Random sets (in particular Boolean models)

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Random sets are models to describe random geometric structures, as they occur in applications in natural or technical sciences. In order to fit a random set model Z to a given structure, more specific assumptions on Z have to be made. A basic and widely used assumption is that Z arises as the union set of particles grown around points of a Poisson point process in space (this is the Boolean model). After a short introduction into the general theory of random sets, the lectures will concentrate on Boolean models in d-dimensional space. Various characteristics of Boolean models will be described (quermass densities, contact distributions) and connections to integral geometry and point process theory will be explained, first under strong invariance assumptions (stationarity and isotropy), but then also without these. The titles of the lectures (of 90 minutes each) are:

1. Random sets, particle processes and Boolean models
2. Stationary and isotropic models; quermass densities and other mean values
3. Stationary and non-stationary random sets; local densities
4. Contact distributions

A random set is simply a set-valued random variable. For a formal definition, one needs a σ-algebra on the class of sets to be considered, e.g. the Borel σ-algebra w.r.t. a natural topology on sets. For random closed sets Z in $\mathbb{R}^d$, the corresponding setup has been described in the classical book of Matheron (1975); this setup will also be the basis for our considerations. The generation of interesting classes of random (closed) sets requires more specific models. A powerful approach makes use of point processes and considers random sets as union sets of point processes on the space of compact sets in $\mathbb{R}^d$ (particle processes). Alternatively, one can start with a (ordinary) point process Φ in $\mathbb{R}^d$, attach random compact sets to each point of Φ and then take the union set. If the underlying process Φ is a Poisson process and the attachment is done independently and with the same distribution, then the resulting random set is called a Boolean model. The relations between random sets, point processes and Boolean models will be the content of the first lecture where we also discuss invariance properties and connections to integral geometry.

For the geometric description of random sets (in particular of Boolean models), mean values of geometric functionals (mean volume, mean surface area, mean Euler number etc.) are of interest. Their definition requires more specific models (e.g. based on convexity) and also some invariance properties like stationarity and/or isotropy. In the second lecture, we discuss different approaches to define such geometric mean values (the quermass densities and generalizations) and we show how the kinematic formulas for intrinsic volumes resp. curvature measures lead to explicit results for Boolean models which are stationary and isotropic.

In Lecture 3, we study first the stationary and non-isotropic case and then consider Boolean models without any invariance assumption. It turns out that the basic formulas for geometric mean values extend in a suitable way, if the kinematic formulas are replaced by translative counterparts. For the non-stationary case this also requires a definition of local densities as Radon-Nikodym derivatives of mixed curvature measures.

Whereas quermass densities can be viewed as intrinsic quantities of Boolean models, contact distributions describe Boolean models Z (and more general random sets) from outside. As the simplest version, we consider the distribution of the distance to Z from a point outside Z, i.e. the instant of the first contact of a growing ball with Z. Various generalizations of this concept are possible: the ball can be replaced by other structuring elements (general convex bodies, e.g. segments) and additional measurements can be taken into account (the direction of contact or even the local geometry of Z at the point of contact). In the final lecture we show how these various contact distributions are connected to Steiner-type formulas and we use a recent general Steiner formula to obtain results for Boolean models with arbitrary compact grains (without any convexity assumption). We also discuss the question whether (or under which conditions) the distribution of the Boolean model Z is determined by contact distributions.

Some References:

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