here N is the (random) number of grains $K \in X$ hitting K_0 and $K_1, ..., K_N$ is a (measurable) enumeration of these grains. Since $\varphi(\emptyset) = 0$, we may replace N by ∞ (with a corresponding enumeration $K_1, K_2, ...$ of X). In addition, we can simplify this formula by using the product particle process $X^k_{\neq} := \{(K_1, ..., K_k) \in X^k : K_i \text{ pairwise different}\}$, and get

$$\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \sum_{(K_1,...,K_k) \in X^k_{\neq}} \varphi\big(K_0 \cap K_1 \cap \cdots \cap K_k\big).$$

Turning now to the expectation, the independence properties of the Poisson process yield

$$\begin{aligned}&\mathbb{E}\, \varphi(Z \cap K_0) \\ &\quad= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \mathbb{E} \sum_{(K_1,...,K_k) \in X^k_{\neq}} \varphi\big(K_0 \cap K_1 \cap \cdots \cap K_k\big) \\ &\quad= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \int_{\mathcal{K}'} \cdots \int_{\mathcal{K}'} \varphi\big(K_0 \cap K_1 \cap \cdots \cap K_k\big) \Theta(\mathrm{d}K_1) \cdots \Theta(\mathrm{d}K_k),\end{aligned}$$

which is our desired result.

Our derivation was not totally correct, since we did not pay attention to integrability requirements when we exchanged expectation and summation. In view of the alternating sign $(-1)^{k+1}$ in the sum, this may be problematic and requires us to impose a further restriction on φ. We call a functional $\varphi : \mathcal{R} \to \mathbb{R}$ **conditionally bounded**, if φ is bounded on each set $\{K \in \mathcal{K} : K \subset K'\}, K' \in \mathcal{K}$. The intrinsic volumes V_j, $j = 0, ..., n$, are examples of additive (and measurable) functionals on $\mathcal{R}$, which are monotonic (and continuous) on $\mathcal{K}$ and therefore conditionally bounded.

Now we can formulate a precise result. We remark that a corresponding theorem holds for Boolean models with grains in $\mathcal{R}$ and $K_0 \in \mathcal{R}$, but this requires additional integrability conditions which we want to avoid here.

expectation **Theorem 9.** *Let Z be a Boolean model with convex grains and let Θ be the intensity measure of the underlying Poisson particle process X on $\mathcal{K}'$. Let*

$\varphi : \mathcal{R} \to \mathbb{R}$ be additive, measurable and conditionally bounded. Then, for each $K_0 \in \mathcal{K}$, the random variable $\varphi(Z \cap K_0)$ is integrable and

$$\mathbb{E}\,\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \int_{\mathcal{K}'} \cdots \int_{\mathcal{K}'} \varphi\big(K_0 \cap K_1 \cap \cdots \cap K_k\big)\Theta(\mathrm{d}K_1)\cdots\Theta(\mathrm{d}K_k). \quad (11)$$ basicdecomp

Proof. Let $c = c_{K_0}$ be an upper bound for $|\varphi|$ on $\{M \in \mathcal{K} : M \subset K_0\}$. Then

$$|\varphi(Z \cap K_0)| \le \sum_{k=1}^{N} \sum_{(K_1,\ldots,K_k)\in X^k_{\neq}} |\varphi\big(K_0 \cap K_1 \cap \cdots \cap K_k\big)|$$
$$\le \sum_{k=1}^{N} \binom{N}{k} c \le c\,2^N = c\,2^{X(\mathcal{F}_{K_0})}.$$

Here, N is again the (random) number of particles in X which intersect K_0. The right-hand side is integrable since

$$\mathbb{E}\,2^{X(\mathcal{F}_{K_0})} = \sum_{k=0}^{\infty} 2^k \mathbb{P}(X(\mathcal{F}_{K_0}) = k)$$
$$= \mathrm{e}^{-\Theta(\mathcal{F}_{K_0})} \sum_{k=0}^{\infty} 2^k \frac{\Theta(\mathcal{F}_{K_0})^k}{k!}$$
$$= \mathrm{e}^{-\Theta(\mathcal{F}_{K_0})}\mathrm{e}^{2\Theta(\mathcal{F}_{K_0})} = \mathrm{e}^{\Theta(\mathcal{F}_{K_0})} < \infty.$$

This yields the integrability of $\varphi(Z \cap K_0)$, but also justifies the interchange of expectation and summation, which we performed in the derivation above.

The equation

$$\mathbb{E} \sum_{(K_1,\ldots,K_k)\in X^k_{\neq}} \psi\big(K_1,\ldots,K_k\big) = \int_{\mathcal{K}'} \cdots \int_{\mathcal{K}'} \psi(K_1,\ldots,K_k)\Theta(\mathrm{d}K_1)\cdots\Theta(\mathrm{d}K_k),$$

which we used for $\psi\big(K_1,\ldots,K_k\big) := \varphi(K_0 \cap K_1 \cap \cdots \cap K_k\big)$ and which we explained with the independence properties of X actually holds for integrable ψ and is formally a consequence of Campbell's theorem (applied to $X^k_{\neq}$) together with the fact that the intensity measure of $X^k_{\neq}$ is the product measure Θ^k. □

If Z is stationary, (3) yields

$$\mathbb{E}\,\varphi(Z \cap K_0) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \gamma^k \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \Phi(K_0, K_1, \ldots, K_k)\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k) \quad (12)$$ stat

with

$$\begin{aligned}\Phi(K_0, K_1, \ldots, K_k)\\ := \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \varphi\big(K_0 \cap x_1 K_1 \cap \cdots \cap x_k K_k\big) \lambda_n(\mathrm{d}x_1) \cdots \lambda_n(\mathrm{d}x_k).\end{aligned}$$

(We remind the reader of the operational notation $xK := K + x$, which we shall frequently use, in the following.)

For example, we can put $\varphi = V_n$ (the volume or Lebesgue measure), where

$$\Phi(K_0, K_1, \ldots, K_k) = V_n(K_0) V_n(K_1) \cdots V_n(K_k),$$

hence we obtain

$$\begin{aligned}\mathbb{E}\, V_n(Z \cap K_0) &= V_n(K_0) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \left(\gamma \int_{\mathcal{K}_0} V_n(K) \mathbb{Q}(\mathrm{d}K)\right)^k \\ &= V_n(K_0) \left(1 - \exp\left(-\gamma \int_{\mathcal{K}_0} V_n(K) \mathbb{Q}(\mathrm{d}K)\right)\right) \\ &= V_n(K_0) \left(1 - \mathrm{e}^{-\overline{V}_n(X)}\right).\end{aligned}$$

For $V_n(K_0) > 0$, we have $\mathbb{E}\, V_n(Z \cap K_0)/V_n(K_0) = p$ and thus we get the formula which we derived already in a more direct way (and in a slightly more general situation, namely for compact grains) in Theorem 8.

As another example, we choose $\varphi = V_{n-1}$ (which is half the surface area). Then we get from the translative integral formula for V_{n-1}, which is explained in [S],

$$\begin{aligned}\Phi(K_0, K_1, \ldots, K_k)\\ = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} V_{n-1}\big(K_0 \cap x_1 K_1 \cap \cdots \cap x_k K_k\big) \lambda_n(\mathrm{d}x_1) \cdots \lambda_n(\mathrm{d}x_k)\\ = \sum_{i=0}^{k} V_n(K_0) \cdots V_n(K_{i-1}) V_{n-1}(K_i) V_n(K_{i+1}) \cdots V_n(K_n),\end{aligned}$$

and so

$$\begin{aligned}\mathbb{E}\, V_{n-1}(Z \cap K_0) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \big(V_{n-1}(K_0) \overline{V}_n(X)^k \\ &\qquad + k V_n(K_0) \overline{V}_{n-1}(X) \overline{V}_n(X)^{k-1}\big) \\ &= V_n(K_0) \overline{V}_{n-1}(X) \mathrm{e}^{-\overline{V}_n(X)} + V_{n-1}(K_0) \left(1 - \mathrm{e}^{-\overline{V}_n(X)}\right).\end{aligned}$$

If we consider here the normalized value $\mathbb{E}\, V_{n-1}(Z \cap K_0)/V_n(K_0)$ (for $V_n(K_0) > 0$), this is no longer independent of K_0 (as it was the case with volume), but is influenced by the shape of the boundary ∂K_0. In order to eliminate these boundary effects, we may replace our sampling window K_0 (with inner points) by $rK_0, r > 0$, and let $r \to \infty$. Then $V_{n-1}(rK_0)/V_n(rK_0) = c/r \to 0$, and we see that $\mathbb{E}\, V_{n-1}(Z \cap rK_0)/V_n(rK_0)$ has a limit which we denote by $\overline{V}_{n-1}(Z)$,

$$\overline{V}_{n-1}(Z) := \lim_{r\to\infty} \frac{\mathbb{E}\, V_{n-1}(Z \cap rK_0)}{V_n(rK_0)},$$

and which satisfies

$$\overline{V}_{n-1}(Z) = \overline{V}_{n-1}(X) \mathrm{e}^{-\overline{V}_n(X)}. \tag{13}$$

surfacedensity

We call $\overline{V}_{n-1}(Z)$ the **surface area density** of Z.

For the other intrinsic volumes V_j, $0 \le j \le n-2$, the situation is not as simple anymore since for them a translative integral formula looks more complicated and the iterated version is even more technical. We will come back to this problem later.

Now we assume that Z is stationary and isotropic. Since then $\mathbb{Q}$ is rotation invariant, we may replace the translation integrals by integrals over rigid motions and obtain (12) with

$$\Phi(K_0, K_1, \dots, K_k) = \int_{G_n} \cdots \int_{G_n} \varphi\big(K_0 \cap g_1 K_1 \cap \cdots \cap g_k K_k\big) \mu_n(\mathrm{d}g_1) \cdots \mu_n(\mathrm{d}g_k).$$

Thus, we can apply Hadwiger's (iterated) kinematic formula (Theorem 1.5 in [S]),

$$\int_{G_n} \cdots \int_{G_n} \varphi\big(K_0 \cap g_1 K_1 \cap \cdots \cap g_k K_k\big) \mu_n(\mathrm{d}g_1) \cdots \mu_n(\mathrm{d}g_k) = \sum_{\substack{m_0,\dots,m_k=0 \\ m_0+\cdots+m_k=kn}}^{n} c^{n,m_1,\dots,m_k}_{n-m_0,n,\dots,n} \varphi_{m_0}(K_0) V_{m_1}(K_1) \cdots V_{m_k}(K_k),$$

with

$$c^{n,m_1,\dots,m_k}_{n-m_0,n,\dots,n} = \frac{n!\kappa_n}{(n-m_0)!\kappa_{n-m_0}} \frac{m_1!\kappa_{m_1}}{n!\kappa_n} \cdots \frac{m_k!\kappa_{m_k}}{n!\kappa_n} = c^n_{n-m_0} c^{m_1}_n \cdots c^{m_k}_n$$

and with the Crofton integrals $\varphi_{m_0}(K_0), m_0 = 0, ..., n$. We obtain

$$
\begin{aligned}
&\mathbb{E}\,\varphi(Z\cap K_0)\\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}\sum_{\substack{m_0,\ldots,m_k=0\\ m_0+\cdots+m_k=kn}}^{n} c_{n-m_0,n,\ldots,n}^{n,m_1,\ldots,m_k}\varphi_{m_0}(K_0)\overline{V}_{m_1}(X)\cdots\overline{V}_{m_k}(X)\\
&= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}\sum_{m=0}^{n} c_{n-m}^{n}\varphi_m(K_0)\sum_{\substack{m_1,\ldots,m_k=0\\ m_1+\cdots+m_k=kn-m}}^{n}\prod_{i=1}^{k} c_n^{m_i}\overline{V}_{m_i}(X)\\
&= \varphi(K_0)\left(1-\mathrm{e}^{-\overline{V}_n(X)}\right)\\
&\qquad + \sum_{m=1}^{n} c_{n-m}^{n}\varphi_m(K_0)\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}\sum_{\substack{m_1,\ldots,m_k=0\\ m_1+\cdots+m_k=kn-m}}^{n}\prod_{i=1}^{k} c_n^{m_i}\overline{V}_{m_i}(X).
\end{aligned}
$$

We notice that, in the last sum, the number s of the indices m_i which are smaller than n ranges between 1 and m. Therefore, we can re-arrange the last two sums and get

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}\sum_{\substack{m_1,\ldots,m_k=0\\ m_1+\cdots+m_k=kn-m}}^{n}\prod_{i=1}^{k} c_n^{m_i}\overline{V}_{m_i}(X)\\
&\quad = \sum_{s=1}^{m}\sum_{r=0}^{\infty}\binom{r+s}{r}\frac{(-1)^{r+s+1}}{(r+s)!}\overline{V}_n(X)^r\sum_{\substack{m_1,\ldots,m_s=0\\ m_1+\cdots+m_s=sn-m}}^{n-1}\prod_{i=1}^{s} c_n^{m_i}\overline{V}_{m_i}(X)\\
&\quad = -\mathrm{e}^{-\overline{V}_n(X)}\sum_{s=1}^{m}\frac{(-1)^s}{s!}\sum_{\substack{m_1,\ldots,m_s=0\\ m_1+\cdots+m_s=sn-m}}^{n-1}\prod_{i=1}^{s} c_n^{m_i}\overline{V}_{m_i}(X)\,.
\end{aligned}
$$

Altogether we obtain the following result.

statisobm

Theorem 10. *Let Z be a stationary and isotropic Boolean model with convex grains, $K_0\in\mathcal{K}$, and let $\varphi:\mathcal{R}\to\mathbb{R}$ be additive, measurable and conditionally bounded. Then,*

$$
\begin{aligned}
\mathbb{E}\,\varphi(Z\cap K_0) &= \varphi(K_0)\left(1-\mathrm{e}^{-\overline{V}_n(X)}\right) - \mathrm{e}^{-\overline{V}_n(X)}\sum_{m=1}^{n} c_{n-m}^{n}\varphi_m(K_0)\\
&\quad\times\sum_{s=1}^{m}\frac{(-1)^s}{s!}\sum_{\substack{m_1,\ldots,m_s=0\\ m_1+\cdots+m_s=sn-m}}^{n-1}\prod_{i=1}^{s} c_n^{m_i}\overline{V}_{m_i}(X)\,. \qquad (14)
\end{aligned}
$$

valBM

In general, the expectation $\mathbb{E}\,\varphi(Z\cap K_0)$ will depend on the shape and even the location of the sampling window K_0. To delete the dependence on the location, we may concentrate on translation invariant φ. But still the dependence on the shape of K_0 remains and is expressed explicitly by (14). In order to obtain a mean value of φ for Z which does not depend on a specific sampling window, it is tempting to proceed as in the case of the surface area density $\overline{V}_{n-1}(Z)$ and consider

$$\lim_{r\to\infty} \frac{\mathbb{E}\,\varphi(Z\cap rK_0)}{V_n(rK_0)},$$

provided this limit exists. As we shall show next, for a translation invariant additive functional φ, this is indeed the case. In fact, a corresponding limit result holds more generally for stationary RACS, which need not be isotropic. We will discuss this and similar results for particle processes in the next two subsections. In 2.4, we come back to Boolean models and apply the above result to $\varphi = V_j$, $j = 0, ..., n-1$.

Hints to the literature. The basic formula (basicdecomp 11) appears in SW00 [SW00].

2.2 Mean values for RACS

The following considerations on mean values of additive functionals for stationary RACS are based on a result for valuations which we explain first. We denote by $C^n := [0,1]^n$ the **unit cube** and by $\partial^+ C^n := \{x = (x_1, ..., x_n) \in C^n : \max_{1\le i\le n} x_i = 1\}$ the **'upper right boundary'** of C^n. Note that $\partial^+ C^n \in \mathcal{R}$.

val **Lemma 1.** *Let $\varphi : \mathcal{R} \to \mathbb{R}$ be additive, translation invariant and conditionally bounded. Then,*

$$\lim_{r\to\infty} \frac{\varphi(rK)}{V_n(rK)} = \varphi(C^n) - \varphi(\partial^+ C^n), \qquad (15)$$ limit

for each $K \in \mathcal{K}$ with $V_n(K) > 0$.

Proof. . The additivity can be used to show

$$\varphi(M) = \sum_{z\in\mathbb{Z}^n} \left(\varphi(M\cap zC^n) - \varphi(M\cap z\partial^+ C^n)\right),$$

for all $M \in \mathcal{R}$ (we omit the details of this slightly lengthy derivation). In particular,

$$\varphi(rK) = \sum_{z\in\mathbb{Z}^n} \left(\varphi(rK\cap zC^n) - \varphi(rK\cap z\partial^+ C^n)\right),$$

for $r > 0$ and our given K, were we may assume $0 \in \operatorname{int} K$.

We define two sets of lattice points,

$$Z_r^1 := \{z \in \mathbb{Z}^n : rK\cap zC^n \neq \emptyset, zC^n \not\subset rK\}$$

and

$$Z_r^2 := \{z \in \mathbb{Z}^n : zC^n \subset rK\}.$$

Then,

$$\lim_{r\to\infty} \frac{|Z_r^1|}{V_n(rK)} = 0, \qquad \lim_{r\to\infty} \frac{|Z_r^2|}{V_n(rK)} = 1.$$

Consequently,

$$\frac{1}{V_n(rK)} \left| \sum_{z\in Z_r^1} \big(\varphi(rK \cap zC^n) - \varphi(rK \cap z\partial^+ C^n)\big) \right|$$
$$\le (|\varphi(C^n)| + |\varphi(\partial^+ C^n)|) \frac{|Z_r^1|}{V_n(rK)} \to 0 \qquad (r \to \infty)$$

and therefore

$$\begin{aligned}\lim_{r\to\infty} \frac{\varphi(rK)}{V_n(rK)} &= \lim_{r\to\infty} \frac{1}{V_n(rK)} \sum_{z\in Z_r^2} \big(\varphi(rK \cap zC^n) - \varphi(rK \cap z\partial^+ C^n)\big) \\ &= (\varphi(C^n) - \varphi(\partial^+ C^n)) \lim_{r\to\infty} \frac{|Z_r^2|}{V_n(rK)} \\ &= \varphi(C^n) - \varphi(\partial^+ C^n).\end{aligned}$$ □

Now we turn to a stationary RACS Z with values in $\mathcal{S}$. In contrast to the case of Boolean models, we need an additional integrability condition here, and we choose

$$\mathbb{E}\, 2^{N(Z\cap C^n)} < \infty, \tag{16}$$ intcond

which, although not optimal, is simple enough and works for all valuations φ. Here, for $K \in \mathcal{R}$, $N(K)$ is the minimal number m of convex bodies $K_1, ..., K_m$ with $K = \bigcup_{i=1}^m K_i$. Condition (16) guarantees that the realizations of Z do not become too complex in structure.

existence **Theorem 11.** *Let Z be a stationary random $\mathcal{S}$-set fulfilling (16) and let $\varphi : \mathcal{R} \to \mathbb{R}$ be additive, translation invariant, measurable and conditionally bounded. Then, for every $K \in \mathcal{K}$ with $V_n(K) > 0$, the limit*

$$\overline{\varphi}(Z) := \lim_{r\to\infty} \frac{\mathbb{E}\,\varphi(Z \cap rK)}{V_n(rK)}$$

exists and satisfies

$$\overline{\varphi}(Z) = \mathbb{E}\left(\varphi(Z \cap C^n) - \varphi(Z \cap \partial^+ C^n)\right).$$

Hence, $\overline{\varphi}(Z)$ is independent of K.

Proof. Consider $M \in \mathcal{K}$ with $M \subset C^n$. For each realization $Z(\omega)$ of Z, we use a representation

$$Z(\omega) \cap M = \bigcup_{i=1}^{N_M(\omega)} K_i(\omega)$$

with $K_i(\omega) \in \mathcal{K}$ and $N_M(\omega) := N(Z(\omega) \cap M)$. The inclusion-exclusion formula yields

$$\varphi(Z(\omega) \cap M) = \sum_{k=1}^{N_M(\omega)} (-1)^{k+1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N_M(\omega)} \varphi\big(K_{i_1}(\omega) \cap \cdots \cap K_{i_k}(\omega)\big).$$

Therefore,

$$\begin{aligned} \mathbb{E}|\varphi(Z \cap M)| &\le c\, \mathbb{E} \sum_{k=1}^{N_M} \binom{N_M}{k} \\ &\le c\, \mathbb{E}\, 2^{N_M} < \infty, \end{aligned} \tag{17}$$

upperest

by (16). Here, c is an upper bound for $|\varphi|$ on $\{M' \in \mathcal{K} : M' \subset C^n\}$ (which exists, since φ is conditionally bounded).

Hence, $\varphi(Z \cap M)$ is integrable for every $M \in \mathcal{K}$, $M \subset C^n$, but then, by the inclusion-exclusion formula, also for all $M \in \mathcal{R}$. Consequently, all expectations, which appear in the theorem, exist. We use this to define a functional

$$\phi : \mathcal{R} \to \mathbb{R}, \qquad \phi(M) := \mathbb{E}\varphi(Z \cap M).$$

ϕ is additive, translation invariant (here we use the stationarity), and conditionally bounded (this follows from (17)). Lemma 1 now yields the asserted result. □

Theorem 11 shows that $\varphi(Z \cap C^n) - \varphi(Z \cap \partial^+ C^n)$ is an unbiased estimator for $\overline{\varphi}(Z)$.

The results, obtained so far, hold in particular for the case $\varphi = V_j$, $j \in \{0, ..., n\}$. Here we write $\overline{V}_j(Z)$ for the corresponding density (for $j = n$ we get the volume density $p = \overline{V}_n(Z)$ again). The following result gives a formula of kinematic type. We recall from [S] that ν is the invariant probability measure on SO_n and we denote by $\mathbb{E}_\nu$ the expectation with respect to ν.

kinematic

Theorem 12. *Let Z be a stationary random $\mathcal{S}$-set fulfilling (16). Let $K \in \mathcal{K}$ and let ϑ be a random rotation with distribution ν and independent of Z. Then*

$$\mathbb{E}_\nu \, \mathbb{E}\, V_j(Z \cap \vartheta K) = \sum_{k=j}^{n} c_{j,n}^{k,n+j-k} V_k(K) \overline{V}_{n+j-k}(Z),$$

for $j = 0, ..., n$.

Proof. As in the proof of Theorem 11, one shows that

$$(x, \vartheta, \omega) \mapsto V_j(Z(\omega) \cap \vartheta K \cap x B^n)$$

is $\lambda \otimes \nu \otimes \mathbb{P}$-integrable. The invariance properties of V_j and Z then show that

$$\mathbb{E}_\nu \, \mathbb{E} \, V_j(Z \cap \vartheta K \cap xrB^n) = \mathbb{E}_\nu \, \mathbb{E} \, V_j(Z \cap (-x)\vartheta K \cap rB^n),$$

for all $x \in \mathbb{R}^n$. Integration over x and Fubini's theorem yield

$$\mathbb{E}_\nu \, \mathbb{E} \int_{\mathbb{R}^n} V_j(Z \cap \vartheta K \cap xrB^n) \, \lambda(\mathrm{d}x) = \mathbb{E} \int_{G_n} V_j(Z \cap gK \cap rB^n) \mu(\mathrm{d}g).$$

Since B^n is rotation invariant, we can replace the integral over $\mathbb{R}^n$ on the left-hand side by an integration over G_n. The principal kinematic formula thus gives

$$\sum_{k=j}^{n} c_{j,n}^{k,n+j-k} \mathbb{E}_\nu \, \mathbb{E} \, V_k(Z \cap \vartheta K) V_{n+j-k}(rB^n) = \sum_{k=j}^{n} c_{j,n}^{k,n+j-k} V_k(K) \mathbb{E} \, V_{n+j-k}(Z \cap rB^n).$$

We divide both sides by $V_n(rB^n)$ and let $r \to \infty$. The left-hand side then converges to

$$\mathbb{E}_\nu \, \mathbb{E} \, V_j(Z \cap \vartheta K),$$

and the right-hand side converges to

$$\sum_{k=j}^{n} c_{j,n}^{k,n+j-k} V_k(K) \overline{V}_{n+j-k}(Z). \qquad \square$$

The expectation $\mathbb{E}_\nu$ can be omitted if $j = n$ or $j = n-1$ or $K = B^n$ or if Z is isotropic.

Theorem 12 is useful for two reasons. First, it describes the bias, if the value $V_j(Z \cap \vartheta K)$, for a randomly rotated sampling window K, is used as an estimator for $\overline{V}_j(Z)$. Second, it provides us with a further unbiased estimator, if we solve the corresponding (triangular) system of linear equations with unknowns $\overline{V}_j(Z), j = 0, ..., n$. For example, if $K = B^n$, we get

$$\overline{V}_j(Z) = \sum_{i=j}^{n} a_{ij} \mathbb{E} \, V_i(Z \cap B^n), \qquad j = 0, ...n,$$

with given constants a_{ij}. Hence,

$$\sum_{i=j}^{n} a_{ij} V_i(Z \cap B^n)$$

is an unbiased estimator for $\overline{V}_j(Z)$.

Hints to the literature. Also for this section, [SW00] is the main reference. The slightly more general version of Theorem 12 was taken from [We97c].

2.3 Mean values for particle processes

For a stationary particle process X, Theorem decomposition 3 immediately allows us to define a mean value $\overline{\varphi}(X)$, for any translation invariant, measurable function $\varphi : \mathcal{C}' \to \mathbb{R}$ (which is either nonnegative or $\mathbb{Q}$-integrable), namely by

$$\overline{\varphi}(X) := \gamma \int_{\mathcal{C}_0} \varphi(K)\mathbb{Q}(\mathrm{d}K).$$

The following alternative representations of $\overline{\varphi}(X)$ follow with standard techniques (Campbell's Theorem, majorized convergence).

densities **Theorem 13.** *Let X be a stationary particle process and $\varphi : \mathcal{C}' \to \mathbb{R}$ translation invariant, measurable and $\mathbb{Q}$-integrable (or nonnegative). Then,*
(a) *for all Borel sets $A \subset \mathbb{R}^n$ with $0 < \lambda_n(A) < \infty$,*

$$\overline{\varphi}(X) = \frac{1}{\lambda_n(A)} \mathbb{E} \sum_{K \in X, c(K) \in A} \varphi(K),$$

(b) *for all $K_0 \in \mathcal{K}$ with $V_n(K_0) > 0$,*

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{V_n(rK_0)} \mathbb{E} \sum_{K \in X, K \subset rK_0} \varphi(K),$$

(c) *for all $K_0 \in \mathcal{K}$ with $V_n(K_0) > 0$,*

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{V_n(rK_0)} \mathbb{E} \sum_{K \in X, K \cap rK_0 \neq \emptyset} \varphi(K)$$

(if we assume, in addition that $\int_{\mathcal{C}_0} |\varphi(K)| V_n(K + B^n)\mathbb{Q}(\mathrm{d}K) < \infty$).

As in the case of RACS, we get further results for additive and conditionally bounded functionals φ on $\mathcal{R}$ and processes X with particles in $\mathcal{R}'$, satisfying a certain integrability condition. We assume

$$\int_{\mathcal{R}_0} 2^{N(K)} V_n(K + B^n)\mathbb{Q}(\mathrm{d}K) < \infty. \qquad (18)$$ intcond2

The following result is proved similarly to Lemma val 1 and Theorem existence 11.

val2 **Theorem 14.** *Let X be a stationary process of particles in $\mathcal{R}'$ satisfying (intcond2 18). Let $\varphi : \mathcal{R}' \to \mathbb{R}$ be translation invariant, additive, measurable and conditionally bounded. Then φ is $\mathbb{Q}$-integrable and*

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{V_n(rK_0)} \mathbb{E} \sum_{K \in X} \varphi(K \cap rK_0),$$

for all $K_0 \in \mathcal{K}$ with $V_n(K_0) > 0$. Moreover,

$$\overline{\varphi}(X) = \mathbb{E} \sum_{K \in X} \left(\varphi(K \cap C^n) - \varphi(K \cap \partial^+ C^n) \right).$$

The choice $\varphi = V_j$ provides us therefore with a number of alternative representations of the quermass densities $\overline{V}_j(X)$. We also get an analogue of Theorem 12. The proof is even simpler here and only requires Campbell's Theorem and the principal kinematic formula. If we consider processes X with convex particles, we can even skip condition (18).

kinematic2 **Theorem 15.** *Let X be a stationary process of particles in $\mathcal{K}'$ and $K \in \mathcal{K}$. Let ϑ be a random rotation with distribution ν and independent of Z. Then*

$$\mathbb{E}_\nu \, \mathbb{E} \sum_{M \in X} V_j(M \cap \vartheta K) = \sum_{k=j}^{n} c_{j,n}^{k,n+j-k} V_k(K) \overline{V}_{n+j-k}(X),$$

for $j = 0, ..., n$.

Again, the expectation $\mathbb{E}_\nu$ can be omitted in any of the cases $j = n$, $j = n-1$, $K = B^n$ or if X is isotropic.

As in the case of RACS, these results produce various (unbiased) estimators for $\overline{V}_j(X)$.

Hints to the literature. Again, [SW00] is the main reference and Theorem 15 was taken from [We97c].

2.4 Quermass densities of Boolean models

We return now to Boolean models Z and assume first that Z is stationary and isotropic. We apply Theorem 10 to $\varphi = V_j$. Since $(V_j)_m = 0$, for $m > n - j$, and $(V_j)_m = c_{j,n}^{n-m,m+j} V_{m+j}$, for $m = 0, ..., n-j$, by the Crofton formula (see [S], we obtain the following result.

statisobm2 **Theorem 16.** *Let Z be a stationary and isotropic Boolean model with convex grains. Then,*

$$\overline{V}_n(Z) = 1 - \mathrm{e}^{-\overline{V}_n(X)}$$

and

$$\overline{V}_j(Z) = \mathrm{e}^{-\overline{V}_n(X)} \left(\overline{V}_j(X) - \sum_{s=2}^{n-j} \frac{(-1)^s}{s!} \, c_j^n \sum_{\substack{m_1,\ldots,m_s=j+1 \\ m_1+\cdots+m_s=(s-1)n+j}}^{n-1} \prod_{i=1}^{s} c_n^{m_i} \overline{V}_{m_i}(X) \right),$$

for $j = 0, \ldots, n-1$.

Although these formulas still look very technical, they are quite useful for practical applications. Of course, these applications mostly appear in the planar or spatial situation. Therefore, we discuss these cases shortly. We use A and U, for the area and the boundary length in the plane, V, S and M, for the volume, the surface area and the (additively extended) mean width in three-dimensional space, and χ for the Euler characteristic.

planarbm **Corollary 1.** *For a stationary and isotropic Boolean model Z in $\mathbb{R}^2$ with convex grains, we have*

$$\begin{aligned}
\overline{A}(Z) &= 1 - \mathrm{e}^{-\overline{A}(X)},\\
\overline{U}(Z) &= \mathrm{e}^{-\overline{A}(X)}\overline{U}(X),\\
\overline{\chi}(Z) &= \mathrm{e}^{-\overline{A}(X)}\left(\gamma - \frac{1}{4\pi}\overline{U}(X)^2\right).
\end{aligned}$$

spatialbm **Corollary 2.** *For a stationary and isotropic Boolean model Z in $\mathbb{R}^3$ with convex grains, we have*

$$\begin{aligned}
\overline{V}(Z) &= 1 - \mathrm{e}^{-\overline{V}(X)},\\
\overline{S}(Z) &= \mathrm{e}^{-\overline{V}(X)}\overline{S}(X),\\
\overline{M}(Z) &= \mathrm{e}^{-\overline{V}(X)}\left(\overline{M}(X) - \frac{\pi^2}{32}\overline{S}(X)^2\right),\\
\overline{\chi}(Z) &= \mathrm{e}^{-\overline{A}(X)}\left(\gamma - \frac{1}{4\pi}\overline{M}(X)\overline{S}(X) + \frac{\pi}{384}\overline{S}(X)^3\right).
\end{aligned}$$

Since we now have several possibilities to estimate the densities of Z on the left-hand side, these equations allow the estimation of the particle means and therefore of the intensity γ. An important aspect is that the formulas hold for Boolean models Z with grains in $\mathcal{R}$ as well (under an additional integrability assumption). If the grains K obey $\chi(K) = 1$, $\mathbb{Q}$-almost surely (in the plane this follows, for example, if the grains are all simply connected), then the results hold true without any change. Otherwise, γ has to be replaced by $\overline{\chi}(X)$ (and then we do not get an estimation of the intensity itself).

What changes if we skip the isotropy assumption? For a stationary Boolean model Z (again we assume convex grains, for simplicity) and $\varphi = V_j$, we can use (stat 12) as a starting point and apply the iterated translative formula (Theorem 3.1 in [S]). We obtain

$$\begin{aligned}
&\mathbb{E}\, V_j(Z \cap K_0)\\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}\gamma^k \sum_{\substack{m_0,\dots,m_k=j\\ m_0+\dots+m_k=kn+j}}^{n} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} V^{(j)}_{m_0,\dots,m_k}(K_0,\dots,K_k)\\
&\qquad\qquad \times \mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k).
\end{aligned}$$

Again, we replace K by rK, normalize by $V_n(rK_0)$ and let $r \to \infty$. Then, due to the homogeneity properties of mixed functionals (see [S]), all summands on the right-hand side with $m_0 < n$ disappear asymptotically. For $m_0 = n$, we can use the decomposition property of mixed functionals and get, with essentially the same arguments as in the isotropic case,

$$
\begin{aligned}
\overline{V}_j(Z) &= \lim_{r\to\infty} \frac{\mathbb{E}\, V_j(Z\cap rK_0)}{V_n(rK_0)} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!}\gamma^k \sum_{\substack{m_1,\dots,m_k=j\\ m_1+\cdots+m_k=(k-1)n+j}}^{n} \int_{\mathcal{K}_0}\cdots\int_{\mathcal{K}_0} V^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k) \\
&\qquad\qquad \times \mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k) \\
&= \sum_{s=1}^{n-j}\sum_{r=0}^{\infty}\binom{r+s}{r}\frac{(-1)^{r+s+1}}{(r+s)!}\overline{V}_n(X)^r\gamma^s \\
&\times \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\cdots+m_s=(s-1)n+j}}^{n-1} \int_{\mathcal{K}_0}\cdots\int_{\mathcal{K}_0} V^{(j)}_{m_1,\dots,m_s}(K_1,\dots,K_s)\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_s) \\
&= -\mathrm{e}^{-\overline{V}_n(X)}\sum_{s=1}^{n-j}\frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j\\ m_1+\cdots+m_s=(s-1)n+j}}^{n-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X) \\
&= \mathrm{e}^{-\overline{V}_n(X)}\left(\overline{V}_j(X) - \sum_{s=2}^{n-j}\frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1\\ m_1+\cdots+m_s=(s-1)n+j}}^{n-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X)\right).
\end{aligned}
$$

Here the mixed densities of X are defined as

$$
\overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X) := \gamma^s \int_{\mathcal{K}_0}\cdots\int_{\mathcal{K}_0} V^{(j)}_{m_1,\dots,m_s}(K_1,\dots,K_s)\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_s).
$$

Hence, we arrive at the following result.

statbm **Theorem 17.** *Let Z be a stationary Boolean model with convex grains. Then,*

$$
\overline{V}_n(Z) = 1 - \mathrm{e}^{-\overline{V}_n(X)},
$$
$$
\overline{V}_{n-1}(Z) = \mathrm{e}^{-\overline{V}_n(X)}\overline{V}_{n-1}(X),
$$

and

$$
\begin{aligned}
&\overline{V}_j(Z) \\
&= \mathrm{e}^{-\overline{V}_n(X)}\left(\overline{V}_j(X) - \sum_{s=2}^{n-j}\frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1\\ m_1+\cdots+m_s=(s-1)n+j}}^{n-1} \overline{V}^{(j)}_{m_1,\dots,m_s}(X,\dots,X)\right),
\end{aligned}
$$

for $j = 0,\dots,n-2$.

For $n = 2$, only the formula for the Euler characteristic changes and we have

$$
\begin{aligned}
\overline{A}(Z) &= 1 - \mathrm{e}^{-\overline{A}(X)}, \\
\overline{U}(Z) &= \mathrm{e}^{-\overline{A}(X)}\overline{U}(X), \\
\overline{\chi}(Z) &= \mathrm{e}^{-\overline{A}(X)}\left(\gamma - \overline{A}(X,X^*)\right),
\end{aligned}
$$

where

$$A(X, X^*) := \gamma^2 \int_{\mathcal{K}_0} \int_{\mathcal{K}_0} A(K, M^*) \mathbb{Q}(\mathrm{d}K) \mathbb{Q}(\mathrm{d}M).$$

Here, we made use of the fact that the mixed functional $V_{1,1}^{(0)}(K, M)$ in the plane equals the mixed area of K and the reflection M^* of M. It is obvious that the formulas can no longer be used directly for the estimation of γ. Hence, we need more (local) information for the statistical analysis of nonisotropic Boolean models and this will be discussed in Section 3.

We emphasize again, that the formulas for the quermass densities hold also for grains in $\mathcal{R}$. For convex grains, there are further methods to estimate γ. One possibility is to use the capacity functional, as we have mentioned already. A further one is to use associated points (the so-called tangent count). We describe the basis of this method shortly. We associate a boundary point $z(K)$ with each particle $K \in X$, in a suitable way, and then count the number $\chi^+(Z \cap K_0)$ of associated points which are visible in $Z \cap K_0$ (these are the associated points which lie in K_0 and are not covered by other particles). The associated points $z(K), K \in X$, build a stationary Poisson process $\tilde{\Phi}$ with the same intensity as the process Φ of center points, namely γ. The associated points which are not covered by other particles, build a stationary point process Φ' which is obtained from $\tilde{\Phi}$ by a thinning and has intensity $\gamma \mathrm{e}^{-\overline{V}_n(X)}$. This follows from Slivnyak's theorem. The probability that a given associated point $x \in \tilde{\Phi}$ is not covered by other particles equals the probability that an arbitrary point $x \in \mathbb{R}^d$ is not covered by any particle, hence that $x \notin Z$. Since $\mathbb{P}(x \notin Z) = 1 - p$, the thinning probability is $\mathrm{e}^{-\overline{V}_n(X)}$. Therefore,

$$\mathbb{E}\chi^+(Z \cap K_0) = V_n(K_0)\gamma \mathrm{e}^{-\overline{V}_n(X)}. \tag{19}$$

tangentcount

In the plane, one can choose the "lower left tangent point" as associated point (the left most boundary point in the lower part of a particle with a horizontal tangent) and then has to count these points in $Z \cap K_0$. Together with the formula for the area density, this yields a simple estimator for γ.

Hints to the literature. The formulas for the quermass densities of stationary and isotropic Boolean models have a long history, beginning with results by Miles and Davy and including contributions by Kellerer, Weil and Wieacker, and Zähle. More details and references can be found in [SW00]. Theorem 17 was proved in [We90]. The formula (19) on the 'tangent count' is classical (see [SKM95]). Molchanov and Stoyan [MS94] (see also [Mo97]) have shown that, in the case of convex grains with interior points, the point process of visible tangent points (together with another quantity, the covariance function of Z) determines the grain distribution $\mathbb{Q}$ (and γ) uniquely.

2.5 Ergodicity

3 Directional data, local densities, nonstationary Boolean models

In this section, we extend the results on mean values of RACS, particle processes and Boolean models in various directions. First, we consider functionals which reflect the directional behavior of a random structure. Then, we give local interpretations of the quermass densities and other mean values by using curvature measures and their generalizations. Finally, we extend the basic formulas for Boolean models to the nonstationary case.

3.1 Directional data and associated bodies

Motion invariant functionals like the intrinsic volumes may give some information on the shape of a convex body K (e.g. if we consider the isoperimetric quotient of K) but they do not give any information about the orientation. It is therefore reasonable that, for stationary nonisotropic Boolean models Z, the densities $\overline{V}_j(Z), j = 0, ..., n$, are not sufficient to estimate the intensity γ. Therefore, we consider additive functionals φ now, which better reflect the orientation. Two different but related approaches exist here.

First, we observe that the intrinsic volumes are mixed volumes (with the unit ball B^n),

$$V_j(K) = \binom{n}{j} \frac{1}{\kappa_{n-j}} V(K\,[j], B^n\,[n-j]), \qquad j = 0, ..., n.$$

Therefore, an obvious generalization is to consider functionals

$$V(K\,[j], M\,[n-j]),$$

for $j \in \{0, ..., n\}$ and $M \in \mathcal{K}$. For example, if M is a segment s, the mixed volume $V(K\,[n-1], s\,[1])$ is (proportional to) the $(n-1)$-volume of the projection of K orthogonal to s.

We note that we can also work with mixed translative functionals here since

$$\binom{n}{j} V(K\,[j], M\,[n-j]) = V^{(0)}_{j,n-j}(K, M^*).$$

For fixed M, the functional $K \mapsto V^{(0)}_{j,n-j}(K, M)$ is additive, translation invariant and continuous on $\mathcal{K}$ and therefore has a unique additive extension to $\mathcal{R}$ existence which is measurable and conditionally bounded. By Theorem 11, the density

$$\overline{V}^{(0)}_{j,n-j}(Z, M) := \lim_{r\to\infty} \frac{\mathbb{E} V^{(0)}_{j,n-j}(Z \cap rK_0, M)}{V_n(rK_0)}$$

exists. Also, densities of general mixed functionals exist for X, either obtained as a limit or simply as integrals with respect to $\mathbb{Q}$,

$$\begin{aligned} &V^{(0)}_{m_1,\dots,m_s,n-j}(X,\dots,X,M) \\ &\qquad = \gamma^s \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} V^{(0)}_{m_1,\dots,m_s,n-j}(K_1,\dots,K_s,M)\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k), \end{aligned}$$

for $s = 1,\dots,n-1$, and $m_i \in \{j,\dots,n\}$ with $m_1 + \cdots + m_s = (s-1)n + j$. Since $K \mapsto V^{(0)}_{j,n-j}(K,M)$ satisfies an iterated translative integral formula, similarly to the one for $K \mapsto V_j(K)$, we obtain the following result, similarly to the proof of Theorem 17.

mixvolbm **Theorem 18.** *Let Z be a stationary Boolean model with convex grains. Then,*

$$\begin{aligned} \overline{V}^{(0)}_{j,n-j}(Z,M) = \mathrm{e}^{-\overline{V}_n(X)} \Bigg(&\overline{V}^{(0)}_{j,n-j}(X,M) \\ &- \sum_{s=2}^{n-j} \frac{(-1)^s}{s!} \sum_{\substack{m_1,\dots,m_s=j+1 \\ m_1+\cdots+m_s=(s-1)n+j}}^{n-1} \overline{V}^{(0)}_{m_1,\dots,m_s,n-j}(X,\dots,X,M) \Bigg), \end{aligned}$$

for $j = 0,\dots,n-1$ and $M \in \mathcal{K}$.

For $n = 2$, the resulting formulas read (after simple modifications)

$$\begin{aligned} \overline{A}(Z) &= 1 - \mathrm{e}^{-\overline{A}(X)}, \\ \overline{A}(Z,M) &= \mathrm{e}^{-\overline{A}(X)}\overline{A}(X,M), \qquad M \in \mathcal{K}, \\ \overline{\chi}(Z) &= \mathrm{e}^{-\overline{A}(X)}\left(\gamma - \overline{A}(X,X^*)\right). \end{aligned} \tag{20}$$ bm2

For $M = B^n$, the second formula reduces to the equation for $\overline{U}(Z)$ given earlier.

Before we discuss (20) further, we mention a second approach to directional data. This is based on measure- or function-valued functionals which describe convex bodies K uniquely (up to translation). One such functional is the **area measure** $S_{n-1}(K,\cdot)$, a Borel measure on the unit sphere S^{n-1} which described the surface area of K in boundary points with prescribed outer normals (for a more detailed description of area measures, curvature measures and support measures, we refer to Section 2 of [S]). The reason that we consider only the $(n-1)$st area measure and not the other lower order ones lies in the fact that $S_{n-1}(K,\cdot)$ satisfies a (simple) translative formula (which is not the case for the other area measures),

$$\int_{\mathbb{R}^n} S_{n-1}(K \cap xM,\cdot)\,\lambda_n(\mathrm{d}x) = V_n(M)S_{n-1}(K,\cdot) + V_n(K,\cdot)S_{n-1}(M,\cdot). \tag{21}$$ surftrans

The iteration of (21) is obvious.

For each Borel set $B \subset S^{n-1}$ the real functional $K \mapsto S_{n-1}(K, B)$ satisfies the conditions of Theorem 11, it is additive, translation invariant, measurable and conditionally bounded (in general, for given B, it is not continuous on $\mathcal{K}$, but the measure-valued map $K \mapsto S_{n-1}(K, \cdot)$ is continuous in the weak topology). Thus, the densities $\overline{S}_{n-1}(Z, B)$ and $\overline{S}_{n-1}(X, B)$ exist and define finite Borel measures $\overline{S}_{n-1}(Z, \cdot)$ and $\overline{S}_{n-1}(X, \cdot)$ on S^{n-1}. For $\overline{S}_{n-1}(Z, \cdot)$ the nonnegativity may not be obvious, in fact one has to show first that the additive extension of the area measure to sets $K \in \mathcal{R}$ is nonnegative, since it is the image of the Hausdorff measure on ∂K under the spherical image map (for the latter, see [Sch93, p. 78]). The following generalization of (13) is now easy to obtain,

$$\overline{S}_{n-1}(Z, \cdot) = \mathrm{e}^{-\overline{V}_n(X)} \overline{S}_{n-1}(X, \cdot). \tag{22}$$

areamdensity

In fact, (22) is equivalent to the case $j = n - 1$ of Theorem 18, which reads

$$\overline{V}^{(0)}_{n-1,1}(Z, M) = \mathrm{e}^{-\overline{V}_n(X)} \overline{V}^{(0)}_{n-1,1}(X, M),$$

for $M \in \mathcal{K}$. The connection follows from the equation

$$V^{(0)}_{n-1,1}(K, M) = \int_{S^{n-1}} h(M, -u) S_{n-1}(K, \mathrm{d}u),$$

which is a classical formula for mixed volumes of convex bodies K, M. Here, $h(M, \cdot)$ is the support function of M. Since area measures have centroid 0, we may replace $h(M, \cdot)$ by the **centred support function** $h^*(M, \cdot)$ (see [S], for details) and obtain

$$V^{(0)}_{n-1,1}(K, M) = \int_{S^{n-1}} h^*(M, -u) S_{n-1}(K, \mathrm{d}u). \tag{23}$$

supportint

A classical result from convex analysis tells us that the differences of centred support functions are dense in the Banach space of continuous functions f on S^{n-1}, which are centred in the sense that

$$\int_{S^{n-1}} u f(u)\, \sigma_{n-1}(\mathrm{d}u) = 0.$$

Therefore, the collection

$$\{V^{(0)}_{n-1,1}(K, M) : M \in \mathcal{K}\}$$

uniquely determines the measure $S_{n-1}(K, \cdot)$ and vice versa.

The (centred) support function $h^*(K, \cdot)$ is another directional functional of K, it determines K up to a translation and fulfills the assumptions of Theorem 11, namely it is translation invariant, additive, measurable and conditionally bounded (it is even continuous on $\mathcal{K}$). We therefore also have a density $\overline{h}(Z, \cdot)$ (we suppress the * here). Since $h^*(K, \cdot)$ also satisfies an iterated translative formula (see Theorem 3.2 in [S]), we obtain a further formula for Z, either by copying the proof of the previous results (Theorem 17 or Theorem 18) or by using (23) in Theorem 18.

Theorem 19. *Let Z be a stationary Boolean model with convex grains. Then,*

$$\overline{h}(Z,\cdot) = \mathrm{e}^{-\overline{V}_n(X)}\Bigg(\overline{h}(X,\cdot) - \sum_{s=2}^{n-1}\frac{(-1)^s}{s!}\sum_{\substack{m_1,\dots,m_s=2\\ m_1+\dots+m_s=(s-1)n+1}}^{n-1}\overline{h}_{m_1,\dots,m_s}(X,\dots,X,\cdot)\Bigg). \tag{24}$$

Here,

$$\overline{h}_{m_1,\dots,m_s}(X,\dots,X,\cdot) := \gamma^s\int_{\mathcal{K}_0}\cdots\int_{\mathcal{K}_0} h^*_{m_1,\dots,m_s}(K_1,\dots,K_s,\cdot)\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k), \tag{25}$$

where $h^*_{m_1,\dots,m_s}(K_1,\dots,K_s,\cdot)$ is the mixed (centred) support function occuring in [S], Theorem 3.2.

For $n=2$, the formula reduces to

$$\overline{h}(Z,\cdot) = \mathrm{e}^{-\overline{A}(X)}\overline{h}(X,\cdot), \tag{26}$$

which does not give us further information since it is essentially equivalent to (22). In fact, both $S_{n-1}(K,\cdot)$ and $h^*(K;\cdot)$ determine K uniquely, up to translation, and therefore they also determine each other. In the plane, this connection can be made even more precise since

$$S_1(K,\cdot) = h^*(K,\cdot) + (h^*(K,\cdot))'', \tag{27}$$

in the sense of Schwartz distributions. If we look at (20) again, we can write it now as

$$\begin{aligned}\overline{A}(Z) &= 1-\mathrm{e}^{-\overline{A}(X)},\\ \overline{h}(Z,\cdot) &= \mathrm{e}^{-\overline{A}(X)}\overline{h}(X,\cdot),\\ \overline{\chi}(Z) &= \mathrm{e}^{-\overline{A}(X)}\left(\gamma - \overline{A}(X,X^*)\right).\end{aligned} \tag{28}$$

These three formulas actually suffice to obtain an estimator for γ, if the left-hand densities are estimated. Namely, the first formula determines $\mathrm{e}^{-\overline{A}(X)}$, so the second gives us $\overline{h}(X,\cdot)$. From (27), we deduce

$$\overline{S}_1(X,\cdot) = \overline{h}(X,\cdot) + (\overline{h}(X,\cdot))''$$

and get $\overline{S}_1(X,\cdot)$. Equation (23) transfers to X as

$$\overline{V}^{(0)}_{n-1,1}(X,X) = \int_{S^{n-1}}\overline{h}(X,-u)\overline{S}_{n-1}(X,\mathrm{d}u), \tag{29}$$

hence

$$\overline{A}(X, X^*) = \frac{1}{2} \int_{S^1} \overline{h}(X, -u) \overline{S}_1(X, \mathrm{d}u)$$

is determined and so we get γ. Again, we emphasize that these results hold true for Boolean models Z with grains in $\mathcal{R}$ (under suitable additional integrability conditions).

Let us shortly discuss the case $n = 3$. From Theorem 18, we get four formulas for the mean values of mixed volumes. The first one is

$$\overline{V}(Z) = 1 - \mathrm{e}^{-\overline{V}(X)},$$

which determines $\mathrm{e}^{-\overline{V}(X)}$. The second one can be replaced by its measure version (22)

$$\overline{S}_2(Z, \cdot) = \mathrm{e}^{-\overline{V}(X)} \overline{S}_2(X, \cdot),$$

and this gives us $\overline{S}_2(X, \cdot)$. By (23) and another denseness argument, the third equation for $\overline{V}^{(0)}_{1,2}(Z, M), M \in \mathcal{K}$, is equivalent to (and therefore can be replaced by)

$$\overline{h}(Z, \cdot) = \mathrm{e}^{-\overline{V}(X)} \left(\overline{h}(X, \cdot) - \overline{h}_{2,2}(X, X, \cdot) \right).$$

From explicit formulas for $h_{2,2}(P, Q, \cdot)$, in the case of polytopes P, Q (see Theorem 3.2 in [S]), we obtain from (25) (and approximation of convex bodies by polytopes),

$$\overline{h}_{2,2}(X, X, \cdot) = \int_{S^2} \int_{S^2} f(u, v, \cdot) \overline{S}_2(X, \mathrm{d}u) \overline{S}_2(X, \mathrm{d}v)$$

with a given geometric function f on $(S^2)^3$. Thus, $\overline{h}_{2,2}(X, X, \cdot)$ is determined by $\overline{S}_2(X, \cdot)$ and so the third equation gives us $\overline{h}(X, \cdot)$. The fourth equation reads

$$\overline{\chi}(Z) = \mathrm{e}^{-\overline{V}_n(X)} \left(\gamma - \overline{V}^{(0)}_{1,2}(X, X) + \overline{V}^{(0)}_{2,2,2}(X, X, X) \right).$$

We already know that

$$\overline{V}^{(0)}_{1,2}(X, X) = \int_{S^2} \overline{h}(X, -u) \overline{S}_2(X, \mathrm{d}u)$$

and similarly we get

$$\overline{V}^{(0)}_{2,2,2}(X, X, X) = \int_{S^2} \overline{h}_{2,2}(X, X, u) \overline{S}_2(X, \mathrm{d}u).$$

Hence, both densities are determined by the quantities, which we already have. Therefore, we obtain γ.

For higher dimensions, the corresponding formulas become more and more complex and the question is open, whether the densities of mixed volumes of Z determine the intensity γ.

The above considerations show that, for $n = 2$ and $n = 3$, the densities for mixed volumes of Z, respectively their measure- and function-valued counterparts determine γ uniquely. Of course, the mean value formulas can be used to construct corresponding estimators for mean values of X and thus for γ. However, the question still remains whether these lead to applicable procedures in practice. For $n = 2$, this is the case as we shall show in a moment. First, we mention a third method to describe the directional behavior of random sets and particle processes, the method of associated bodies.

It has been a quite successful idea to associate convex bodies with the directional data of a random structure. In particular, this often allows to apply classical geometric inequalities (like the isoperimetric inequality) to these associated bodies and use this to formulate and solve extremal problems for the random structures. Such associated convex bodies even exist for random structures like processes of curves (fibre processes) or line processes. We only discuss here the case of Boolean models but hope that the general principle will be apparent.

The principle of associated bodies is based on the fact that certain mean measures of random structures are area measures of a convex body, uniquely determined up to translation, and similarly certain mean functions of random structures are (centred) support functions of a convex body, again determined up to translation. More precisely, any finite Borel measure ρ on S^{n-1} which has centroid 0 (and is not supported by a great sphere) is the area measure of a unique convex body K, $\rho = S_{n-1}(K, \cdot)$. This is Minkowski's existence theorem. Also, a continuous function h on S^{n-1} is the support function of a convex body K, $h = h(K, \cdot)$, if the positive homogeneous extension of h is convex on $\mathbb{R}^n$. The first principle can be used in nearly all situations in Stochastic Geometry, where measures on S^{n-1} occur. The second principle is only helpful in certain cases since the required convexity often fails. Using the first principle, we can define a convex body $B(Z)$ (the **Blaschke body** of Z) by

$$S_{n-1}(B(Z), \cdot) = \overline{S}_{n-1}(Z, \cdot)$$

and in the same way a Blaschke body $B(X)$ of X. Then, (22) becomes a formula between convex bodies,

$$B(Z) = \mathrm{e}^{-\overline{V}_n(X)} B(X).$$

For a Boolean model with convex grains, the functions $\overline{h}_{m_1,\dots,m_s}(X, \dots, X, \cdot)$ in Theorem 19 are in fact support functions of a mixed body $M_{m_1,\dots,m_s}(X, \dots, X)$, but due to the alternating sign in (24) the function $\overline{h}(Z, \cdot)$ on the left-hand side is in general not a support function.

An exception is the case $n = 2$, where the convexity follows from (26). Here, $\overline{h}(Z, \cdot) = h(B(Z), \cdot)$ (and $\overline{h}(X, \cdot) = h(B(X), \cdot)$). The main statistical problem in the analysis of stationary, nonisotropic Boolean models Z in $\mathbb{R}^2$ thus consists in the estimation of the Blaschke body $B(Z)$. The latter and the volume fraction p immediately yield $B(X)$. Since $\overline{A}(X, X^*) = A(B(X), B(X)^*)$, simple formulas for the mixed area can then be used together with an empirical value for $\overline{\chi}(Z)$ to obtain γ. Several approaches for the estimation of $B(Z)$ (respectively $\overline{S}_1(Z, \cdot) = S_1(B(Z), \cdot)$ have been described in the literature. One uses the idea of convexification of a nonconvex set, another one is based on contact distributions (these will be discussed in Section 4), a third one counts intersections with directed lines.

Hints to the literature. Densities of mixed volumes were studied in [We01]. Densities for area measures and support functions appear in [We94], [We97c]. The intensity analysis for $n = 2$ was given in [We95], the case $n = 3$ was discussed in [We99]. In [We01], a corresponding result was claimed also for $n = 4$. The given proof is however incomplete, since there is a term $\overline{V}^{(0)}_{2,2}(X, X)$ missing in the formula for $\overline{\chi}(Z)$ (there are also some constants missing in the formulas for $\overline{\chi}(Z)$, for $n = 2, 3, 4$). Mean and Blaschke bodies for RACS and particle processes have been studied in [We97a], [We97b]. The use of associated convex bodies in stochastic geometry is demonstrated in Section 4.5 of [SW00].

3.2 Local densities

The mean values $\overline{\varphi}(Z)$ of additive, translation invariant functionals φ for a stationary RACS Z have been introduced as limits

$$\overline{\varphi}(Z) = \lim_{r\to\infty} \frac{\mathbb{E}\,\varphi(Z \cap rK_0)}{V_n(rK_0)}$$

over increasing sampling windows rK_0, $r \to \infty$. For a prospective extension to nonstationary RACS Z it would be helpful to have a local interpretation of these mean values. This is possible in the cases we treated so far, namely for the quermass densities $\overline{V}_j(Z)$ and the densities $\overline{S}_{n-1}(Z, \cdot)$ and $\overline{h}(Z, \cdot)$. The following considerations will also make clear why we speak of densities here. The basic idea is to replace the intrinsic volume V_j by its local counterpart, the curvature measure $\Phi_j(K, \cdot)$. These measures are defined for $K \in \mathcal{K}$, but have an additive extension to sets in the convex ring (which may be a finite signed measure). Since curvature measures are locally defined, they even extend to sets in $\mathcal{S}$ as signed Radon measures (set functions which are defined on bounded Borel sets). Hence, for a random $\mathcal{S}$-set Z, we are allowed to write $\Phi_j(Z, \cdot)$ and this is a random signed Radon measure (as always in these lectures, we skip some technical details like the measurability and integrability of $\Phi(Z, \cdot)$). If Z is stationary, $\Phi_j(Z, \cdot)$ is stationary, hence the (signed Radon) measure $\mathbb{E}\,\Phi_j(Z, \cdot)$ is translation invariant. Consequently, this measure is a multiple of λ_n, $\mathbb{E}\,\Phi_j(Z, \cdot) = d_j\,\lambda_n$. As it turns out, d_j equals the quermass density $\overline{V}_j(Z)$.

density **Theorem 20.** *Let Z be a stationary random $\mathcal{S}$-set fulfilling (16) and $j \in \{0, ..., n\}$. Then,*

$$\mathbb{E}\,\Phi_j(Z,\cdot) = \overline{V}_j(Z)\,\lambda_n.$$

Proof. We copy the proof of Theorem 12 and modify it appropriately, but leave out some of the more technical details.

Let $A \subset \mathbb{R}^d$ be a bounded Borel set and K a convex body containing A in its interior. The stationarity of Z and the fact that the curvature measures are locally defined and translation covariant implies

$$\mathbb{E}\int_{\mathbb{R}^n} \Phi_j(Z\cap K\cap xrB^n, A)\,\lambda_n(\mathrm{d}x) = \mathbb{E}\int_{\mathbb{R}^n} \Phi_j(Z\cap yK\cap rB^n, yA)\,\lambda(\mathrm{d}y).$$

On the left-hand side, we can replace the integration over $\mathbb{R}^n$ by the invariant integral over G_n and apply the principal kinematic formula for curvature measures. We get

$$\sum_{k=j}^{n} c_{j,n}^{k,n+j-k}\,\mathbb{E}\,\Phi_k(Z,A)V_{n+j-k}(rB^n)$$

(here we have used that $\Phi_k(Z\cap K, A) = \Phi_k(Z,A)$). On the right-hand side, we apply the principal translative formula for curvature measures (see Section 3.1 of [S]) and use the fact that the mixed measures $\Phi_{k,n+j-k}^{(j)}(Z\cap rB^n, K; \mathbb{R}^n\times A)$ vanish, for $k > j$, since $A \subset \operatorname{int} K$. The remaining summand, with $k = j$, is $\Phi_j(Z\cap rB^n, \mathbb{R}^n)\,\lambda_n(A)$. Hence we obtain

$$\sum_{k=j}^{n} c_{j,n}^{k,n+j-k}\,\mathbb{E}\,\Phi_k(Z,A)V_{n+j-k}(rB^n) = \mathbb{E}\,\Phi_j(Z\cap rB^n, \mathbb{R}^n)\,\lambda_n(A).$$

Dividing both sides by $V_n(rB^n)$ and letting $r\to\infty$, we obtain

$$\mathbb{E}\,\Phi_j(Z,A) = \overline{V}_j(Z)\,\lambda_n(A).$$ □

For the directional densities $\overline{S}_{n-1}(Z,\cdot)$ and $\overline{h}(Z,\cdot)$, we obtain similar results. In the first case, we use the support measure $\Xi_{n-1}(K,\cdot)$, $K\in\mathcal{K}$, (see [S]) and show that

$$\mathbb{E}\,\Xi_{n-1}(Z,\cdot\times A) = \frac{1}{2}\overline{S}_{n-1}(Z,A)\,\lambda_n, \tag{30}$$ suppmeas

for each Borel set $A\subset S^{n-1}$ (the factor 1/2 here comes from the different normalizations: $\Xi_{n-1}(K,\mathbb{R}^n\times S^{n-1}) = V_{n-1}(K)$, which is half the surface area $S_{n-1}(K,S^{n-1})$). (30) can be deduced as in the proof above and is based on the translative formula

$$\int_{\mathbb{R}^n} \Xi_{n-1}(K\cap xM, (B\cap xC)\times A)\,\lambda_n(\mathrm{d}x)$$
$$= \Xi_{n-1}(K, B\times A)\,\lambda_n(M\cap C) + \Xi_{n-1}(M, C\times A)\,\lambda_n(K\cap B),$$

for $K, M \in \mathcal{K}$ and Borel sets $B, C \subset \mathbb{R}^n$, $A \subset S^{n-1}$, which follows from Theorem 1.2.7 in [SW92].

For $\overline{h}(Z, \cdot)$, we use the **support kernel** $\rho(K; u, \cdot)$, $K \in \mathcal{K}, u \in S^{n-1}$,

$$\rho(K; u, \cdot) := \Phi^{(0)}_{1,n-1}(K, u^+; \cdot \times \beta(u)),$$

here $u^+ := \{x \in \mathbb{R}^n : \langle x, u \rangle \geq 0\}$ and $\beta(u)$ is a ball in ∂u^+ of $(n-1)$-volume 1. The extension properties and the translative integral formula for mixed measures then show that

$$\mathbb{E}\, \rho(Z; u, \cdot) = \overline{h}(Z, u)\, \lambda_n,$$

again with a similar proof as above.

These results show also that the densities can be considered as Radon-Nikodym derivatives, in particular we have

$$\overline{V}_j(Z) = \lim_{r \to 0} \frac{\mathbb{E}\, \Phi_j(Z, rB^n)}{V_n(rB^n)}, \qquad j = 0, ..., n-1,$$

(note, however, that the relation $\overline{V}_j(Z) = \lim_{r\to 0} \mathbb{E}\, V_j(Z \cap rB^n)/V_n(rB^n)$ is wrong).

For particle processes, we only formulate the corresponding results (and concentrate on convex particles). The proofs are similar but simpler because of Campbell's theorem.

density2 **Theorem 21.** *Let X be a stationary process of convex particles. Then,*

$$\mathbb{E} \sum_{K \in X} \Phi_j(K, \cdot) = \overline{V}_j(X)\, \lambda_n, \qquad j = 0, ..., n,$$

$$\mathbb{E} \sum_{K \in X} \Xi_{n-1}(K, \cdot \times A) = \frac{1}{2} \overline{S}_{n-1}(X, A)\, \lambda_n, \qquad A \subset S^{n-1},$$

$$\mathbb{E} \sum_{K \in X} \rho(K; u, \cdot) = \overline{h}(X, u)\, \lambda_n, \qquad u \in S^{n-1}.$$

Hints to the literature. The interpretation of quermass densities as Radon-Nikodym derivatives goes back to Weil and Wieacker [WW84], Weil [We84] and, for sets of positive reach, to Zähle [Za86]. The derivative interpretation of $\overline{S}_{n-1}(Z, \cdot)$ and $\overline{h}(Z, \cdot)$, and the corresponding results in Theorem 21 can be found at different places in the literature, in particular see [GW02].

3.3 Nonstationary Boolean models

In this subsection, we discuss extensions of the previous results to nonstationary Boolean models Z. It is rather obvious that we have to require some regularity of Z, because with arbitrary intensity measures Θ of the underlying

Poisson particle process X rather pathological union sets Z can arise. Starting with a Boolean model with compact grains, our basic assumption is that Θ allows a decomposition

$$\Theta(A) = \int_{\mathcal{C}_0} \int_{\mathbb{R}^n} \mathbf{1}_A(x+K) f(K,x)\, \lambda_n(\mathrm{d}x)\mathbb{Q}(\mathrm{d}K), \qquad A \in \mathcal{B}(\mathcal{C}'), \tag{31}$$

with a nonnegative and measurable function f on $\mathcal{C}_0 \times \mathbb{R}^n$ and a probability measure $\mathbb{Q}$ on $\mathcal{C}_0$. Theorem 3 shows that (31) means that Θ is absolutely continuous to a translation invariant measure Ω on $\mathcal{C}'$. Such a measure Ω is not uniquely determined and consequently the representation (31) for Θ is not unique, in general. There is however an important case, where uniqueness holds, and we will concentrate on this case, although most of the results in this subsection hold true in the more general situation (31).

Theorem 22. *Let Θ be a locally finite measure on $\mathcal{C}'$ and suppose that*

$$\Theta(A) = \int_{\mathcal{C}_0} \int_{\mathbb{R}^n} \mathbf{1}_A(x+K) f(x)\, \lambda_n(\mathrm{d}x)\mathbb{Q}(\mathrm{d}K), \qquad A \in \mathcal{B}(\mathcal{C}'), \tag{32}$$

with a nonnegative and measurable function f on $\mathbb{R}^n$ and a probability measure $\mathbb{Q}$ on $\mathcal{C}_0$. Then $\mathbb{Q}$ is uniquely determined and f is determined up to a set of λ_n-measure 0.

If we consider the image measure $\tilde{\Theta}$ of Θ under $K \mapsto (c(K), K - c(K))$, the theorem presents the obvious fact that a decomposition of the form $\tilde{\Theta} = (\int f \mathrm{d}\lambda_n) \otimes \mathbb{Q}$ is unique.

If Θ is now the intensity measure of a Poisson particle process X and Z the corresponding Boolean model, we call f the **intensity function** of Z and $\mathbb{Q}$ the **distribution of the typical grain**. We still have a simple interpretation of Z which can also be used for simulations. If we distribute points $x_1, x_2, \dots$ according to a Poisson process with intensity function f and then add independent (convex) particles $Z_1, Z_2, \dots$ with common distribution $\mathbb{Q}$, the resulting union set is equivalent (in distribution) to Z. The Boolean model Z is stationary (with intensity γ), if and only if $f \equiv \gamma$.

How strong are conditions like (31) or (32)? Since $\mathcal{C}$ and $\mathbb{R}^n$ are both Polish spaces and since Θ is assumed to be locally finite, a general decomposition principle in measure theory shows that a representation

$$\Theta(A) = \int_{\mathcal{C}_0} \int_{\mathbb{R}^n} \mathbf{1}_A(x+K) \rho(K, \mathrm{d}x)\mathbb{Q}(\mathrm{d}K), \qquad A \in \mathcal{B}(\mathcal{C}'),$$

always exists, where ρ is a kernel (i.e, for each $K \in \mathcal{C}_0$, $\rho(K, \cdot)$ is a measure on $\mathbb{R}^n$ and, for each $C \in \mathcal{B}(\mathbb{R}^n)$, $\rho(\cdot, B)$ is a (nonnegative) measurable function on $\mathcal{C}_0$) and $\mathbb{Q}$ is a probability measure. Condition (31) thus requires, in addition, that the kernel ρ is absolutely continuous, that means, there are versions of ρ and $\mathbb{Q}$ (remember that ρ and $\mathbb{Q}$ are not uniquely determined), such that

$\rho(K,\cdot)$ has a density $f(K,\cdot)$ with respect to λ_n, for each $K \in \mathcal{C}_0$. In contrast to this, (Thetadecomp3 32) is a much stronger condition since it requires that $\rho(K,\cdot)$ is in addition independent of $K \in \mathcal{C}_0$.

Now we concentrate on Boolean models with convex grains again. Our next goal is to define densities of additive functionals like V_j, S_{n-1}, h^* and the mixed functionals $V^{(j)}_{m_1,\dots,m_k}$ for the Poisson process X and the Boolean model Z. Since X and Z are not assumed to be stationary anymore, we cannot expect a constant density but a quantity depending on the location z in space. The results of Section 3.2 motivate us to introduce densities as Radon-Nikodym derivatives of appropriate random measures with respect to λ_n. The main task is therefore to show that the random measures are absolutely continuous. Here, condition (Thetadecomp3 32) is of basic importance. For the mixed functionals $V^{(j)}_{m_1,\dots,m_k}$, the corresponding random measures are measures on $(\mathbb{R}^n)^k$ and consequently the Radon-Nikodym derivatives will be taken with respect to λ_n^k and will be functions of k variables $z_1,\dots,z_k$.

nonstat1 **Theorem 23.** *Let X be a Poisson process of convex particles with intensity measure satisfying (Thetadecomp3 32). Then, for $j = 0,\dots,n$, $k \in \mathbb{N}$, and $m_1,\dots,m_k \in \{j,\dots,n\}$ with $m_1 + \dots + m_k = (k-1)n + j$, the signed measure*

$$\mathbb{E} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}} \varPhi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\cdot)$$

is locally finite and absolutely continuous with respect to λ_n^k.

Its Radon-Nikodym derivative $\overline{V}^{(j)}_{m_1,\dots,m_k}(X,\dots,X;\cdot)$ fulfils λ_n^k-almost everywhere

$$\begin{aligned}
&\overline{V}^{(j)}_{m_1,\dots,m_k}(X,\dots,X;z_1,\dots,z_k)\\
&= \int_{\mathcal{K}_0}\cdots\int_{\mathcal{K}_0}\int_{(\mathbb{R}^n)^k} f(z_1-x_1)\cdots f(z_k-x_k)\\
&\qquad\qquad \times \varPhi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;\mathrm{d}(x_1,\dots,x_k))\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k)\\
&= \lim_{r\to 0}\frac{1}{(V_n(rK))^k}\\
&\quad \times \mathbb{E}\sum_{(K_1,\dots,K_k)\in X^k_{\neq}} \varPhi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k;(z_1+rK)\times\cdots\times(z_k+rK)),
\end{aligned}$$

for each $K \in \mathcal{K}$ with inner points.

For $k = 1$, we thus obtain the local quermass density $\overline{V}_j(X;\cdot)$ with

$$\begin{aligned}
\overline{V}_j(X;z) &= \int_{\mathcal{K}_0}\int_{\mathbb{R}^n} f(z-x)\varPhi_j(M,\mathrm{d}x)\mathbb{Q}(\mathrm{d}M)\\
&= \lim_{r\to 0}\frac{1}{(V_n(rK))^k}\mathbb{E}\sum_{M\in X}\varPhi_j(M,(z+rK)), \qquad z \in \mathbb{R}^n.
\end{aligned}$$

For $j = n$, we get the volume density (or volume fraction)

$$V_n(X; z) = \int_{\mathcal{K}_0} \int_M f(z - x)\, \lambda(\mathrm{d}x)\mathbb{Q}(\mathrm{d}M).$$

This equation (with the proof given below) holds true for Poisson processes X on $\mathcal{C}'$ as well.

Apart from integrability considerations which are of a more technical nature, the basic tool in the proof of Theorem nonstat1 23 is Campbell's theorem again which gives us

$$\mathbb{E} \sum_{(K_1,\dots,K_k)\in X^k_{\neq}} \Phi^{(j)}_{m_1,\dots,m_k}(K_1, ..., K_k; B)$$
$$= \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{(\mathbb{R}^n)^k} \Phi^{(j)}_{m_1,\dots,m_k}(K_1 + x_1, ..., K_k + x_k; B) f(x_1) \cdots f(x_k)$$
$$\times\ \lambda_n(\mathrm{d}x_1) \cdots \lambda_n(\mathrm{d}x_k)\mathbb{Q}(\mathrm{d}K_1) \cdots \mathbb{Q}(\mathrm{d}K_k),$$

since Θ^k is the intensity measure of $X^k_{\neq}$. The main step is then to show that

$$\int_{(\mathbb{R}^n)^k} \Phi^{(j)}_{m_1,\dots,m_k}(K_1 + x_1, ..., K_k + x_k; B) f(x_1) \cdots f(x_k)\, \lambda_n(\mathrm{d}x_1) \cdots \lambda_n(\mathrm{d}x_k)$$
$$= \int_B \int_{(\mathbb{R}^n)^k} f(z_1 - x_1) \cdots f(z_k - x_k) \Phi^{(j)}_{m_1,\dots,m_k}(K_1, ..., K_k; \mathrm{d}(x_1, ..., x_k))$$
$$\times\ \lambda_n(\mathrm{d}z_1) \cdots \lambda_n(\mathrm{d}z_k),$$

which follows from the translation covariance of the mixed measures and a simple change of variables.

Densities for other (mixed) functionals which occurred in previous sections, in the stationary case for X, can be treated in a similar way.

Now we consider the union set Z.

nonstatbm **Theorem 24.** *Let Z be a Boolean model with convex grains, where the intensity measure of the underlying Poisson process X satisfies (Thetadecomp3 32), and let $j \in \{0, ..., n\}$. Then, the signed Radon measure*

$$\mathbb{E}\, \Phi_j(Z, \cdot)$$

is locally finite and absolutely continuous with respect to λ_n. Its Radon-Nikodym derivative $\overline{V}_j(Z; \cdot)$ fulfills

$$\overline{V}_j(Z; z) = \lim_{r\to 0} \frac{1}{V_n(rK)} \mathbb{E}\, \Phi_j(Z, z + rK)$$

for λ_n-almost all $z \in \mathbb{R}^n$ and each $K \in \mathcal{K}$ with inner points.

Moreover, we have, for λ_n-almost all $z \in \mathbb{R}^n$,

$$\overline{V}_n(Z;z) = 1 - \mathrm{e}^{-\overline{V}_n(X;z)}$$

and

$$\overline{V}_j(Z,z) = e^{-\overline{V}_n(X;z)} \Bigg(\overline{V}_j(X;z) + \sum_{k=2}^{n-j} \frac{(-1)^{k+1}}{k!} \sum_{\substack{m_1,\dots,m_k=j \\ m_1+\dots+m_k=(k-1)n+j}}^{n-1} \overline{V}^{(j)}_{m_1,\dots,m_k}(X,\dots,X;z,\dots,z) \Bigg),$$

for $j = 0,\dots,n-1$.

Proof. The assertions on the volume density hold in greater generality. In fact, for a RACS Z, we have

$$\mathbb{E}\,\lambda_n(Z \cap A) = \int_A \mathbb{P}(z \in Z)\,\lambda_n(\mathrm{d}z), \qquad A \in \mathcal{B}(\mathbb{R}^n).$$

Hence, the measure $\mathbb{E}\,\lambda_n(Z \cap \cdot)$ is absolutely continuous and has density $\overline{V}_n(Z,z) := \mathbb{P}(z \in Z)$. For a Boolean model Z with compact grains,

$$\begin{aligned} \mathbb{P}(z \in Z) &= 1 - \mathbb{P}(\{z\} \cap Z = \emptyset) = 1 - \mathrm{e}^{-\Theta(\mathcal{F}_{\{z\}})} \\ &= 1 - \exp\left(-\int_{\mathcal{C}_0}\int_{\mathbb{R}^n} \mathbf{1}_{x+K}(z) f(x)\,\lambda_n(\mathrm{d}x)\mathbb{Q}(\mathrm{d}K)\right) \\ &= 1 - \exp\left(-\int_{\mathcal{C}_0}\int_K f(z-y)\,\lambda_n(\mathrm{d}y)\mathbb{Q}(\mathrm{d}K)\right) \\ &= 1 - \mathrm{e}^{-\overline{V}_n(X;z)}. \end{aligned} \tag{33}$$

voldens

Now we consider convex grains and $j \in \{0,\dots,n-1\}$. Again, we leave out the technical parts concerning the local finiteness etc. We apply Theorem 9 with $\varphi(K) := \Phi_j(K,B)$, B a fixed bounded Borel set, and choose a convex body K_0 with B in its interior. Because of (32), we obtain

expectation Thetadecomp3

$$\begin{aligned} \mathbb{E}\,\Phi_j(Z,B) &= \mathbb{E}\,\Phi_j(Z \cap K_0, B) \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \int_{\mathcal{K}_0} \cdots \int_{\mathcal{K}_0} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Phi_j(K_0 \cap x_1K_1 \cap \cdots \cap x_kK_k, B) \\ &\qquad \times f(x_1)\cdots f(x_k)\,\lambda_n(\mathrm{d}x_1)\cdots\lambda_n(\mathrm{d}x_k)\mathbb{Q}(\mathrm{d}K_1)\cdots\mathbb{Q}(\mathrm{d}K_k). \end{aligned}$$

The iterated translative formula for curvature measures, in its version (49) from [S], shows that

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \Phi_j(K_0 \cap x_1K_1 \cap \cdots \cap x_kK_k, B) f(x_1)\cdots f(x_k)\lambda_n(\mathrm{d}x_1)\cdots\lambda_n(\mathrm{d}x_k)$$
$$= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{1}_B(z) f(x_1)\cdots f(x_k)\Phi_j(K_0 \cap x_1K_1 \cap \cdots \cap x_kK_k, \mathrm{d}z) \times \lambda_n(\mathrm{d}x_1)\cdots\lambda_n(\mathrm{d}x_k)$$
$$= \sum_{\substack{m_1,\dots,m_k=j \\ m_1+\cdots+m_k=(k-1)n+j}}^{n} \int_B \int_{(\mathbb{R}^n)^k} f(z-x_1)\cdots f(z-x_k) \times \Phi^{(j)}_{m_1,\dots,m_k}(K_1,\dots,K_k; d(x_1,\dots,x_k))\,\lambda_n(\mathrm{d}z).$$

Therefore, we obtain from Theorem 23

$$\mathbb{E}\,\Phi_j(Z,B) = \int_B \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \sum_{\substack{m_1,\dots,m_k=j \\ m_1+\cdots+m_k=(k-1)n+j}}^{n} \overline{V}^{(j)}_{m_1,\dots,m_k}(X,\dots,X; z,\dots,z) \right) \lambda_n(\mathrm{d}z).$$

The remaining part of the proof follows now the standard procedure, since the decomposition properties for mixed measures carry over to the local densities. □

Hints to the literature. Nonstationary Boolean models were first studied in detail by Fallert [Fa92], [Fa96], who also defined quermass densities and proved the basic Theorems 23 and 24. Corresponding results for mixed volumes are given in [We01].

3.4 Sections of Boolean models

If Z is a Boolean model in $\mathbb{R}^n$ and L any affine subspace (with dimension in $\{1,\dots,n-1\}$), then the intersection $Z \cap L$ is a Boolean model. This follows from the fact that the intersections $K \cap L$, $K \in X$, build a Poisson process again, as can be easily checked. It is therefore natural to ask, how the characteristic quantities of Z and $Z \cap L$ are connected. Formulas of this kind are helpful since they can be used to estimate quantities of Z from lower dimensional sections. Such estimation problems are typical for the field of stereology (see ???).

It is, in general, quite difficult to get information about the grain distribution $\mathbb{Q}$ from sections. An exception is a Boolean model Z with balls as grains. Here, an integral equation connecting the radii distributions of Z and $Z \cap L$ can be given and used for an inversion (which is however unstable and leads to an ill-posed problem). For the quermass densities, the Crofton formulas from integral geometry can be used. These lead to formulas which hold for RACS Z and particle processes X and therefore can be used for Boolean models as well. We only state some results here, the proofs are quite similar to the ones given previously for intersections of Z (or X) with sampling windows K_0.

crofton1 **Theorem 25.** *Let Z be a stationary random $\mathcal{S}$-set fulfilling (16) and let X be a stationary process of particles in $\mathcal{K}'$. Let $L \in \mathcal{L}_k^n$ be a k-dimensional linear subspace, $k \in \{0, ..., n\}$, and let ϑ be a random rotation with distribution ν and independent of Z and X. Then*

$$\mathbb{E}_\nu \, \overline{V}_j(Z \cap \vartheta L) = c_{j,n}^{k,n+j-k} \overline{V}_{n+j-k}(Z)$$

and

$$\mathbb{E}_\nu \, \overline{V}_j(X \cap \vartheta L) = c_{j,n}^{k,n+j-k} \overline{V}_{n+j-k}(X),$$

for $j = 0, ..., k$.

The cases $k = n$ and $k = 0$ are trivial and not of interest. If $j = k$ or if Z is isotropic, the expectation $\mathbb{E}_\nu$ can be omitted.

Hints to the literature. Crofton formulas are classical in stochastic geometry (see [SKM95] or [SW00]).

4 Contact distributions

In the previous sections, we mostly discussed properties of Boolean models Z which are based on intrinsic geometric quantities of the set Z. A corresponding statistical analysis of Z would require observations of Z from inside, for example by counting pixel points of Z in a bounded sampling window K_0, by considering curvatures in boundary points of Z, etc. In this last section, we shall consider quantities which are based on observations outside Z, for example by measuring the distance $d(x, Z)$ from a given point $x \notin Z$ to Z. The distribution of $d(x, Z)$ is a particular case of a contact distribution. Such contact distributions are often easier to estimate in practice and will also give us interesting information about Z.

Contact distributions and their generalizations will be treated in the first three subsections. The last subsection contains additional material on Boolean models. Various aspects which are important but could not be mentioned so far are addressed here, although in a quite short, summarizing form. We also shall mention some open problems on Boolean models.

4.1 Contact distribution with structuring element

Contact distributions can be introduced for arbitrary RACS Z. In their general form, they are based on a given convex body $B \in \mathcal{K}'$ with $0 \in B$, the **structuring element**, and describe the distribution of the 'B-distance' from a point $x \notin Z$ to the set Z.

To give a more precise definition, we start with

$$d_B(x, A) := \inf\{r \geq 0 : (x + rB) \cap A \neq \emptyset\},$$

for $x \in \mathbb{R}^n$ and $A \in \mathcal{F}$. In this situation, we also call B a **gauge body**. The set on the right-hand side may be empty (for example, if $0 \in \partial B$ or if B is lower dimensional), then $d_B(x, Z) = \infty$. We also put $d_B(A, x) := d_{B^*}(x, A)$ and $d_B(x, y) := d_B(x, \{y\}), y \in \mathbb{R}^n$. If the gauge body has inner points and is symmetric with respect to 0, then $(x, y) \mapsto d_B(x, y)$ is a metric on $\mathbb{R}^n$ and hence $(\mathbb{R}^n, d_B)$ is a Minkowski space. Of course, for $B = B^n$ we obtain the Euclidean metric, we then write $d(x, A)$ instead of $d_{B^n}(x, A)$, etc.

Now we consider a RACS Z and $x \in \mathbb{R}^n$ and define the **contact distribution** H_B at x by

$$\begin{aligned} H_B(x, r) &:= \mathbb{P}(x + rB \cap Z \neq \emptyset \mid x \notin Z) \\ &= \mathbb{P}(d_B(x, Z) \leq r \mid x \notin Z), \qquad r \geq 0. \end{aligned}$$

Thus, $H_B(x, \cdot)$ is the distribution function of the B-distance $d_B(x, Z)$ of x to Z, conditional to the event that $x \notin Z$ (it is of course also possible to consider the unconditional distribution function, but the above definition is the widely used one). Actually, $H_B(x, \cdot)$ is the contact distribution function and the contact distribution would be the corresponding probability measure on $[0, \infty]$, but it is now common use in the literature not to distinguish between these two notions. If $\mathbb{P}(x \notin Z) > 0$, then one can express $H_B(x, \cdot)$ in terms of the (local) volume density, namely

$$\begin{aligned} H_B(x, r) &= \mathbb{P}(x + rB \cap Z \neq \emptyset \mid x \notin Z) \\ &= 1 - \frac{\mathbb{P}(x \notin Z + rB^*)}{\mathbb{P}(x \notin Z)} \\ &= \frac{\mathbb{P}(x \in Z + rB^*) - \mathbb{P}(x \in Z)}{1 - \mathbb{P}(x \in Z)} \\ &= \frac{\overline{V}_n(Z + rB^*, x) - \overline{V}_n(Z, x)}{1 - \overline{V}_n(Z, x)}. \end{aligned}$$

The condition $\mathbb{P}(x \notin Z) > 0$ is always satisfied if Z is stationary and non-trivial, i.e. $Z \neq \mathbb{R}^n$ with positive probability.

Now we assume that Z is a Boolean model (with compact grains). Then, voldens (33) implies

$$\begin{aligned} H_B(x, r) &= 1 - \frac{\overline{V}_n(Z + rB^*, x)}{\overline{V}_n(Z, x)} \\ &= \frac{e^{-\overline{V}_n(X;x)} - e^{-\overline{V}_n(X + rB^*;x)}}{e^{-\overline{V}_n(X;x)}} \\ &= 1 - \exp\left(- \int_{\mathcal{C}_0} \int_{(K + rB^*) \setminus K} f(x - y)\, \lambda_n(dy) \mathbb{Q}(dK) \right). \qquad (34) \end{aligned}$$ voldens2

We first discuss the stationary situation and assume convex grains. Hence, $f \equiv \gamma$ and $H_B(x, r)$ is independent of x. We put $H_B(r) := H_B(0, r)$. Then,

$$
\begin{aligned}
H_B(r) &= 1 - \exp\left(-\gamma \int_{\mathcal{K}_0} V_n((K + rB^*) \setminus K)\mathbb{Q}(\mathrm{d}K)\right) \\
&= 1 - \exp\left(-\gamma \sum_{i=0}^{n-1} \binom{n}{i} r^{n-i} \int_{\mathcal{K}_0} V(\underbrace{K, ..., K}_{i}, \underbrace{B^*, ..., B^*}_{n-i})\mathbb{Q}(\mathrm{d}K)\right) \\
&= 1 - \exp\left(-\sum_{i=0}^{n-1} r^{n-i} \overline{V}^{(0)}_{i,n-i}(X, B)\right),
\end{aligned}
$$

where we used the expansion of $V_n(K+rB^*)$ into mixed volumes (resp. mixed functionals). We thus get the following result.

contdis

Theorem 26. *Let Z be a stationary Boolean model with convex grains and B a gauge body. Then*

$$
H_B(r) = 1 - \exp\left(-\sum_{i=0}^{n-1} r^{n-i} \overline{V}^{(0)}_{i,n-i}(X, B)\right), \qquad \text{for } r \geq 0.
$$

Interesting special cases are $B = B^n$ (the **spherical contact distribution**) and $B = [0, u], u \in S^{n-1}$, (the **linear contact distribution**). For the spherical contact distribution, we get

$$
H_{B^n}(r) = 1 - \exp\left(-\sum_{i=0}^{n-1} r^{n-i} \kappa_{n-i} \overline{V}_i(X)\right),
$$

the result we already obtained at the end of Section 1 (for the capacity functional). For $H_{[0,u]}$, we observe that

$$
V(\underbrace{K, ..., K}_{i}, \underbrace{[0, u], ..., [0, u]}_{n-i}) = 0, \qquad \text{for } i = 0, ..., n-2,
$$

whereas $V(K, ..., K, [0, u])$ is proportional to the $(n-1)$-volume of the orthogonal projection of K onto the hyperplane orthogonal to u,

$$
V(K, ..., K, [0, u]) = \frac{2}{n} V_{n-1}(K \mid u^\perp).
$$

Therefore,

$$
H_{[0,u]}(r) = 1 - \exp\left(-\frac{2\gamma r}{n} \int_{\mathcal{K}_0} V_{n-1}(K \mid u^\perp)\mathbb{Q}(\mathrm{d}K)\right).
$$

From the formula

$$
V_{n-1}(K \mid u^\perp) = \frac{1}{2} \int_{S^{n-1}} |\langle x, u\rangle| S_{n-1}(K, \mathrm{d}x),
$$

we obtain

$$\begin{aligned} H_{[0,u]}(r) &= 1 - \exp\left(-\frac{r}{n}\int_{S^{n-1}} |\langle x,u\rangle| S_{n-1}(B(X),\mathrm{d}x)\right) \\ &= 1 - \exp\left(-rh(\Pi B(X),u)\right), \end{aligned}$$

with the projection body $\Pi B(X)$ of the Blaschke body $B(X)$ of X. The linear contact distribution $H_{[0,u]}$ therefore determines the support value $h(\Pi B(X),u)$. If we know $H_{[0,u]}$ in all directions $u \in S^{n-1}$, the convex body $\Pi B(X)$ is determined. The corresponding integral transform

$$h(\Pi B(X),u) = \frac{1}{n}\int_{S^{n-1}} |\langle x,u\rangle| S_{n-1}(B(X),\mathrm{d}x)$$

is called the **cosine transform**. It is injective on even measures, which means that $B(X)$ and therefore the mean area measure $\overline{S}_{n-1}(X,\cdot) = S_{n-1}(B(X),\cdot)$ are determined, provided they are symmetric. A sufficient condition for the latter is of course that all particles $K \in X$ are centrally symmetric. As we mentioned in Section 3.1, for the estimation of $\overline{S}_{n-1}(X,\cdot)$ in the nonsymmetric case modifications of the linear contact distribution are necessary.

Now we consider the nonstationary case (but still with convex grains). Then, voldens2 (34) shows that we need a general Steiner-type formula for

$$\int_{(K+rB^*)\setminus K} f(x-y)\,\lambda_n(\mathrm{d}y).$$

This can be given using relative support measures $\Theta_{j;n-j}(K;B^*;\cdot)$ and results in

$$\begin{aligned} &\int_{(K+rB^*)\setminus K} f(x-y)\,\lambda_n(\mathrm{d}y) \\ &\quad = \sum_{j=0}^{n-1}\binom{n-1}{j}\int_0^r\int_{\mathbb{R}^n\times\mathbb{R}^n} t^{n-1-j} f(x-tu-y)\Theta_{j;n-j}(K;B^*;\mathrm{d}(y,u))\mathrm{d}t. \end{aligned}$$

The definition of relative support measures requires that K and B^* are in general relative position, which means that for all support sets $K(u), B^*(u)$ and $(K+B^*)(u)$, $u \in S^{n-1}$, of K, B^* and $K+B^*$ we have

$$\dim(K+B^*)(u) = \dim K(u) + \dim B^*(u).$$

The condition of general relative position is satisfied, for example, if one of the bodies K, B^* is strictly convex.

contdis2 **Theorem 27.** *Let Z be a Boolean model with convex grains satisfying* Thetadecomp3 (32) *and let B be a gauge body. Assume that K and B^* are in general relative position, for $\mathbb{Q}$-almost all K. Then*

$$H_B(x,r) = 1 - \exp\left(-\int_0^r \lambda_B(x,t)\mathrm{d}t\right), \qquad \textit{for } r \geq 0,$$

with

$$\lambda_B(x,t) := \sum_{j=0}^{n-1} \binom{n-1}{j} t^{n-1-j} \int_{\mathcal{K}_0} \int_{\mathbb{R}^n \times \mathbb{R}^n} f(x - tu - y) \\ \times \Theta_{j;n-j}(K; B^*; \mathrm{d}(y,u)) \mathbb{Q}(\mathrm{d}K).$$

If $B = B^n$, the measure $\Theta_{j;n-j}(K; B^*; \cdot)$ is the (ordinary) support measure $\Theta_j(K, \cdot)$.

Hints to the literature. For stationary (or stationary and isotropic) Boolean models, the results of this section are classical and can be found in [SKM95] and [SW00]. For the nonstationary case, [Hu00] and [HL00] are the main references. A survey on contact distributions is given in [HLW02b].

4.2 Generalized contact distributions

The results of the last section can be generalized in various directions.

First we can replace the point x by a convex body L and thus measure the B-distance from L to Z, provided that $L \cap Z = \emptyset$. Hence, we define

$$d_B(L, A) := \inf\{r \geq 0 : (L + rB) \cap A \neq \emptyset\},$$

for $A \in \mathcal{F}$, and

$$\begin{aligned} H_B(L, r) &:= \mathbb{P}(L + rB \cap Z \neq \emptyset \mid L \cap Z = \emptyset) \\ &= \mathbb{P}(d_B(L, Z) \leq r \mid L \cap Z = \emptyset). \end{aligned}$$

Then we have to work with a corresponding more general notion of general relative position, use mixed relative support measures $\Theta_{i,j;n-i-j}(L, K^*; B; \cdot)$, and get the following result.

Theorem 28. *Let Z be a Boolean model with convex grains satisfying (32) and let B be a gauge body. Let $L \in \mathcal{K}'$ be such that L, K and B^* are in general relative position, for $\mathbb{Q}$-almost all K. Then*

$$H_B(L, r) = 1 - \exp\left(-\int_0^r \lambda_B(L, t)\mathrm{d}t\right), \qquad \text{for } r \geq 0,$$

with

$$\lambda_B(L,t) := \sum_{i,j,k=0}^{n-1} \binom{n-1}{i,j,k} t^k \int_{\mathcal{K}_0} \int_{\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n} f(x + tu + y) \\ \times \Theta_{i,j;k+1}(L, K^*; B; \mathrm{d}(x,y,u)) \mathbb{Q}(\mathrm{d}K).$$

Here we used the multinomial coefficient

$$\binom{n-1}{i,j,k} := \frac{(n-1)!}{i!j!k!},$$

for $i,j,k \in \mathbb{N}_0$ with $i+j+k=n-1$ (and 0 else).

If Z is stationary, we can use

$$\Theta_{i,j;k+1}(L,K^*;B;\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n) = nV(L\,[i],K^*\,[j],B\,[k+1])$$

and get

$$H_B(L,r) = 1-\exp\left(\sum_{i,j,k=0}^{n-1}\binom{n}{i,j,k+1} r^{k+1}\overline{V}(L\,[i],X^*\,[j],B\,[k+1])\right).$$

As a further generalization, we may include directions. As one can show, if $L\cap Z=\emptyset$ (and under the condition of general relative position), with probability one the pair of points $(p_B(L,Z),p_B(Z,L))\in\partial L\times\partial Z$ which realizes the B-distance,

$$d_B(L,Z)=d_B(p_B(L,Z),p_B(Z,L)),$$

is uniquely determined. Let $u_B(L,Z)\in S^{n-1}$ be the direction from $p_B(L,Z)$ to $p_B(Z,L)$, thus

$$p_B(Z,L)=p_B(L,Z)+d_B(L,Z)u_B(L,Z).$$

We now may even add some geometric information $l_B(L,Z)=\rho_Z(p_B(Z,L))$ in the boundary point $p_B(Z,L)$, which is translation covariant and 'local' in the sense that it depends only on Z in an arbitrarily small neighborhood of $p_B(Z,L)$. For example, in the plane and for a Boolean model with smooth convex grains, $\rho_Z(p_B(Z,L))$ could be the curvature of Z in $p_B(Z,L)$. A quite general version of Theorem 28 then yields the formula

$$\begin{aligned}
&\mathbb{E}\left(\mathbf{1}\{d_B(L,Z)<\infty\}g(d_B(L,Z),u_B(L,Z),p_B(L,Z),l_B(L,Z))\,|\,Z\cap L=\emptyset\right)\\
&=\sum_{i,j,k=0}^{n-1}\binom{n-1}{i,j,k}\int_0^\infty t^k(1-H_B(L,t))\int_{\mathcal{K}_0}\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n} g(t,u,x,\rho_{y+K^*}(0))\\
&\qquad\times f(x+tu+y)\Theta_{i,j;k+1}(L,K^*;B;\mathrm{d}(x,y,u))\mathbb{Q}(\mathrm{d}K)\mathrm{d}t,
\end{aligned}$$

for any measurable nonnegative function g.

Hints to the literature. The results of this section were taken from [HLW02a] (see also [HLW02b]).

4.3 Characterization of convex grains

Hints to the literature. The results in this section can be found in [HLW04].

4.4 Miscellaneous results, open problems

Hints to the literature.

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