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Analytical investigation of oscillations in intersecting flows of pedestrian and vehicle traffic

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In two intersecting many-particle streams, one can often find the emergence of oscillatory patterns. Here, we investigate the interaction of pedestrians with vehicles, when they try to cross a road. A numerical study of this coupled pedestrian-vehicle delay problem has been presented in a previous paper. Here, we focus on the analytical treatment of the problem, which requires us to use a simplified car-following model. Our analytical results for the transition to oscillatory pedestrian and traffic flows and the average waiting times are well supported by numerical evaluations and give a detailed picture of the collective dynamics emerging when pedestrians try to cross a road. The mathematical expressions allow one to identify the dependence on model parameters such as the vehicle or pedestrian arrival rate, and the safety factor of pedestrian gap acceptance. We also calculate a formula for the vehicle time gap distribution, which corresponds to the departure time contribution of a $M/D/1$ queue characterized by Poissonian distributed Markovian arrivals, 1 service channel, and deterministic departures.

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I. INTRODUCTION

Pattern formation is a widespread feature of driven many-particle systems. In particular, oscillatory patterns are found in fluids, granular materials, colloidal systems, and traffic flows. A typical example is the stop-and-go waves in traffic flows on freeways caused by a delayed adaptation to changing traffic conditions [1–5]. Emergent oscillations have been discovered in such different systems as the density oscillator [6], ticking hour glass [7], RNA polymerase traffic on DNA [8], pedestrians passing a bottleneck [9,10], or ants [11]. Oscillatory patterns have also been found in two intersecting pedestrians streams [12] or simulations of colloidal systems [13].

Although the subject is rather old [14–16], the crossing of vehicle streams by pedestrians has recently attracted an increasing interest, also among physicists [17–21]. However, the problem of interactions between vehicles and pedestrians, when pedestrians are trying to cross a road, has not yet been sufficiently understood. The mathematical investigation of this problem will be the subject of this paper. Numerical studies have shown a transition from crossing the road one by one or in small groups to coupled oscillations of pedestrian and vehicle flows, if pedestrians use small gaps to cross the road [21]. In the following, the dynamics of this phenomenon and the parameter dependence of the transition point will be investigated analytically.

Our paper is organized as follows. Section II formulates the model for the pedestrian and vehicle behavior and their interactions. Moreover, we calculate a formula for an idealized vehicle time gap distribution. In Sec. III, we will derive analytical results on the dynamic behavior of interacting pedestrian and vehicle flows. Moreover, we will compare these results with numerical evaluations of computer simulations of the underlying model. Our analytical formulas for the transition point and the waiting times of pedestrians and cars are well compatible with numerically determined data. Finally, we will summarize and discuss our results in Sec. IV, which are relevant for many systems with intersecting flows or competing processes.

II. FORMULATION OF THE MODEL

A. Vehicle behavior

In our simplified model of vehicle dynamics, cars are treated as moving objects of length l_0. We assume a constant arrival flow Q_{arr} of vehicles and that new cars try to enter the investigated road section with a probability $q=Q_{\text{arr}}dt$ per time step dt. This implies an exponential time gap distribution, which is modified by vehicle-vehicle interactions (see Sec. II B). In fact, a vehicle with the speed v following a leading vehicle with speed v_* is assumed to decelerate as $dv/dt=-a$, if $v>0$ and

$$\Delta x < l_0 + d_0 + \frac{v^2}{2a} - \frac{{v_*}^2}{2a}, \tag{1}$$

where Δx denotes the distance between the two vehicles, l_0 the vehicle length, and d_0 the preferred minimum bumper-to-bumper distance among cars. This condition guarantees accident-free driving [22]. For a $>$ sign in Eq. (1), the vehicle accelerates as $dv/dt=a$, delayed by the reaction time T, until the maximum (free) speed v_0 is reached. For an = sign in Eq. (1), the velocity is not changed, i.e., $dv/dt=0$. The above continuous car-following model may be called the constant-deceleration-delayed-acceleration model and has some similarities with the slow-to-start cellular automation model [23]. A model similar to Eq. (1) has also been used in the study of cooperation in a toy autobahn model [24].

We assume that pedestrians enter the street at the crossing point O, when they consider it safe (see Sec. II C). Moreover, crossing the road takes a time period τ. In order to avoid accidents with pedestrians, vehicles decelerate as $dv/dt=-a$ if necessary. We consider two different deceleration rules.

(a) Careful drivers. The closest car to a pedestrian on the street decelerates, if the distance $d(t)=-x(t)$ to the crossing point O is within the range

$$0 \leqslant d(t) \leqslant d_0 + \frac{v^2}{2a} \tag{2}$$

where d_0 is the safety distance that a car should keep from a crossing pedestrian. We assumed this safety distance to be identical to the minimum bumper-to-bumper distance among vehicles appearing in (1).

(b) Aggressive drivers. The closest car starts to decelerate at the time t_0 determined so that the distance to the pedestrian corresponds to the safety distance $d(t_n+\tau)=d_0$ at the time $t_n+\tau$ when the last (the nth) pedestrian on the street (entering at time t_n) leaves the road after the crossing time τ.

After the last pedestrian has left the street, i.e., at time $t_n+\tau$, the car accelerates as $dv/dt=a$, until it has reached its desired velocity v_0 again. The characteristic distance between stopped vehicles in a queue is the vehicle length l_0 plus the minimum bumper-to-bumper distance d_0, which defines the jam density

$$\rho_{\text{jam}} := \frac{1}{l_0 + d_0}. \tag{3}$$

In the following, we will assume that a car starts to accelerate after its leader is delayed by the reaction time T. This implies that the following car has reached the position of the leading car in the queue after a time period $T + \sqrt{2(d_0+l_0)/a}$ and that the distance to the leading car is $l_0 + d_0 + Tv_0$, when the following car has reached its maximum velocity v_0. Therefore, the outflow from a traffic jam starts with a value of $[T+\sqrt{2(d_0+l_0)/a}]^{-1}$ and eventually reaches the characteristic (maximum) value

$$Q_{\text{out}} := \left(T + \frac{l_0 + d_0}{v_0} \right)^{-1}, \tag{4}$$

while the traffic jam (queue) resolves upstream with the characteristic speed

$$c := \frac{l_0 + d_0}{T} = \frac{1}{\rho_{\text{jam}} T} \tag{5}$$

due to the distance l_0+d_0 between queued cars and the delay T in acceleration. Moreover, when a vehicle is stopped at point $x(t)=-d_0$, the forming traffic jam behind it propagates upstream with the velocity [25]

$$C := \left(\frac{\rho_{\text{jam}}}{Q_{\text{arr}}} - \frac{1}{v_0} \right)^{-1}, \tag{6}$$

which depends on the vehicle arrival rate Q_{arr}.

The proposed simple car-following model essentially reflects the features of the section-based, fluid-dynamic traffic-flow model proposed in Ref. [25], with the only difference that the acceleration and braking processes require time periods of $T+v_0/a$ and T/v_0, respectively. Apart from scaling time and space variables in order to get rid of two more model parameters, it is hard to think of any further simplification of the above vehicle model without sacrificing fundamental properties of traffic flows such as the constant outflow from traffic jams and the characteristic jam resolution speed [26]. Nevertheless, it may be interesting to study the limit $a\to\infty$ of unlimited acceleration possibilities, which eliminates acceleration and deceleration times. More realistic variants of the above car-following model, however, should distinguish different acceleration and deceleration strengths a and b, which have been set equal here for the sake of simplicity. A stochastic variant of this model describing a fluctuating acceleration behavior would also be interesting to study.

B. Idealized vehicle distance distribution

In our vehicle simulations, we have generated vehicles with initial velocity $v=0$ at the upstream boundary of the simulation stretch according to the exponential time gap distribution $Q_{\text{arr}}e^{-Q_{\text{arr}}T'}$, where T' denotes the actual time gap. However, according to our car-following model, vehicles had gained at least their preferred distance $D=l_0+d_0+v_0T$, when they reached the maximum speed v_0. According to theoretical considerations, this changed the effective time-gap distribution at the crossing point to

$$P(T') = Q_{\text{arr}} T_0 \delta(T' - T_0) + (1 - Q_{\text{arr}} T_0) Q_{\text{arr}} e^{-Q_{\text{arr}}(T' - T_0)} \Theta(T' - T_0) \tag{7}$$

with $T_0=D/v_0$ (see the Appendix), when no vehicles at the entry point were dropped. That is, a fraction $Q_{\text{arr}}T_0$ of vehicles will follow with the desired time gap T_0, while the rest has an exponentially distributed, larger time gap $T' > T_0$. $\delta(y)$ denotes Dirac's delta function, while the Heaviside function $\Theta(y)$ is 1 for $y \geqslant 0$ and 0 otherwise.

Our exponentially distributed vehicle generation mechanism sometimes causes a virtual queue of vehicles at the upstream boundary, which can be avoided by generating vehicles according to the resulting time gap distribution (7). In fact, our implementation of the boundary conditions corresponds to a $M/D/1$ queuing system [27,28], i.e., to a queue with Poissonian distributed Markovian arrivals (where the time gaps between successive arrivals are exponentially distributed), while the service rate $1/T_0$ is assumed to be deterministic. (The "1" stands for one "channel," i.e., no parallel service.)

Now, let P_0 be the probability that no vehicle is waiting in the queue to be served, i.e., to enter the road. The probability of releasing the next vehicle with a time gap $T'=T_0$ is then given by the probability $(1-P_0)$ of having queued vehicles waiting to enter, plus the probability $P_0(1-e^{-Q_{\text{arr}}T_0})$ that we have the no-queue case and a vehicle arrives during the service time T_0. In cases with no queue where the time gap T' of the next arriving vehicle is greater than T_0, we have an exponential time gap distribution $Q_{\text{arr}}e^{-Q_{\text{arr}}T'}/e^{-Q_{\text{arr}}T_0}$, where

$e^{-Q_{\mathrm{arr}}T_0}$ is the normalization factor of the conditional probability of finding time gaps larger than T_0. Altogether, we obtain the time gap distribution

$$P(T') = [(1-P_0) + P_0(1-e^{-Q_{\mathrm{arr}}T_0})]\delta(T'-T_0) + P_0 Q_{\mathrm{arr}} e^{-Q_{\mathrm{arr}}T'}\Theta(T'-T_0). \tag{8}$$

Demanding

$$\frac{1}{Q_{\mathrm{arr}}} = \int_0^\infty dT'\, T'\, P(T') = T_0 + \frac{P_0}{Q_{\mathrm{arr}}} e^{-Q_{\mathrm{arr}}T_0}, \tag{9}$$

i.e., that the vehicle flow Q_{arr} and, therefore, the average time gap remains unchanged, we find

$$P_0 = (1 - Q_{\mathrm{arr}}T_0)e^{Q_{\mathrm{arr}}T_0}. \tag{10}$$

This implies the idealized vehicle time-gap distribution (7), which will be necessary to evaluate the expected waiting time of pedestrians for a suitable time gap to cross the road (see the Appendix).

C. Pedestrian behavior

We will assume that pedestrians enter the sidewalk of the street at the crossing point O with probability $p=\lambda\, dt$ per time step dt, i.e., λ denotes the arrival rate of pedestrians. If there is no sufficient gap in the vehicle stream to cross, they accumulate around point O, but they start immediately to enter the road at time t, if $v(t)=0$ (i.e., if the vehicle velocity is zero) or if

$$d(t) > d_0 \quad \text{and} \quad \Delta t(t) := \frac{d(t)}{v(t)} \geqslant \sigma\tau \tag{11}$$

[i.e., if the distance $d(t)$ is larger than the preferred safety distance d_0 and the time gap Δt is large enough to cross the road]. Here, Δt is the time to collision of the nearest approaching vehicle and σ a safety factor applied by pedestrians. τ is the time period required for a pedestrian to cross (one lane of) the road. We may distinguish two limiting cases of gap selection, i.e., interactions with approaching vehicles.

(1) Careful pedestrians assume that cars may not decelerate and approach with their desired velocity v_0. They cross the road only if the car at no time comes closer than the preferred safety distance d_0, which implies the following choice of the safety factor:

$$\sigma = \sigma_1 := 1 + \frac{d_0}{v_0\tau}. \tag{12}$$

(2) Daring pedestrians enter the road if a car with velocity v_0 would not come closer than the preferred safety distance d_0, if it decelerated as $dv/dt=-a$ in order to avoid an accident. This implies the reduced safety factor

$$\sigma = \sigma_2 := 1 + \frac{d_0}{v_0\tau} - \frac{a\tau}{2v_0} = \sigma_1 - \frac{a\tau}{2v_0}. \tag{13}$$

In this case, a single pedestrian can force a car to stop, namely, when entering at a vehicle distance $d(t)=d_0 + v_0{}^2/(2a)$.

Realistic values of the safety factor σ are expected to be above σ_2.

For the following analysis, we will identify the time point $t=0$ with the time when the first pedestrian(s) who cause(s) a vehicle to decelerate enter(s) the road. The entering time of the next entering pedestrian is denoted by t_1, the entering time of the kth following pedestrian by t_k, and the entering time of the last (nth) following pedestrian before the car passes point O by t_n.

D. Simulated dynamic behavior of interacting vehicle and pedestrian flows

Simulations of vehicles interacting with pedestrians crossing a street have recently shown an interesting phenomenon. While for large enough values of the safety factor σ, pedestrians cross the road one by one or in small groups, one finds alternating pedestrian and vehicle streams if the safety factor is smaller than some critical value σ_0. This value can be exactly calculated for the above model [see Eq. (33)], which shows qualitatively the same dynamic behavior as the variant of the intelligent driver model (IDM) model studied in a previous publication [21]. Representative simulation results for the above proposed pedestrian and vehicle model are displayed in Fig. 1. The parameter values used in this paper are $a=1\ \mathrm{m/s^2}, \tau=2\ \mathrm{s}, T=0.9\ \mathrm{s},\ l_0=4\ \mathrm{m},\ d_0=2\ \mathrm{m}$, and $v_0=15\ \mathrm{m/s}$, and our numerical investigation focuses on careful drivers.

The reason for the observed oscillations is that pedestrians can force vehicles to stop, if they choose small time gaps Δt. However, if vehicles are stopped, they have to wait until there is a gap of period τ or larger in the pedestrian stream, before they can accelerate again. During this waiting time, a vehicle queue is formed, which can become very long, dependent on the vehicle arrival rate. Pedestrians cannot cross the road again, before this queue is completely dissolved, at least if

$$\sigma\tau > \frac{d_0 + Tv_0}{v_0} = T + \frac{d_0}{v_0}, \tag{14}$$

i.e., if the time gap between successive vehicles having left the queue is too short for pedestrians to enter the street, and if

$$\sigma\tau > \frac{l_0 + 2d_0 - (a/2)[\sqrt{2(l_0+d_0)/a} - T]^2}{a[\sqrt{2(l_0+d_0)/a} - T]}, \tag{15}$$

i.e., if the time gap with respect to the second car in the queue at the time $\sqrt{2(l_0+d_0)/a} > T$ (when the back of the first vehicle has passed the crossing point O) is not large enough for pedestrians to enter the street.

In summary, we may have alternating time periods in which pedestrians can cross the road and time periods in which cars can pass point O. In the following sections, based on statistical approaches, we will try to estimate the time period until a sufficiently large gap in the vehicle flow occurs to allow pedestrians a crossing of the road. Likewise, we will calculate the time period until queued vehicles find a large enough gap between crossing pedestrians, allowing them to accelerate again. Analytical results can be only gained for simple models as the one proposed above. Nevertheless, we expect qualitatively similar relationships for a broad class of other traffic models.

III. ANALYTICAL RESULTS AND COMPARISON WITH COMPUTER SIMULATIONS

A. Dynamics of vehicles reacting to pedestrians

Let t_0 be the time point when the car starts to decelerate in response to a crossing pedestrian. According to Secs. II A and II C, we find that the time to collision evolves in time according to

$$\Delta t(t) = \frac{d(t)}{v(t)} = \frac{d(0) - v_0 t}{v_0} = \frac{d(0)}{v_0} - t \quad \text{if } t < t_0. \tag{16}$$

For careful drivers, i.e., case (a), the start time of deceleration can be determined as

$$t_0 = \frac{d(0) - d_0}{v_0} - \frac{v_0}{2a}. \tag{17}$$

This yields the time to collision

$$\begin{aligned}\Delta t(t) &= \frac{d_0 + v_0{}^2/(2a) - v_0(t - t_0) + a(t - t_0)^2/2}{v_0 - a(t - t_0)} \\ &= \frac{v_0}{2a} - \frac{t - t_0}{2} + \frac{d_0}{v_0 - a(t - t_0)} \quad \text{if } t \geqslant t_0 \end{aligned}\tag{18}$$

(see Fig. 2) and the vehicle velocity

$$v(t_0 + \tau) = v_0 - a(\tau - t_0) = \frac{v_0}{2} - a\tau + a\frac{d(0) - d_0}{v_0} \tag{19}$$

after the pedestrian has crossed the road. If the vehicle velocity at the beginning of the braking maneuver is $v(t_0) < v_0$, one just has to replace v_0 by $v(t_0)$. For aggressive drivers, i.e., case (b), we find

$$t_0 = \tau - \sqrt{\frac{2v_0\tau}{a} - 2\frac{d(0) - d_0}{a}}, \tag{20}$$

$$\begin{aligned}\Delta t(t) &= \frac{d(t_0) - v_0(t - t_0) + a(t - t_0)^2/2}{v_0 - a(t - t_0)} \\ &= \frac{v_0}{2a} - \frac{t - t_0}{2} + \frac{d(t_0) - v_0{}^2/(2a)}{v_0 - a(t - t_0)} \quad \text{if } t \geqslant t_0, \end{aligned}\tag{21}$$

and

$$\begin{aligned} v(t_0 + \tau) &= v_0 - a(\tau - t_0) \\ &= v_0 - \sqrt{2av_0\tau - 2a[d(0) - d_0]}. \end{aligned}\tag{22}$$

That is, the greater the initial distance, the later will the vehicle start to decelerate and the larger will the resulting velocity be. Note that, according to the gap acceptance rules of pedestrians outlined in Sec. II C, the shortest distance to a moving vehicle at which pedestrians enter the road, is given by $\sigma\tau v$.

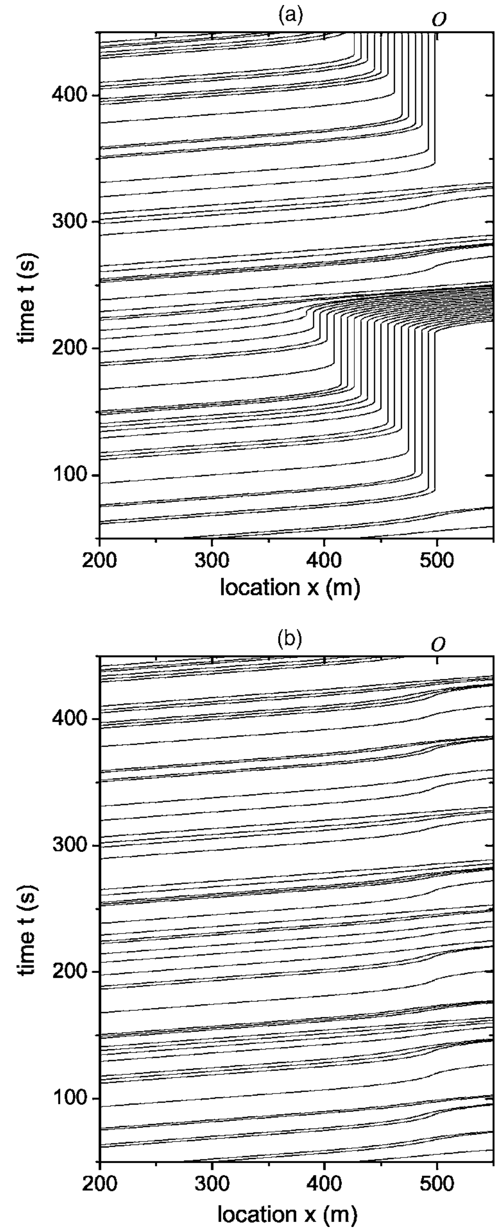

FIG. 1. (a) Representative space-over-time plot of vehicle trajectories for careful drivers and the pedestrian safety factor $\sigma = 1.05$. Pedestrians may stop cars, which causes vehicle queues. These suppress the crossing of newly arriving pedestrians until the vehicle queue has completely dissolved. (b) Representative space-over-time plot of vehicle trajectories for the larger safety factor $\sigma = 1.25$, for which pedestrians use large gaps only. As a consequence, pedestrians do not stop cars completely when they cross the street, and no vehicle queues are formed.

B. Average delay to vehicles

Let us denote by $v_{\min}$ the minimum velocity before the car accelerates again. If only one pedestrian obstructs the car, we have $v_{\min} = v(\tau)$, as calculated above. The time delay to the car compared to a movement with the free velocity v_0 can be calculated as the distance $2(v_0 - v_{\min})^2/(2a)$ traveled less, divided by the desired velocity v_0, which results in

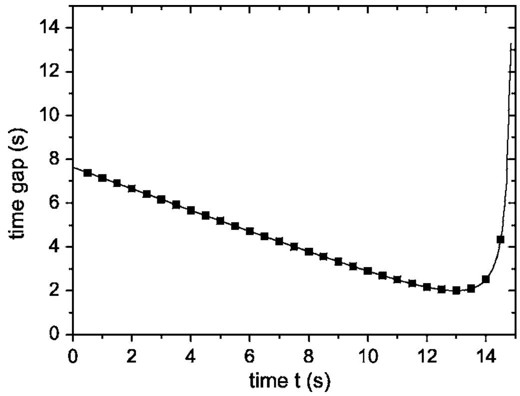

FIG. 2. Time-dependent time to collision $\Delta t(t)=d(t)/v(t)$ for careful drivers [see Eq. (18)], when pedestrians would enter the road with probability $p=1$ and $\sigma=1.05$ (symbols, numerically determined values; solid line, analytical formulas). Due to the braking maneuver, the time to collision goes down in the beginning, but it grows again later on, as the vehicle comes to rest at the finite distance $d(t)=d_0$ to the pedestrian.

$$\Delta t_{\mathrm{br}} = \frac{(v_0 - v_{\min})^2}{a v_0}. \tag{23}$$

If the vehicle is stopped, the time lost by the acceleration and deceleration process amounts to v_0/a. On top of this, we have to add the average waiting time t_{w}. This can be obtained as follows: If Δt_1 denotes the waiting time of the first stopped vehicle, the number of vehicles queuing up behind it until the first car in the queue starts to accelerate is given by $\rho_{\mathrm{jam}} C \Delta t_1$. The delay of the last vehicle in the queue is the queue length $l = C\Delta t_1$, divided by the queue resolution speed c. As the waiting time between the first and the last vehicle in the queue progresses approximately linearly, their cumulative waiting time is given by

$$\frac{\rho_{\mathrm{jam}} C \Delta t_1}{2}\left(\Delta t_1 + \frac{C\Delta t_1}{c}\right) = \frac{\rho_{\mathrm{jam}} C (\Delta t_1)^2}{2}\left(1 + \frac{C}{c}\right). \tag{24}$$

Moreover, up to the time point when the queue formed within the stopping time Δt_1 has resolved, another $\rho_{\mathrm{jam}} l C/(c-C)$ vehicles have joined the queue [cf. formula (1.48) in Ref. [25]]. While the waiting time of the first of these additional vehicles is approximately $l/c = C\Delta t_1/c$ (as the one of the last vehicle in the first part of the queue), the waiting time of the last vehicle is basically zero, which implies a cumulative waiting time of

$$\frac{\rho_{\mathrm{jam}} C \Delta t_1}{2}\left(\frac{C}{c-C}\frac{C\Delta t_1}{c} + 0\right) = \frac{\rho_{\mathrm{jam}} C (\Delta t_1)^2}{2}\frac{C^2}{c^2 - cC}. \tag{25}$$

Adding this to Eq. (24) gives the cumulative waiting time

$$t_{\mathrm{c}} = \frac{\rho_{\mathrm{jam}}}{2}(\Delta t_1)^2 \frac{cC}{c-C}, \tag{26}$$

which grows quadratically in Δt_1 (see Fig. 3).

Finally, dividing this result by the total number $C\Delta t_1[1 + C/(c-C)]$ of vehicles yields a very simple relationship for

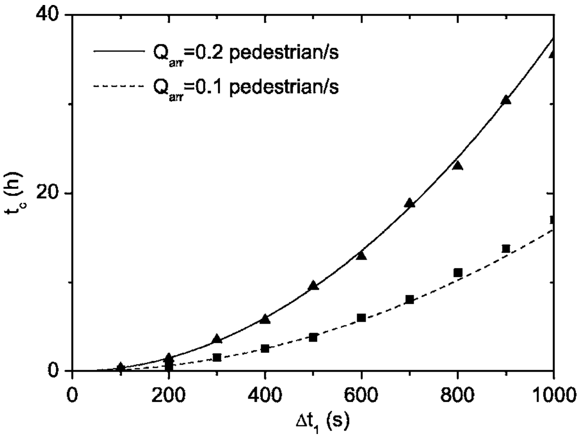

FIG. 3. Average of the cumulative waiting times t_{c} of vehicles as a function of the time period Δt_1 the first vehicle in the queue has to wait, for different values of the vehicle arrival rate Q_{arr} [see Eq. (6)] (symbols, numerically determined values; parabolic curves, analytical formula).

the average waiting time, which is just given as the average waiting time of the first and the last queued vehicles:

$$t_{\mathrm{w}} = \frac{\Delta t_1}{2}. \tag{27}$$

However, the estimation of the waiting time Δt_1 of the first stopped vehicle is rather difficult (see Sec. III E).

C. Determination of the transition point to alternating flows

The long vehicle and pedestrian queues required for pronounced oscillations in the pedestrian and vehicle flows can only occur if vehicles can be completely stopped by pedestrians. This cannot happen, if the safety factor σ of pedestrians is large enough. For small values of σ, however, there exists a time point t_-, after which the safety criterion (11) prohibits a further entering of pedestrians into the road. This time point is given by the earlier time satisfying the critical safety condition $\Delta t(t_{\mp}) = \sigma\tau$. Together with the expressions for the times to collision in Sec. III A, this eventually implies

$$t_{\mp} - t_0 = \frac{v_0}{a} - \sigma\tau \mp \sqrt{(\sigma\tau)^2 - \frac{2d_0}{a}} \tag{28}$$

for careful drivers. t_+ is the first time point at which pedestrians may reenter the road again, as the time to collision $\Delta t(t)$ increases close to the crossing point [see formula (18)]. The car reaches its minimum possible velocity a time period τ after t_-, i.e., after the latest entering pedestrian has left the road at time $t_- + \tau$. With Eq. (28) this implies

$$v(t_- + \tau) = a\tau(\sigma - 1) + \sqrt{(a\sigma\tau)^2 - 2ad_0} \tag{29}$$

for careful drivers. For aggressive drivers, we have to replace d_0 by $d(t_0) - v_0^2/(2a)$. To exclude stopped vehicles, on the one hand, this minimum velocity should be positive, i.e.,

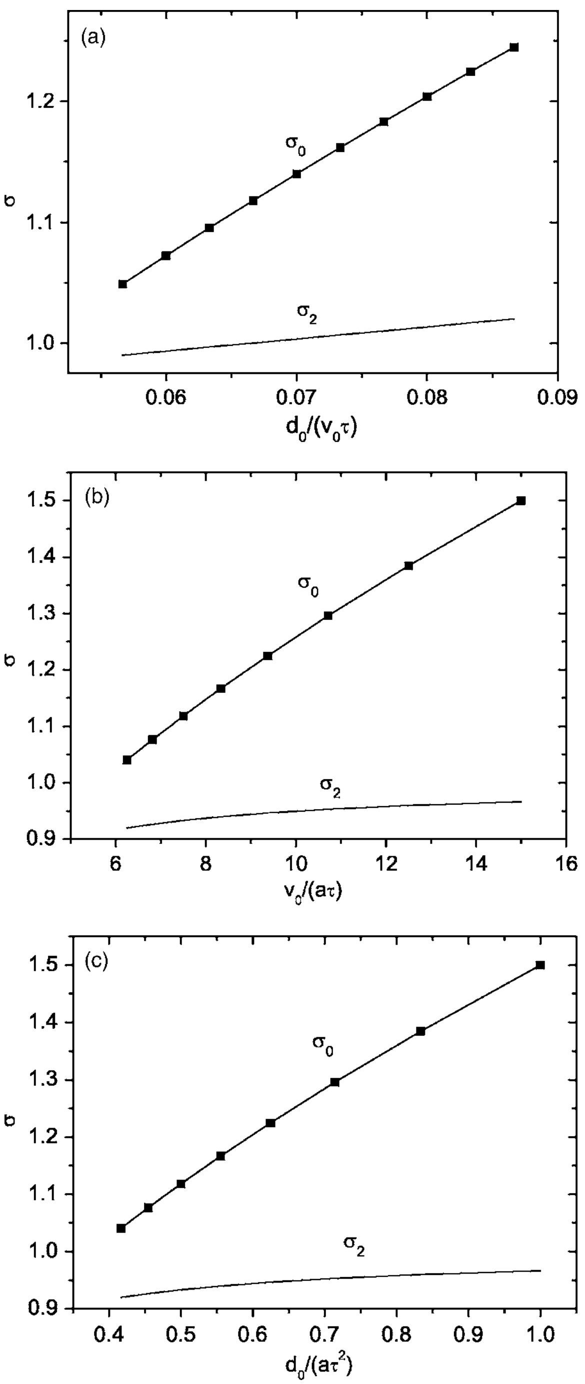

FIG. 4. Transition point σ_0 to alternating vehicle and pedestrian flows as a function of the dimensionless parameters (a) $x_1 = d_0/(v_0\tau)$ obtained for $d_0 \in [1.6\text{ m}, 2.5\text{ m}]$, (b) $x_2 = v_0/(a\tau)$ obtained for $a \in [0.5\text{ m/s}^2, 1.2\text{ m/s}^2]$, and (c) $x_3 = d_0/(a\tau^2)$ obtained for $a \in [0.5\text{ m/s}^2, 1.2\text{ m/s}^2]$ in comparison with the lower limit σ_2 of reasonable safety factors [see Eq. (13)] (symbols, numerically determined values; solid lines, analytical formula). Note that the value of σ_0 is constant for $x_3 = x_1 x_2 = d_0/(a\tau^2) = \text{const}$.

$$\left(\sigma - \frac{1}{2}\right) a\tau^2 > d_0. \tag{30}$$

On the other hand, vehicles could also be stopped by new pedestrians entering the road at a time $t \geqslant t_+$ that lies before the time $t_- + \tau$ at which the last pedestrian has left the road. Therefore, in order to avoid the stopping of vehicles by multiple crossing pedestrians, we have to demand

$$t_+ - t_- = 2\sqrt{(\sigma\tau)^2 - \frac{2d_0}{a}} > \tau, \tag{31}$$

which results in

$$\sigma > \sqrt{\frac{2d_0}{a\tau^2} + \frac{1}{4}}. \tag{32}$$

Together with condition (30) we find that a careful driver cannot be stopped completely under the condition

$$\sigma > \sigma_0 = \max\left(\frac{d_0}{a\tau^2} + \frac{1}{2}, \sqrt{\frac{2d_0}{a\tau^2} + \frac{1}{4}}\right). \tag{33}$$

At the value $\sigma = \sigma_0$, we expect a transition from continuous pedestrian and vehicle flows to alternating flows (see Fig. 4).

D. Calculation of earlier acceleration

Due to the statistical arrival of pedestrians with a rate $\lambda = p/dt$, it is likely that the time point $t_n \leqslant t_-$ of the last (nth) pedestrian entering the road is smaller than the latest *possible* entering time t_-. We are, therefore, interested in calculating the mean value $\langle t_- - t_n \rangle = t_- - \langle t_n \rangle$ of the time gap $t_- - t_n$, where n is an arbitrary integer number. For this, let $K = t_-/dt$ be the number of time steps between the first entering pedestrian and t_-. As the probability that no pedestrian enters in a time step is given by $r = (1-p)$, $(1-p)^K$ is the probability that nobody enters between $t=0$ and $t=t_-$, and $p(1-p)^{K-k}$ the probability that the last pedestrian enters at time $t_- - (K-k)dt = kdt$. The expected value of $t_- - t_n$ is

$$\begin{aligned}\frac{\langle t_- - t_n \rangle}{dt} &= K(1-p)^N + p\sum_{k=1}^{K} (K-k)(1-p)^{K-k} \\ &= Kr^K + (1-r)r\frac{d}{dr}\sum_{k=1}^{K} r^{K-k} \\ &= \frac{r(1-r^K)}{1-r} = (1-p)\frac{1-(1-p)^K}{p}\end{aligned} \tag{34}$$

(see Fig. 5). Therefore, if a vehicle is not stopped, instead of at time $t_- + \tau$, on average it already starts to accelerate at the earlier time

$$\langle t_n \rangle + \tau = t_- + \tau - (1-p)dt\frac{1-(1-p)^K}{p} \approx t_- + \tau - \frac{1-e^{-\lambda t_-}}{\lambda}, \tag{35}$$

where the last step of this calculation is based on Eq. (37) below. With this result, we can now estimate the expected value $\langle v_{\min} \rangle$ of the minimum vehicle velocity $v_{\min}$ entering Eq. (23):

$$\langle v_{\min}\rangle - v(t_- + \tau) = a(t_- - \langle t_n\rangle) = a(1 - e^{-\lambda t_-})/\lambda. \quad (36)$$

The higher velocity compared to $v(t_-+\tau)$ given by Eq. (29) originates from the earlier car acceleration, i.e., the shorter deceleration time.

E. Estimation of the waiting time of the first vehicle

If a vehicle is stopped by crossing pedestrians after a deceleration time v_0/a, it will have to wait until a time gap of duration τ in the pedestrian flow occurs. A gap of length $\tau = N\,dt$ or greater occurs with probability

$$(1-p)^N = (1-p)^{\tau/dt} = [\underbrace{(1-p)^{1/dt}}_{=e^{-\lambda}}]^\tau = e^{-\lambda\tau}, \quad (37)$$

i.e., gap sizes are exponentially distributed, as expected. Here, we have assumed $\ln(1-p) \approx -p$, but the required small values of $p=\lambda\,dt$ can be reached by sufficiently small choice of the time steps dt. In fact, in the following considerations, we will study the limit $dt \to 0$. Therefore, we have used the value $dt=0.001$ s in our computer simulations.

Now, let k_i denote the size of the ith gap $T_i = t_i - t_{i-1}$ (i.e., the number of time steps dt with no pedestrian arrival). Then, the expected value for the time period until a time gap of length $\tau = N\,dt$ or greater starts is given by

$$\sum_{n=0}^{\infty}\sum_{k_1=0}^{N}\cdots\sum_{k_n=0}^{N}(k_1+1+\cdots+k_n+1)(1-p)^{k_1}p\cdots(1-p)^{k_n}p(1-p)^N$$
$$\approx \sum_{n=0}^{\infty}\lambda^n\int_0^\tau dT_1\cdots\int_0^\tau dT_n(T_1+\cdots+T_n)e^{-\lambda(T_1+\cdots+T_n)}e^{-\lambda\tau} = -e^{-\lambda\tau}\sum_{n=0}^{\infty}\lambda^n\frac{d}{d\lambda}\prod_{i=1}^{n}\left(\int_0^\tau dT_i e^{-\lambda T_i}\right)$$
$$= -e^{-\lambda\tau}\sum_{n=0}^{\infty}\lambda^n\frac{d}{d\lambda}\left(\frac{1}{\lambda^n}(1-e^{-\lambda\tau})^n\right) = -e^{-\lambda\tau}\sum_{n=0}^{\infty}n(1-e^{-\lambda\tau})^n\left(\frac{\tau e^{-\lambda\tau}}{1-e^{-\lambda\tau}}-\frac{1}{\lambda}\right) = \left(\frac{1}{\lambda}-\frac{\tau e^{-\lambda\tau}}{1-e^{-\lambda\tau}}\right)e^{-\lambda\tau}s\frac{d}{ds}\sum_{n=0}^{\infty}s^n \quad \text{with } s = 1-e^{-\lambda\tau}$$
$$= \frac{1}{\lambda}[e^{\lambda\tau}-(1+\lambda\tau)] \approx \frac{\lambda\tau^2}{2}+\cdots. \quad (38)$$

That means the average waiting time for a gap of size τ or larger starts to grow linearly with the pedestrian arrival rate $\lambda = p/dt$ and quadratically with τ as long as these values are small, but it grows exponentially with $\lambda\tau$, when this value is large.

Note, however, that the waiting time is reduced by the gap between the time $M\,dt := v_0/a$ when the vehicle is stopped and the time $t_n \leqslant t_-$ at which the last pedestrian has entered the street before. Analogously to Sec. III D, we can calculate the expected value of this time gap as

$$\langle v_0/a - t_n\rangle = (1-p)\frac{1-(1-p)^M}{p/dt} = \frac{1-e^{-\lambda v_0/a}}{\lambda}, \quad (39)$$

since we have $(1-p)\to 1$ in the limit $dt \to 0$. As a consequence, the expected value $\langle \Delta t_1\rangle$ of the time period Δt_1 the first vehicle in the queue has to wait can be estimated as

$$\langle \Delta t_1\rangle = \frac{1}{\lambda}[e^{\lambda\tau}-(1+\lambda\tau)] - \frac{1-e^{-\lambda v_0/a}}{\lambda}$$
$$= \frac{1}{\lambda}(e^{\lambda\tau}+e^{-\lambda v_0/a}-2-\lambda\tau) \quad (40)$$

(see Fig. 6).

F. Average delay to pedestrians

After a time interval Δt_1, i.e., a time period τ after the last pedestrian has entered the road, the first vehicle in the queue can accelerate again. The time period available to pedestrians for crossing the road is $\Delta t_1 + v_0/a$, as the time period v_0/a required to stop the vehicle is usable as well. When the vehicle has started to move again, no pedestrian will be able to cross the road until the last vehicle of the queue has passed point O (at least if $\sigma\tau > T$). This time period can be calculated as [[25]

$$\Delta t_2 = C\,\Delta t_1\frac{1+c/v_0}{c-C}. \quad (41)$$

The expected value of Δt_2 is

$$\langle \Delta t_2\rangle = C\langle \Delta t_1\rangle\frac{1+c/v_0}{c-C} + \sqrt{\frac{2d_0}{a}} \quad (42)$$

(see Fig. 7), where we have also taken into account the additional amount $\sqrt{2d_0/a}$ required by a vehicle to get from $x = -d_0$ to point O.

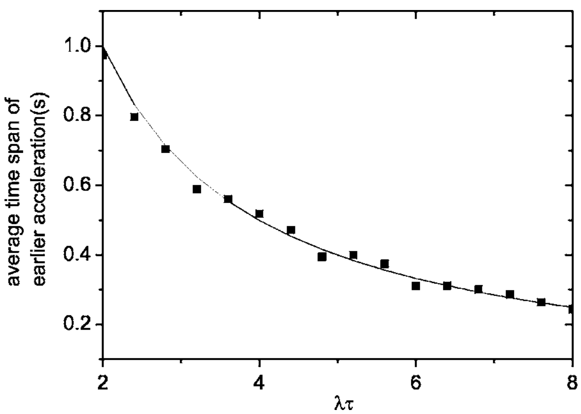

FIG. 5. Average time span $t_{-}-\langle t_n\rangle$ between the latest *possible* entering of the street by a pedestrian and the time point when the last pedestrian *actually* enters the street as a function of the scaled pedestrian arrival rate $\lambda\tau$ [see Eq. (36)] (symbols, numerically determined values; solid line, analytical formula).

After the last vehicle in the queue has passed, pedestrians have a chance to find a suitable gap of size $\sigma\tau$ or larger. A lower bound of the expected waiting time $\langle\Delta t_3\rangle$ for the occurrence of such a gap is calculated in the Appendix. In Fig. 8, we compare the resulting expression

$$\langle T'_{>}\rangle = \frac{1}{Q_{\rm arr}}\left(\frac{e^{Q_{\rm arr}(T_{*}-T_0)}}{1-Q_{\rm arr}T_0} - (1+Q_{\rm arr}T_{*})\right) \tag{43}$$

corresponding to Eq. (A7) with numerical results, where $T_0 = 1/Q_{\rm out}=T+(l_0+d_0)/v_0$ and $T_{*}=\sigma\tau+(l_0+d_0)/v_0$. This formula gives the expected waiting time $\langle\Delta t_3\rangle$ provided that the pedestrian arrives exactly at the time when a vehicle passes the crossing point and there are no vehicle time gaps smaller than T_0. Otherwise, it is an approximation, which neglects (1) the effect that pedestrians tend to arrive at the sidewalk *between* two vehicles (so that there is an incomplete intervehicle time gap, which must be added), and (2) the fact that the gaps of vehicles approaching the last, already accelerating vehicles in a queue may be smaller than T_0. These two

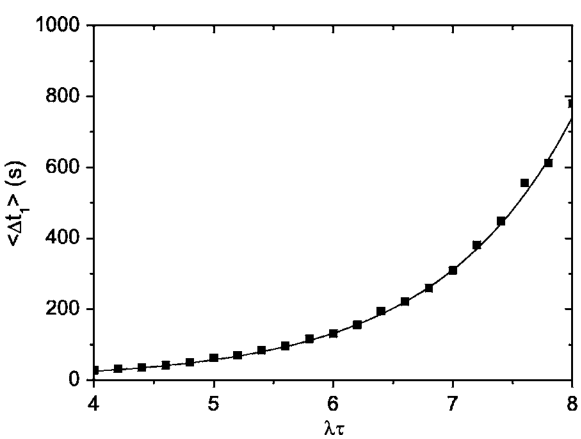

FIG. 6. Average waiting time $\langle\Delta t_1\rangle$ of the first vehicle in the queue as a function of the scaled pedestrian arrival rate $\lambda\tau$ (symbols, numerically determined values; solid line, analytical formula).

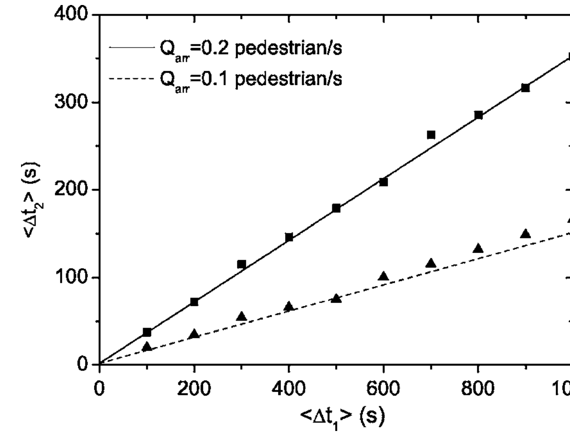

FIG. 7. Average time $\langle\Delta t_2\rangle$ needed to dissolve a vehicle queue as a function of the average time $\langle\Delta t_1\rangle$ for which the first vehicle has been waiting, for various values of the vehicle arrival rate $Q_{\rm arr}$ [see formula (6)] (symbols, numerically determined values; straight lines, analytical results).

effects increase the waiting time, i.e., $\langle\Delta T'_{>}\rangle \leqslant \langle\Delta t_3\rangle$.

During the waiting time $(\Delta t_2+\Delta t_3)$ of pedestrians, the expected number of arriving pedestrians is $\lambda(\Delta t_2+\Delta t_3)$. According to our model, all of these pedestrians will use the next occuring gap of size $\sigma\tau$ or larger to cross the street. We can assume that the waiting time of the last crossing pedestrian is approximately zero, while it is approximately $(\Delta t_2 +\Delta t_3)$ for the first one (when the pedestrian arrival rate is high enough). Therefore, the average delay can be approximated as $(\Delta t_2+\Delta t_3)/2$, and the cumulative delay of all waiting pedestrians amounts to

$$\frac{\lambda(\Delta t_2+\Delta t_3)^2}{2}. \tag{44}$$

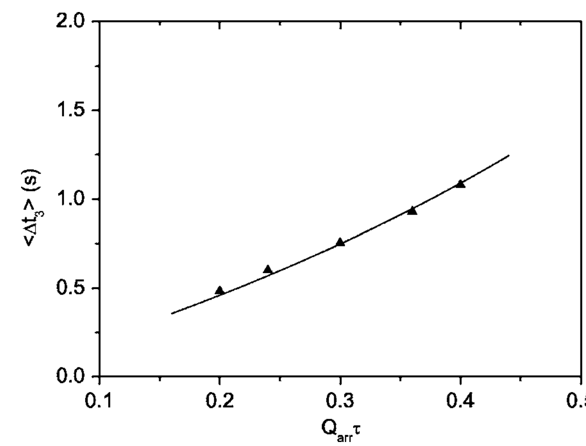

FIG. 8. Average waiting time $\langle\Delta t_3\rangle$ until a pedestrian enters the road after a vehicle queue has completely dissolved, as a function of the scaled vehicle arrival rate $Q_{\rm arr}\tau$ for $\sigma=1.05$. Our numerical simulation assumes the special case that a pedestrian arrives just at time $t=t_0$ when a vehicle of the queue passes the crossing point and vehicle time gaps are not smaller than T_0. The average time delay to this pedestrian is represented by triangles and compared to the analytical results of formula (43) (solid line).

IV. SUMMARY AND DISCUSSION

In this paper, we have proposed the continuous-in-space constant-deceleration-delayed-acceleration car-following model, in order to allow for the analytical calculation of the interactions between vehicles and crossing pedestrians under conditions of statistically distributed arrival times. Although the model is not made to reproduce all currently known properties of traffic flows, it does reflect some essential features such as accident-free driving, a constant outflow from traffic jams and a characteristic queue resolution speed.

We have distinguished two interaction modes between pedestrians and vehicles. (i) When pedestrians prefer large safety factors $\sigma > \sigma_0$, vehicles are not stopped, and pedestrians cross between moving vehicles either one by one or in small groups. (ii) When pedestrians keep small safety factors $\sigma < \sigma_0$, they may stop vehicles, which usually causes vehicle queues. Once a large enough gap between successive pedestrian arrivals occurs, cars will move again and prevent the crossing of pedestrians, until the last vehicle in the queue has passed the crossing point. This oscillatory dynamics with alternating flows of cars and pedestrians tends to be inefficient and related with long waiting times [21]. Empirical observations confirm the existence of such oscillatory patterns. Therefore, we are presently preparing for an empirical study of this phenomenon by means of a special laser-detector device.

In this contribution, we have calculated the threshold σ_0 between the oscillating and nonoscillating regimes. It turned out to be a function of $d_0/(a\tau^2)$ only, i.e., independent of the pedestrian or vehicle arrival rates λ, and Q_{arr}, the vehicle length l_0, the free vehicle velocity v_0, or the preferred time gap T, while the car deceleration a, the desired minimum distance d_0, and the crossing time τ matter. We have also calculated the expected waiting times of pedestrians and vehicles as a function of the arrival rates. The difficult step in gaining these results was the calculation of the first overcritical time gap and its expected value. This also required the determination of the vehicle gap distribution for deterministic, i.e., nonfluctuating vehicle interactions, while variations in the arrival times were taken into account [see Eq. (7)]. The formulas for the waiting time distributions can serve to judge under which conditions pedestrian and vehicle streams should be controlled (terminated) by traffic lights and when a self-organized crossing of streets is more efficient. Beyond this, our approach is generally expected to be useful for a better understanding of intersecting flows and certain conflicting processes. For example, a similar gap acceptance problem is found in lane-changing maneuvers, so that our formulas may help to calculate analytical formulas for lane-changing rates.

Regarding the choice of the behavior and parameters of cars drivers (careful or aggressive) and pedestrians (careful or daring), one may assume an evolutionary perspective: Due to a learning process during many vehicle-pedestrian interactions, an optimal behavior should emerge on the long run. It is, however, not yet clear whether there exists a state which is optimal for both, drivers and pedestrians. If not, one may consider the pedestrian-vehicle interactions as an example for a social dilemma [29], and the outcome may depend on details of the interactions. For example, if pedestrians would tend to use safety factors $\sigma < \sigma_0$, car drivers may react to this by an aggressive approaching behavior. This would make it difficult for pedestrians to stop vehicles. However, cars could still be successfully stopped if pedestrians learned to enter a road exactly with a time gap of $\sigma_2\tau$. In conclusion, there are always strategies to produce or avoid alternating pedestrian and vehicle flows, but the outcome depends always on the parameters of both, pedestrian and driver behavior. The determination of the optimal behavioral parameters and the evaluation of interactive parameter adaptations of pedestrians and vehicles will be left for a future study.

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APPENDIX: CALCULATION OF THE EXPECTED WAITING TIME FOR A SUITABLE GAP

Let $P(T')$ be the distribution density function of vehicle time gaps T'. Moreover, let

$$Q = \text{Prob}(T' \leqslant T_*) = \int_0^{T_*} dT' \, P(T') \tag{A1}$$

be the probability of finding a time gap $T' \leqslant T_*$ and

$$\overline{T'} := \langle T' \rangle_{T' < T_*} = \frac{1}{Q} \int_0^{T_*} dT' \, T' \, P(T') \tag{A2}$$

the expected value of time gaps that are smaller than T_*. Then, given that a car has just passed, the expected time until the first gap T' greater than T_* occurs is given by the expression

$$\langle T'_> \rangle = \sum_{n=0}^{\infty} n \overline{T'} Q^n (1 - Q), \tag{A3}$$

as an arbitrary number n of smaller gaps may occur with probability Q each, before a large enough gap occurs with probability $(1-Q)$. Here, we have used that the expected lengths $\overline{T'}$ of short gaps $T' \leqslant T_*$ just add up due to the assumption of independently and identically distributed time gaps T'. One can calculate

$$\langle T'_{>}\rangle = (1-Q)\overline{T'}\sum_{n=0}^{\infty} nQ^n$$

$$= (1-Q)\overline{T'}Q\frac{d}{dQ}\left(\frac{1}{1-Q}\right) = \frac{Q\overline{T'}}{1-Q} = \frac{\int_0^{T_*} dT' T' P(T')}{\int_0^{\infty} dT' P(T')}. \qquad \text{(A4)}$$

Inserting the vehicle time gap distribution (7) eventually gives

$$1 - Q = P(T' > T_*) = (1 - Q_{\mathrm{arr}}T_0)e^{-Q_{\mathrm{arr}}(T_* - T_0)} \qquad \text{(A5)}$$

and

$$Q\overline{T'} = \frac{1}{Q_{\mathrm{arr}}}[1 - (1 - Q_{\mathrm{arr}}T_0)(1 + Q_{\mathrm{arr}}T_*)\mathrm{e}^{-Q_{\mathrm{arr}}(T_* - T_0)}]. \qquad \text{(A6)}$$

This implies

$$\langle T'_{>}\rangle = \frac{1}{Q_{\mathrm{arr}}}\left(\frac{e^{Q_{\mathrm{arr}}(T_* - T_0)}}{1 - Q_{\mathrm{arr}}T_0} - (1 + Q_{\mathrm{arr}}T_*)\right). \qquad \text{(A7)}$$

The required minimum time gap for the crossing of a pedestrian between two successive vehicles is $T_* = \sigma\tau + (l_0 + d_0)/v$, while the preferred time gap between successive vehicles is $T_0 = T + (l_0 + d_0)/v$. Note that formula (38) for the expected waiting time of vehicles for a large enough gap in the pedestrian stream corresponds to the special case $T_0 = 0$ with $\lambda = Q_{\mathrm{arr}}$ and $T_* = \tau$.

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