Interactions between elliptic PDE’s and convex geometry

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Università degli Studi di Firenze

The Cologne Conference
on Nonlinear Differential Equations

On the occasion of Bernd Kawohl’s sixtieth birthday

Outline

▶ Convexity type properties of solutions of elliptic PDE's.
▶ Geometric inequalities enlightening the connections of PDE's with convex geometry.
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Let $u$ be the solution of a boundary value problem for an elliptic operator in an open set $\Omega \subset \mathbb{R}^n$. Assume that $\Omega$ has some geometric property related to convexity, e.g.:

▶ $\Omega$ is a bounded convex set;
▶ $\Omega$ is a convex ring, i.e. $\Omega = \Omega_2 \setminus \Omega_1$, $\Omega_1$ and $\Omega_2$ open, bounded, convex and $\Omega_2 \supset \Omega_1$;
▶ $\Omega$ is the exterior of a bounded convex set.

Question: How do these geometric features of $\Omega$ reflect on $u$? In general one may expect that $u$ is:

▶ convex, or concave (too much!);
▶ power-concave or log-concave;
▶ quasi-concave (i.e. $u$ has convex level sets).
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Summer Course, Cortona, 1993
Lecturers: G. Buttazzo and B. Kawohl
“It was twenty years ago today...”

(Courtesy of G. Buttazzo)
From my personal memories of the summer course
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_Happiness is warm gun_

[Lennon-McCartney, 1968]
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_Happiness is warm paper!
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Harmonic functions – the pioneering work of Gabriel

(1955) proved that if $G = G(x; y)$ is the Green function for the Laplace operator in a convex domain, then, for every fixed $y_0$ in the interior of $\Omega$, $x \mapsto G(x; y_0)$ has convex level sets. To prove this result he introduced the technique based on the quasi-concavity function, which has been subsequently used to prove many other results of the same form.
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Quasi-concavity of harmonic functions in convex rings

Thm. (Gabriel, 1957)

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open bounded convex sets, such that $\Omega_2 \supset \Omega_1$.

Let $u$ be the unique solution of

\[ \begin{align*}
\Delta u &= 0 \quad \text{in} \quad \Omega_2 \setminus \Omega_1, \\
               u &= 1 \quad \text{on} \quad \partial \Omega_1, \\
               u &= 0 \quad \text{on} \quad \partial \Omega_2.
\end{align*} \]

Then all the level sets of $u$ are convex.

The previous result can be extended to the case when the outer domain $\Omega_2$ is the whole space $\mathbb{R}^n$.

In this case the boundary condition on $\partial \Omega_2$ is replaced by a decay condition at infinity.
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The capacitary potential of a convex set

Thm. Let $\Omega \subset \mathbb{R}^n$ (n > 2) be an open, bounded, convex set, and let $u$ be the unique solution of

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\Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n \setminus \Omega, \\
u &= 1 \quad \text{on} \quad \partial \Omega, \\
\lim_{|x| \to \infty} |x|^{n-2} u(x) &= 0.
\end{align*}$$

Then all the level sets of $u$ are convex. $u$ is called the capacitary potential (or function) of $\Omega$.

Indeed the electrostatic capacity (or 2-capacity) of $\Omega$, $\text{Cap}(\Omega)$, can be expressed as

$$\text{Cap}(\Omega) = \int_{\mathbb{R}^n \setminus \Omega} |\nabla u|^2 \, dx = c_n \lim_{|x| \to \infty} |x|^{n-2} u(x).$$
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Extend $u \equiv 1$ in $\Omega$.

For $0 \leq t \leq 1$ set

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\Omega_t = \{ x \in \mathbb{R}^n | u(x) \geq t \},
\]

\[
\Omega^* = \text{convex hull of } \Omega_t,
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Define $u^*$ as the function such that

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If one can prove:

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By the Maximum Principle $0 \leq u \leq 1$. Extend $u \equiv 1$ in $\Omega$. For $0 \leq t \leq 1$ set

\[\Omega_t = \{x \in \mathbb{R}^n \mid u(x) \geq t\}, \quad \Omega_t^* = \text{convex hull of } \Omega_t.\]

Define $u^*$ as the function such that

\[\{x \in \mathbb{R}^n \mid u^*(x) \geq t\} = \Omega_t^* \Rightarrow u^* \geq u \text{ in } \mathbb{R}^n.\]

If one can prove:

\[u \geq u^* \Rightarrow u \equiv u^* \Rightarrow u \text{ is quasi-concave.}\]
$u^* \leq u$
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Note: $u^*$ coincides with $u$ on $\partial \Omega$ (as $\Omega$ is convex), and at infinity.
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\[ \Delta u^* \geq 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}. \]
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Fix \( t \in [0, 1] \) and \( \mu \in [0, 1] \), and consider

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\[
(1 - \mu)\Omega_t + \mu\Omega_t = \{(1 - \mu)x_0 + \mu x_1 \mid x_0, x_1 \in \Omega_t\}.
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Then

$$\Omega^*_t = \text{convex hull of } \Omega_t = \bigcup_{0 \leq \mu \leq 1} \Omega_{t,\mu}.$$
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Let $u_{t, \mu}^*$ be the function such that

$$\{x \in \mathbb{R}^n \mid u_{t, \mu}^*(x) \geq t\} = \Omega_{t, \mu}, \quad \forall \ t \in [0, 1].$$
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Let \( u^*_\mu \) be the function such that

\[ \{x \in \mathbb{R}^n | u^*_\mu(x) \geq t\} = \Omega_{t, \mu}, \quad \forall \ t \in [0, 1]. \]

Then

\[ u^*(x) = \sup_{0 \leq \mu \leq 1} u^*_\mu(x). \]
The crucial lemma and the conclusion of the proof

Lemma. For every $\mu\in[0,1]$ $\Delta u^\mu \geq 0$ in $\mathbb{R}^n \setminus \Omega$, in the viscosity sense.

The proof is based on the formula $u^\mu(x) = \sup \{ \min \{ u(x_0), u(x_1) \} | (1 - \mu)x_0 + \mu x_1 = x \}$.

Consequently we also have $\Delta u^\mu \geq 0$ in $\mathbb{R}^n \setminus \Omega$, as $u^\mu(x) = \sup_{0 \leq \mu \leq 1} u^\mu(x)$ is the supremum of subsolutions. □
The crucial lemma and the conclusion of the proof

**Lemma.** For every $\mu \in [0, 1]$

$$\Delta u^*_\mu \geq 0 \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \text{ in the viscosity sense.}$$
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**Lemma.** For every $\mu \in [0, 1]$

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The crucial lemma and the conclusion of the proof

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u^*(x) = \sup_{0 \leq \mu \leq 1} u^*_\mu(x)
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\[\square\]
A variant: two domains in place of one

Let $\Omega_0, \Omega_1 \subset \mathbb{R}^n$ be open, bounded and convex.

Fix $\mu \in [0, 1]$ and set $\Omega_{\mu} = (1 - \mu)\Omega_0 + \mu\Omega_1$.

Let $u_0, u_1$ and $u_{\mu}$ be the equilibrium potentials of $\Omega_0$, $\Omega_1$ and $\Omega_{\mu}$ respectively.

Define $u^*_{\mu}$ as the function such that

$$\{u^*_{\mu} \geq t\} = (1 - \mu)\{u_0 \geq t\} + \mu\{u_1 \geq t\},$$

for every $t \in [0, 1]$.

Lemma. $\Delta u^*_{\mu} \geq 0$ in $\mathbb{R}^n$ in the viscosity sense.

As $u^*_{\mu} \equiv u_{\mu}$ on $\partial \Omega_{\mu}$ and at infinity:

Corollary. $u_{\mu} \geq u^*_{\mu}$ in $\mathbb{R}^n \setminus \Omega_{\mu}$. 

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Let \( u_0, u_1 \) and \( u_\mu \) be the equilibrium potentials of \( \Omega_0, \Omega_1 \) and \( \Omega_\mu \) respectively. Define \( u^*_\mu \) as the function such that

\[
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A variant: two domains in place of one

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$$\Omega_\mu = (1 - \mu)\Omega_0 + \mu\Omega_1.$$  

Let $u_0, u_1$ and $u_\mu$ be the equilibrium potentials of $\Omega_0$, $\Omega_1$ and $\Omega_\mu$ respectively. Define $u^{\ast}_\mu$ as the function such that

$$\{u^{\ast}_\mu \geq t\} = (1 - \mu)\{u_0 \geq t\} + \mu\{u_1 \geq t\},$$

for every $t \in [0, 1]$.

**Lemma.**

$$\Delta u^{\ast}_\mu \geq 0 \text{ in } \mathbb{R}^n \setminus \overline{\Omega}_\mu$$

*in the viscosity sense.*
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Let $\Omega_0, \Omega_1 \subset \mathbb{R}^n$ be open, bounded and convex. Fix $\mu \in [0, 1]$ and set

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**Lemma.**

$$\Delta u^*_\mu \geq 0 \quad in \ \mathbb{R}^n \ \setminus \ \overline{\Omega}_\mu$$

*in the viscosity sense.*

As $u^*_\mu \equiv u_\mu$ on $\partial \Omega_\mu$ and at infinity:
A variant: two domains in place of one

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As $u^*_\mu \equiv u_\mu$ on $\partial\Omega_\mu$ and at infinity:

Corollary.

$$u_\mu \geq u^*_\mu \quad \text{in } \mathbb{R}^n \setminus \overline{\Omega}_\mu.$$
An inequality for the capacity

As

\[ u^* \mu(x) = \sup \left\{ \min \{ u_0(x_0), u_1(x_1) \} \mid (1 - \mu) x_0 + \mu x_1 = x \right\} \]

the previous result implies:

\[ u \mu((1 - \mu)x_0 + \mu x_1) \geq \min \{ u_0(x_0), u_1(x_1) \}, \forall x_0, x_1. \]

\[ \lim_{|x| \to \infty} |x|^{n-2} u \mu(x) \geq \min \left\{ \lim_{|x| \to \infty} |x|^{n-2} u_0(x), \lim_{|x| \to \infty} |x|^{n-2} u_1(x) \right\}. \]

As, for every \( \Omega \),

\[ \text{Cap}(\Omega) = c_n \lim_{|x| \to \infty} |x|^{n-2} u(x) \]

we get the inequality

\[ \text{Cap}((1 - \mu)\Omega_0 + \mu \Omega_1) \geq \min \{ \text{Cap}(\Omega_0), \text{Cap}(\Omega_1) \}. \]
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\[ u_\mu^*(x) = \sup\{\min\{u_0(x_0), u_1(x_1)\} \mid (1 - \mu)x_0 + \mu x_1 = x\}, \]

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Thm. Let $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, $n \geq 2$, be open, bounded and convex.

For every $\mu \in [0, 1]$, $\text{Cap}(\Omega_\mu) \geq \min\{\text{Cap}(\Omega_0), \text{Cap}(\Omega_1)\}$.

Using the homogeneity of capacity (order = $n - 2$), an easy argument gives the stronger inequality

\[
\left[ \text{Cap}(\Omega_\mu) \right]_n^{1/(n-2)} \geq (1 - \mu)\left[ \text{Cap}(\Omega_0) \right]_n^{1/(n-2)} + \mu\left[ \text{Cap}(\Omega_1) \right]_n^{1/(n-2)}.
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The first proof of this result is due to Borell (1983).

The above inequality bears a strong similarity to

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| (1 - \mu)\Omega_0 + \mu\Omega_1 |_n^{1/n} \geq (1 - \mu)|\Omega_0|_n^{1/n} + \mu|\Omega_1|_n^{1/n},
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**Thm.** Let \( \Omega_0, \Omega_1 \subset \mathbb{R}^n, n \geq 2, \) be open, bounded and convex. For every \( \mu \in [0, 1] \)

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The Brunn-Minkowski inequality

**Theorem.** For every $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, $n \geq 2$, compact and convex, and for every $\mu \in [0, 1]$,

$$| (1 - \mu) \Omega_0 + \mu \Omega_1 |_n \geq (1 - \mu) | \Omega_0 |_n + \mu | \Omega_1 |_n. $$

Equality holds if and only if $\Omega_0$ and $\Omega_1$ are homothetic (i.e., they coincide up to a translation and a dilation).

**Remarks.**
- BM inequality is one of the cornerstones in Convex Geometry, i.e., the theory of convex bodies (compact convex sets).
- In particular, it is a special case of a family of inequalities called the Aleksandrov-Fenchel inequalities.
- It is related to other geometric and analytic inequalities. It easily implies the isoperimetric inequality (at least for convex sets). It admits a functional form, called Prékopa-Leindler inequality, which is related to the (inverse) Young's convolution inequality.
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More on Brunn-Minkowski inequality

- It can be extended to measurable sets (with some caution), though equality conditions (convexity + homothety) remain the same.

- It admits many "relative" inequalities.

**Definition.** A functional $F$ defined on the class of (open/closed) bounded and convex sets, homogeneous of order $\alpha \neq 0$ with respect to dilations, is said to verify a BM inequality if

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In other words: $F^{1/\alpha}$ is concave.

Examples from convex geometry: volume (Lebesgue measure), perimeter, intrinsic volumes, mixed volumes ...

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Extensions

Quasi-concavity of harmonic functions in convex rings and in the exterior of convex domains, (and BM inequalities for related functionals, when they exist) have been re-proved and/or extended by a number of authors.

▶ Lewis (1977) – extension of quasi-concavity to the $p$-Laplace operator.

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The first Dirichlet eigenvalue of $\Delta$

For $\Omega \subset \mathbb{R}^n$, let $\lambda_1(\Omega)$ be the first eigenvalue of the Laplace operator with Dirichlet boundary condition, and let $u$ be the corresponding eigenfunction, positive and suitably normalized:

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\begin{align*}
\Delta u &= -\lambda_1(\Omega)u, \\
    u &> 0, \quad \text{in } \Omega, \\
    u &= 0 \quad \text{on } \partial \Omega, \\
    \|u\|_{L^2(\Omega)} &= 1.
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$\Rightarrow$ Brascamp and Lieb (1976).

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The Brascamp-Lieb method (I)

The starting point is a parabolic problem:

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\begin{align*}
  &v_t = \Delta v 	ext{ in } \Omega \times \mathbb{R} \\
  &v(x, 0) = \phi(x) \quad \forall x \in \Omega \\
  &v(x, t) = 0 \quad \forall (x, t) \in \partial \Omega \times \mathbb{R},
\end{align*}
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where \( \phi \) is positive, log-concave, and vanishes on \( \partial \Omega \).

Thm. If \( \Omega \) is convex, then for every fixed \( t \)
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On the other hand

\[ v(x, t) = \sum_{i=1}^{\infty} \alpha_i e^{-\lambda_i t} u_i(x), \quad \forall x \in \Omega, \quad t \in (0, \infty) \]

where the \( \lambda_i \)'s are the eigenvalues of \( \Delta \) on \( \Omega \), \( u_i \)'s are the corresponding eigenfunctions and the \( \alpha_i \)'s are suitable coefficients (\( \alpha_1 > 0 \), by the initial condition \( v(\cdot, 0) > 0 \)).

Hence \( u(x) = \alpha_1 \lim_{t \to \infty} e^{\lambda_1 t} v(x, t) \forall x \in \Omega \), i.e. \( u \) is log-concave as point-wise limit of log-concave functions.

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As a by-product of the previous argument, the following inequality is achieved:

$$\lambda_1((1-s)\Omega_0 + s\Omega_1) \leq (1-s)\lambda_1(\Omega_0) + s\lambda_1(\Omega_1),$$

for every $\Omega_0$, $\Omega_1$, and for every $s \in [0,1]$.

Using again homogeneity of $\lambda_1$ (order = $-1/2$) we get a BM inequality for $\lambda_1$:

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$$[\lambda_1((1 - s)\Omega_0 + s\Omega_1)]^{-\frac{1}{2}} \geq (1 - s)[\lambda_1(\Omega_0)]^{-\frac{1}{2}} + s[\lambda_1(\Omega_1)]^{-\frac{1}{2}}.$$

In this case as well the same idea of the proof of a concavity-type property of the solution is used to prove a BM inequality for the relevant functional.
Extensions to other eigenvalue problems

- Log-concavity for the first (and only) eigenfunction of the $p$-Laplacian was proved by Sakaguchi (1987).
- BM inequality for the corresponding eigenvalue was proved by Cuoghi, Salani and C. (2006).
- BM inequality for the eigenvalue of the Monge-Ampère operator was shown by Salani (2005).
- Log-concavity of the eigenfunction, and BM for the corresponding eigenvalue, in case of Hessian operators (a class of fully non-linear elliptic operators, including Laplace and Monge-Ampère operators), have been established, in the three dimensional case, by Ma and Xu (2008) and Salani (2010).
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Final example: the torsion problem

Let $\Omega \subset \mathbb{R}^n$ be convex and bounded, and let $u$ be the unique solution of

$$\begin{align*}
\Delta u &= -1 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

The function $u$ determines the torsion $\tau(\Omega)$ of $\Omega$ via the formula

$$\tau(\Omega) = \int_{\Omega} |\nabla u|^2 \, dx.$$ 

Thm. ▶ If $\Omega$ is convex, then $\sqrt{u}$ is concave in $\Omega$.

▶ $\tau$ verifies a BM inequality ($\alpha = n + 2$).


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Concluding remarks

For $\lambda$ and $\tau$, BM holds also for non-convex sets, as in the case of the Lebesgue measure. The validity of BM for a variational functional looks strictly related to convexity-type properties of the solution of the relevant Euler-Lagrange equation. This is true also in the negative direction; e.g. the first non-trivial eigenvalue of $\Delta$ with Neumann boundary condition does not verify BM and the corresponding eigenfunction has no convexity properties.

In general BM for a functional $F$ seems to be related to the monotonicity of $F$ w.r.t. set inclusion. This is observed in (almost!) all known examples. Some results in this direction have been proved recently (Hug, Saorín and C., 2012), but only in restricted classes of functionals.
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